Wendell Fleming

Functions of Several Variables

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To Flo

Preface

The purpose of this book is to give a systematic development of differential and integral calculus for functions of several variables. The traditional topics from advanced calculus are included: maxima and minima, chain rule, implicit function theorem, multiple integrals, divergence and Stokes's theorems, and so on. However, the treatment differs in several important respects from the traditional one. Vector notation is used throughout, and the distinction is maintained between n-dimensional euclidean space E^n and its dual. The elements of the Lebesgue theory of integrals are given. In place of the traditional vector analysis in E^3, we introduce exterior algebra and the calculus of exterior differential forms. The formulas of vector analysis then become special cases of formulas about differential forms and integrals over manifolds lying in E^n.

The book is suitable for a one-year course at the advanced undergraduate level. By omitting certain chapters, a one semester course can be based on it. For instance, if the students already have a good knowledge of partial differentiation and the elementary topology of E^n, then substantial parts of Chapters 4, 5, 7, and 8 can be covered in a semester. Some knowledge of linear algebra is presumed. However, results from linear algebra are reviewed as needed (in some cases without proof).

A number of changes have been made in the first edition. Many of these were suggested by classroom experience. A new Chapter 2 on elementary topology has been added. Additional physical applications—to thermodynamics and classical mechanics—have been added in Chapters 6 and 8. Different proofs, perhaps easier for the beginner, have been given for two main theorems (the Inverse Function Theorem and the Divergence Theorem.)

The author is indebted to many colleagues and students at Brown University for their valuable suggestions. Particular thanks are due Hildegarde Kneisel, Scott Shenker, and Joseph Silverman for their excellent help in preparing this edition.

Wendell H. Fleming

Providence, Rhode Island
June, 1976

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Euclidean spaces 1

This book is concerned with the differential and integral calculus of functions of several variables. For this purpose one needs first to know some basic properties of euclidean space of arbitrary finite dimension n. We begin this chapter with a brief review of the real numbers and the elements of vector algebra and geometry of such spaces. Later in the chapter the concepts of neighborhood, open set, and closed set are introduced. These constitute the basis for studying what are called topological properties of n-dimensional space.

Format

The word "Theorem" has been reserved for what the author considers the most important results. Results of lesser depth or interest are labeled "Proposition." The symbol □ indicates the end of the proof of a theorem or proposition. Occasionally part of a proof is left to the reader as a homework exercise. The sections marked with an asterisk (*) may be omitted without disrupting the organization. References are given at the end of the book.

We presume that the reader is acquainted with the most elementary aspects of set theory. The symbols

$$\in, \notin, \cup, \cap, -, \subset$$

stand, respectively, for *is an element of*, *is not an element of*, *union*, *intersection*, *difference*, and *inclusion*. Sets ordinarily are denoted by capital italicized letters. A set is described either by listing its elements or by some property characterizing them. Thus $\{2, 5, 7\}$ is the set whose elements are the three numbers 2, 5, and 7. If S is a set and π a property pertaining to elements of S, then $\{p \in S : \pi\}$ denotes *the set of all* $p \in S$ *with property* π. For example, if $Z = \{1, 2, \ldots\}$ is the set of natural numbers, then $S = \{x \in Z : x = 2y - 1$

for some $y \in Z\}$ is the set of odd natural numbers. The set $\{x \in Z : x^2 = 3\}$ is the empty set. The set $\{x \in Z : x(x-1) = x^2 - x\}$ is all of Z.

When the set S in question is clear from the context, we abbreviate by writing simply $\{p : \pi\}$.

1.1 The real number system

While calculus has been motivated in large part by problems from geometry and physics, its foundations rest upon the idea of number. Therefore a thorough treatment of calculus should begin with a study of the real numbers. The real number system satisfies axioms about arithmetic and order, which express properties of numbers with which everyone is familiar from elementary mathematics.

We list these properties as Axioms I and II.

Axiom I

(a) Any two real numbers have a sum $x + y$ and a product xy, which are also real numbers. Moreover,

Commutative law	$x + y = y + x$,	$xy = yx$,
Associative law	$x + (y + z) = (x + y) + z$,	$x(yz) = (xy)z$,
Distributive law	$x(y + z) = xy + xz$	

for every x, y, and z.

(b) There are two (distinct) real numbers 0 and 1, which are identity elements under addition and multiplication, respectively:

$$x + 0 = x, \qquad x1 = x$$

for every x.

(c) Every real number x has an inverse $-x$ with respect to addition, and if $x \neq 0$, an inverse x^{-1} with respect to multiplication:

$$x + (-x) = 0, \qquad xx^{-1} = 1.$$

Axiom II. There is a relation $<$ between real numbers such that:

(a) For every pair of numbers x and y, exactly one of the following alternatives holds: $x < y$, $x = y$, $y < x$.
(b) $w < x$ and $x < y$ imply $w < y$ (transitive law).
(c) $x < y$ implies $x + z < y + z$ for every z.
(d) $x < y$ implies $xz < yz$ whenever $0 < z$.

From Axioms I and II follow all of the ordinary laws of arithmetic. In algebra any set with two operations (usually called *addition* and *multiplication*) having the properties listed in Axiom I is called a *field*. A field is called *ordered* if there is in it a relation $<$ satisfying Axiom II.

The real numbers form an ordered field. However, this is by no means the only ordered field. For example, the rational numbers also form an ordered field. We recall that x rational means that $x = p/q$, where p and q are integers and $q \neq 0$. Yet another axiom is needed to characterize the real number system. This axiom can be introduced in several ways. Perhaps the simplest of these is Axiom III, stated below in terms of least upper bounds.

In Section 2.3 we state other axioms that turn out to be equivalent to Axiom III. We should warn the reader that Axiom III is more subtle than Axioms I and II, and that one becomes aware only gradually of its implications. However, this axiom is the foundation stone for some of the most important theorems in calculus.

Let S be a nonempty set of real numbers. If there is a number c such that $x \leq c$ for every $x \in S$, then c is called an *upper bound* for S. If c is an upper bound for S and $b \geq c$, then b is also an upper bound for S.

Axiom III. Any set S of real numbers that has an upper bound has a least upper bound.

The least upper bound for S is denoted by $\sup S$. If S has no upper bound, then we set $\sup S = +\infty$.

A number d is a lower bound for S if $d \leq x$ for every $x \in S$. If S has a lower bound, then (Problem 2) S has a greatest lower bound. It is denoted by $\inf S$. If S has no lower bound, then we set $\inf S = -\infty$.

EXAMPLE 1. Let $S = \{1, 2, 3, \ldots\}$, the set of positive integers. Then $\sup S = +\infty$ and $\inf S = 1$.

EXAMPLE 2. Let a and b be real numbers with $a < b$. The sets

$$[a, b] = \{x : a \leq x \leq b\}, \qquad (a, b) = \{x : a < x < b\},$$

$$[a, b) = \{x : a \leq x < b\}, \qquad (a, b] = \{x : a < x \leq b\}$$

are called *finite intervals* with endpoints a and b. The first of these intervals is called *closed*, the second *open*, and the last two, *half-open*. In each instance b is the least upper bound and a is the greatest lower bound.

In the same way, the *semiinfinite intervals*

$$[a, \infty) = \{x : x \geq a\}, \qquad (a, \infty) = \{x : x > a\}$$

are called closed and open, respectively, and have a as greatest lower bound. The corresponding intervals $(-\infty, b]$, $(-\infty, b)$ have b as least upper bound.

Let S be a set that has an upper bound. Example 2 shows that the number $\sup S$ need not belong to S. If $\sup S$ does happen to be an element of S, then it is the largest element of S and we write "max S" instead of "sup S." Similarly, if S is bounded below and $\inf S$ is an element of S, then we write for it "min S."

EXAMPLE 3. Let $S = \{x : x^2 < 2 \text{ and } x \text{ is a rational number}\}$. Then $\sqrt{2} = \sup S$ and $-\sqrt{2} = \inf S$. Since $\sqrt{2}$ is not a rational number, this example shows that the least upper bound axiom would no longer hold if we replaced the real number system by the rational number system.

EXAMPLE 4. Let $S = \{\sin x : x \in [-\pi, \pi]\}$. Then $-1 = \min S$, $1 = \max S$.

The real number system also satisfies the *archimedean property*. This means that for every $\varepsilon > 0, x > 0$ there exists a positive integer m such that $x < m\varepsilon$. To prove it, suppose to the contrary that for some pair ε, x of positive numbers, $m\varepsilon \leq x$ for every $m = 1, 2, \ldots$. Then x is an upper bound for the set $S = \{\varepsilon, 2\varepsilon, 3\varepsilon, \ldots\}$. Let $c = \sup S$. Then $(m + 1)\varepsilon \leq c$ and therefore $m\varepsilon \leq c - \varepsilon$, for each $m = 1, 2, \ldots$. Hence $c - \varepsilon$ is an upper bound for S smaller than $\sup S$, a contradiction. This proves the archimedean property.

We shall not prove that there actually is a system satisfying Axioms I, II, and III. There are two well-known methods of constructing the real number system, starting from the rational numbers. One of them is the method of Dedekind cuts and the other is Cantor's method of Cauchy sequences.

Axioms I, II, and III characterize the real numbers; in other words, any two systems satisfying these three axioms are essentially the same. To put this more precisely in algebraic language, any two ordered fields satisfying Axiom III are isomorphic.

For proofs of these facts, refer to the book by Birkhoff and McLane [2, Chapter III].

PROBLEMS

1. Find the least upper bound $\sup S$ and greatest lower bound $\inf S$ of each of the following sets:
   (a) $\{x : x^2 - 3x + 2 < 0\}$.
   (b) $\{x : x^3 + x^2 - 2x \leq 2\}$.
   (c) $\{\sin x + \cos x : x \in [0, \pi]\}$.
   (d) $\{x \exp x : x < 0\}$. [*Note*: exp denotes the exponential function, $\exp x = e^x$, where e is the base for natural logarithms.]

   State whether $\sup S$ and $\inf S$ are elements of S.

2. Let $T = \{x : -x \in S\}$. Show that $-\sup T = \inf S$.

3. Let x and y be real numbers with $x < y$. Show that there is a rational number z such that $x < z < y$. [*Hint*: By the archimedean property there is a positive integer q such that $q^{-1} < y - x$. Let $z = p/q$, where p is the smallest positive integer such that $qx < p$.]

[*Note*: In this book, "Show that ..." and "Prove that ..." both mean "give a valid mathematical proof."]

1.2 Euclidean E^n

In this book we denote the real number system by E^1. Let us now define the space E^n, whose elements are n-tuples of real numbers. The elements of E^n will be called vectors.

Scalars and vectors

By *scalar* we mean a real number. In elementary mathematics a vector is described as a quantity that has both direction and length. Vectors are illustrated by drawing arrows issuing from a given point **0**. The point at the head of the arrow specifies the vector. Therefore we may (and shall) say that this point *is* the vector. Thus in two dimensions a vector is just a point (x, y) of the plane E^2. Vectors in E^2 are added by the parallelogram law, which amounts to adding corresponding components. Thus

$$(x, y) + (u, v) = (x + u, y + v).$$

The product of (x, y) by a scalar c is the vector (cx, cy). The zero vector is $(0, 0)$.

With this in mind, let us define the space E^n for any positive integer n. The elements of E^n are n-tuples $(x^1, \ldots, x^n)$ of real numbers. For short, we write $\mathbf{x}$ for the n-tuple $(x^1, \ldots, x^n)$. The notation $\mathbf{x} \in E^n$ means "$\mathbf{x}$ is an element of E^n." The elements of E^n are called *vectors*, and also *points*, depending on which term seems more suggestive in the context. Addition and scalar multiplication are defined in E^n as follows. If

$$\mathbf{x} = (x^1, \ldots, x^n), \qquad \mathbf{y} = (y^1, \ldots, y^n)$$

are any two elements of E^n, then

$$\mathbf{x} + \mathbf{y} = (x^1 + y^1, \ldots, x^n + y^n).$$

If $\mathbf{x} \in E^n$ and c is a scalar, then

$$c\mathbf{x} = (cx^1, \ldots, cx^n).$$

The *zero element* of E^n is

$$\mathbf{0} = (0, \ldots, 0).$$

With these definitions E^n satisfies the axioms for a vector space (Appendix A.1). The term "vector" is reserved for elements of E^n rather than those of any space satisfying these axioms.

The superscripts should not be confused with powers of x. For instance, $(x^i)^2$ means the square of the ith entry x^i of the n-tuple $(x^1, \ldots, x^n)$.

If $n = 1$ we identify the 1-tuple $\mathbf{x} = (x)$ with the scalar x. In this case addition and scalar multiplication reduce to ordinary addition and multiplication of real numbers. If $n = 2$ or 3 we usually write (x, y) or (x, y, z), as is commonly done in elementary analytic geometry, rather than (x^1, x^2) or

(x^1, x^2, x^3). Practically all of the theorems are stated and proved for arbitrary dimension n. However, the special cases $n = 2, 3$ will frequently appear in the examples and homework problems.

The notions of vector sum and multiplication by scalars determine the vector-space structure of E^n, but are not enough to define the concepts of distance and angle. These arise by introducing an inner product in E^n. An inner product assigns to each pair $\mathbf{x}$, $\mathbf{y}$ of vectors a scalar, and must have the four properties listed in Problem 2 at the end of the section. The one which we shall use is the *euclidean inner product*, denoted by $\cdot$,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x^i y^i.$$

The vector space E^n with this inner product is called *euclidean n-space*. Other inner products in E^n are considered in Section 2.11.

The *euclidean norm* (or *length*) of a vector $\mathbf{x}$ is

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2}.$$

It is positive, except when $\mathbf{x} = \mathbf{0}$, and satisfies the following two important inequalities. For every $\mathbf{x}, \mathbf{y} \in E^n$,

(1.1) $\quad |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$ $\quad$ (Cauchy's inequality),

(1.2) $\quad |\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ $\quad$ (triangle inequality).

PROOF OF (1.1). If $\mathbf{y} = \mathbf{0}$, then both sides of (1.1) are $\mathbf{0}$. Therefore let us suppose that $\mathbf{y} \neq \mathbf{0}$. For every scalar t,

$$(\mathbf{x} + t\mathbf{y}) \cdot (\mathbf{x} + t\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2t\mathbf{x} \cdot \mathbf{y} + t^2 \mathbf{y} \cdot \mathbf{y},$$

since the inner product is commutative and distributive [Problem 2, parts (a), (b), and (c)]. The left-hand side is $|\mathbf{x} + t\mathbf{y}|^2$, and $\mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2, \mathbf{y} \cdot \mathbf{y} = |\mathbf{y}|^2$. The right-hand side is quadratic in t and has a minimum when

$$t = t_0 = -\frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}.$$

Substituting this expression for t, we find that

$$0 \leq |\mathbf{x} + t_0 \mathbf{y}|^2 = |\mathbf{x}|^2 - \frac{|\mathbf{x} \cdot \mathbf{y}|^2}{|\mathbf{y}|^2},$$

or

$$|\mathbf{x} \cdot \mathbf{y}|^2 \leq |\mathbf{x}|^2 |\mathbf{y}|^2.$$

The last inequality is equivalent to Cauchy's inequality. □

From the proof we see that equality in Cauchy's inequality is equivalent to the fact that $|\mathbf{x} + t_0 \mathbf{y}| = 0$, that is, that $\mathbf{x} + t_0 \mathbf{y} = \mathbf{0}$. Thus, if $\mathbf{y} \neq \mathbf{0}$, $|\mathbf{x} \cdot \mathbf{y}| = |\mathbf{x}||\mathbf{y}|$ *if and only if* $\mathbf{x}$ *is a scalar multiple of* $\mathbf{y}$. If $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}|$, then $\mathbf{x}$ is a *nonnegative* scalar multiple of $\mathbf{y}$ (and conversely).

PROOF OF (1.2). We write, as before,

$$(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y}.$$

From Cauchy's inequality,

$$|\mathbf{x} + \mathbf{y}|^2 \leq |\mathbf{x}|^2 + 2|\mathbf{x}||\mathbf{y}| + |\mathbf{y}|^2,$$

or

$$|\mathbf{x} + \mathbf{y}|^2 \leq (|\mathbf{x}| + |\mathbf{y}|)^2.$$

This is equivalent to the triangle inequality. □

If $\mathbf{y} \neq \mathbf{0}$, equality holds in (1.2) if and only if $\mathbf{x}$ is a nonnegative scalar multiple of $\mathbf{y}$.

Using the fact that $|c\mathbf{x}| = |c||\mathbf{x}|$, one can easily prove by induction on m the following extension of the triangle inequality:

$$\left| \sum_{j=1}^{m} c^j \mathbf{x}_j \right| \leq \sum_{j=1}^{m} |c^j||\mathbf{x}_j| \tag{1.3}$$

for every choice of scalars $c^1, \ldots, c^m$ and of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$. We recall (Appendix A.1) that $\sum_j c^j \mathbf{x}_j$ is called a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_m$.

The *euclidean distance* between $\mathbf{x}$ and $\mathbf{y}$ is $|\mathbf{x} - \mathbf{y}|$. If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors, then

$$\mathbf{x} - \mathbf{z} = (\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z}).$$

Applying (1.2) to $\mathbf{x} - \mathbf{y}$ and $\mathbf{y} - \mathbf{z}$, we have

$$|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|, \tag{1.4}$$

which justifies the name "triangle inequality" (see Figure 1.1).

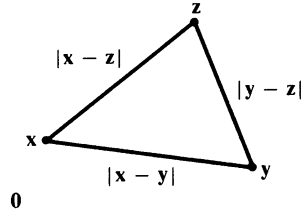


Figure 1.1

If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors, the *angle* θ between $\mathbf{x}$ and $\mathbf{y}$ is defined by the formula

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|}, \qquad 0 \leq \theta \leq \pi. \tag{1.5}$$

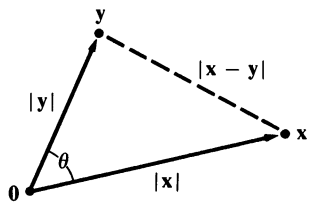


Figure 1.2

This formula agrees in dimensions $n = 2, 3$ with the one in elementary analytic geometry. The vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if $\mathbf{x} \cdot \mathbf{y} = 0$, in other words, if θ is a right angle. We have

$$|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\mathbf{x} \cdot \mathbf{y},$$

or

$$|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}||\mathbf{y}| \cos \theta,$$

which is the law of cosines from trigonometry (Figure 1.2).

Orthonormal bases

E^n is an n-dimensional vector space, and any linearly independent set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ with n elements is a basis for it.

A basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for E^n is called *orthonormal* if $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases} \qquad i, j = 1, \ldots, n.$$

The symbol δ_{ij} was first introduced by the mathematician Kronecker, and consequently is called *Kronecker's delta*. The unit coordinate vectors

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, \ldots, 0), \\ \mathbf{e}_2 &= (0, 1, 0, \ldots, 0), \\ &\vdots \\ \mathbf{e}_n &= (0, 0, \ldots, 0, 1) \end{aligned}$$

form the *standard* orthonormal basis for E^n. We have, for each $\mathbf{x} \in E^n$,

$$\mathbf{x} = (x^1, \ldots, x^n) = \sum_{i=1}^{n} x^i \mathbf{e}_i.$$

For instance, $(2, -1, 3) = 2\mathbf{e}_1 - \mathbf{e}_2 + 3\mathbf{e}_3$.

If $\mathbf{v}$ is any unit vector ($|\mathbf{v}| = 1$), then $\mathbf{x} \cdot \mathbf{v}$ is the *component of* $\mathbf{x}$ with respect to $\mathbf{v}$. Since $\mathbf{x} \cdot \mathbf{e}_i = x^i$, the components of $\mathbf{x}$ with respect to the standard orthonormal basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are $x^1, \ldots, x^n$. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is any orthonormal basis for E^n, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are mutually orthogonal unit vectors. Each $\mathbf{x} \in E^n$ can be uniquely represented as a linear combination

$$\mathbf{x} = c^1 \mathbf{v}_1 + \cdots + c^n \mathbf{v}_n. \tag{1.6}$$

Taking the inner product of each side with $\mathbf{v}_i$ and using the formula $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$, we obtain

$$\mathbf{x} \cdot \mathbf{v}_i = c^i.$$

The coefficients c^i (1.6) are just the components of $\mathbf{x}$ with respect to the orthonormal basis vectors.

PROBLEMS

1. Let $n = 4$, $\mathbf{x} = \mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_4 = (1, -1, 0, 2)$, $\mathbf{y} = 3\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 = (3, -1, 1, 1)$. Find $\mathbf{x} + \mathbf{y}$, $\mathbf{x} - \mathbf{y}$, $|\mathbf{x} + \mathbf{y}|$, $|\mathbf{x} - \mathbf{y}|$, $|\mathbf{x}|$, $|\mathbf{y}|$, $\mathbf{x} \cdot \mathbf{y}$. Verify (1.1) and (1.2) in this example.

2. Prove that the standard euclidean inner product in E^n has the following four properties:
   (a) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.
   (b) $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$.
   (c) $(c\mathbf{x}) \cdot \mathbf{y} = c(\mathbf{x} \cdot \mathbf{y})$.
   (d) $\mathbf{x} \cdot \mathbf{x} > 0$ if $\mathbf{x} \neq \mathbf{0}$.

3. Using Problem 2, show that

$$(\mathbf{w} + c\mathbf{x}) \cdot (\mathbf{y} + d\mathbf{z}) = \mathbf{w} \cdot \mathbf{y} + c\mathbf{x} \cdot \mathbf{y} + d\mathbf{w} \cdot \mathbf{z} + cd\mathbf{x} \cdot \mathbf{z}.$$

4. Show that $\sum_{i=1}^n |x^i| \leq \sqrt{n}|\mathbf{x}|$, for any $\mathbf{x} = (x^1, \ldots, x^n)$. [*Hint*: First suppose that $x^i \geq 0$. Use Equation (1.1) with $y^i = 1$.]

5. Show that $2|\mathbf{x}|^2 + 2|\mathbf{y}|^2 = |\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2$. What does this say about parallelograms (see Figure 1.3)?

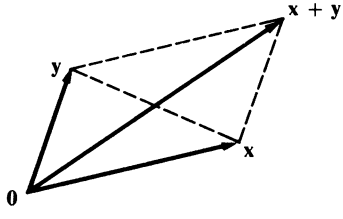


Figure 1.3

6. Show that $|\mathbf{x} + \mathbf{y}||\mathbf{x} - \mathbf{y}| \leq |\mathbf{x}|^2 + |\mathbf{y}|^2$ with equality if and only if $\mathbf{x} \cdot \mathbf{y} = 0$. What does this say about parallelograms?

7. Prove (1.3), using (1.2) and induction on m.

8. Let $n = 4$, and

$$\mathbf{v}_1 = \tfrac{1}{5}(3\mathbf{e}_1 + 4\mathbf{e}_3),$$

$$\mathbf{v}_2 = \tfrac{1}{5}(4\mathbf{e}_2 - 3\mathbf{e}_4),$$

$$\mathbf{v}_3 = (\sqrt{2}/10)(-4\mathbf{e}_1 + 3\mathbf{e}_2 + 3\mathbf{e}_3 + 4\mathbf{e}_4).$$

Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are mutually orthogonal unit vectors. Find a unit vector $\mathbf{v}_4$ such that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ form an orthonormal basis for E^4.

9. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for E^n, and let

$$C = \left\{ \mathbf{x} : \mathbf{x} = \sum_{i=1}^{n} t^i \mathbf{v}_i,\ 0 \leq t^i \leq 1 \quad \text{for} \quad i = 1, \ldots, n \right\}.$$

The set C is an *n-cube*. If each $t^i = 0$ or 1, $\mathbf{x}$ is called a *vertex* of C. What are the possible distances between vertices of C?

10. (*Gram–Schmidt process.*) Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for E^n. Let $\mathbf{v}_1 = |\mathbf{x}_1|^{-1}\mathbf{x}_1$, $\mathbf{y}_2 = \mathbf{x}_2 - (\mathbf{x}_2 \cdot \mathbf{v}_1)\mathbf{v}_1$, $\mathbf{v}_2 = |\mathbf{y}_2|^{-1}\mathbf{y}_2$, $\mathbf{y}_3 = \mathbf{x}_3 - (\mathbf{x}_3 \cdot \mathbf{v}_1)\mathbf{v}_1 - (\mathbf{x}_3 \cdot \mathbf{v}_2)\mathbf{v}_2$, $\mathbf{v}_3 = |\mathbf{y}_3|^{-1}\mathbf{y}_3, \ldots, \mathbf{v}_n = |\mathbf{y}_n|^{-1}\mathbf{y}_n$. Show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for E^n.

11. Let $\mathscr{V}$ be a vector subspace of E^n, of dimension k; and consider its *orthogonal complement*

$$\mathscr{V}^{\perp} = \{\mathbf{y} : \mathbf{y} \cdot \mathbf{x} = 0 \quad \text{for all} \quad \mathbf{x} \in \mathscr{V}\}.$$

(a) Find an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for E^n, such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for $\mathscr{V}$ and $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is a basis for $\mathscr{V}^{\perp}$. [*Hint*: Apply Problem 10 to a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ such that $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a basis for $\mathscr{V}$.]

(b) Show that each $\mathbf{x} \in E^n$ can be written in one and only one way as $\mathbf{x} = \mathbf{y} + \mathbf{z}$ with $\mathbf{y} \in \mathscr{V}$ and $\mathbf{z} \in \mathscr{V}^{\perp}$.

1.3 Elementary geometry of E^n

Such concepts as lines, planes, circles, and spheres in E^2 or E^3 have analogs in E^n for any dimension n. Let us begin with the concept of line in E^n.

Definition. Let $\mathbf{x}_1, \mathbf{x}_2 \in E^n$ with $\mathbf{x}_1 \neq \mathbf{x}_2$. The *line* through $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\{\mathbf{x} : \mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2,\ t \text{ any scalar}\}.$$

If we set $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_2$, then this can be rewritten as

$$\{\mathbf{x} : \mathbf{x} = \mathbf{x}_2 + t\mathbf{z},\ t \text{ any scalar}\}.$$

In the plane E^2 the vector equation $\mathbf{x} = \mathbf{x}_2 + t\mathbf{z}$ becomes

$$x = x_2 + t(x_1 - x_2), \qquad y = y_2 + t(y_1 - y_2),$$

which, in elementary analytic geometry, are called "parametric equations" of the line through (x_1, y_1) and (x_2, y_2).

The *line segment* joining $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\{\mathbf{x} : \mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2,\ t \in [0, 1]\},$$

where $[a, b]$ denotes the set of real numbers t such that $a \leq t \leq b$ (Section 1.1).

For example, if $t = \frac{1}{2}$, then $\mathbf{x}$ is the midpoint of the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$ (Figure 1.4). The points corresponding to $t = \frac{1}{3}, \frac{2}{3}$ trisect the line segment.

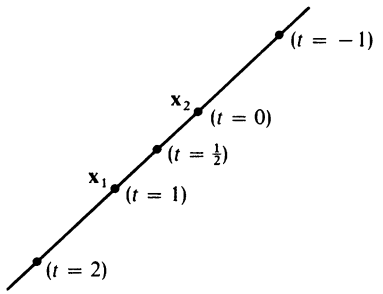


Figure 1.4

Hyperplanes, half-spaces

Given two points $\mathbf{x}_1, \mathbf{x}_2 \in E^n$, consider the set

$$P = \{\mathbf{x} : (\mathbf{x}_1 - \mathbf{x}_2) \cdot (\mathbf{x} - \mathbf{x}_2) = 0\}.$$

Such a set P is called a *hyperplane* through $\mathbf{x}_2$. Geometrically, one can think of P as follows. Let l denote the line through $\mathbf{x}_1$ and $\mathbf{x}_2$, and l' the line through $\mathbf{x}_2$ and $\mathbf{x}$. We call l and l' perpendicular if $(\mathbf{x}_1 - \mathbf{x}_2) \cdot (\mathbf{x} - \mathbf{x}_2) = 0$. Thus P is made up of all lines through $\mathbf{x}_2$ perpendicular to l. If we set $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_2$ and $c = \mathbf{z} \cdot \mathbf{x}_2$, then

$$(\mathbf{x}_1 - \mathbf{x}_2) \cdot (\mathbf{x} - \mathbf{x}_2) = \mathbf{z} \cdot \mathbf{x} - c.$$

Thus the definition of hyperplane can be conveniently formulated as follows:

Definition. A *hyperplane* in E^n is a set of the form $\{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} = c\}$, where $\mathbf{z} \neq \mathbf{0}$ and c are given.

A hyperplane is a point for $n = 1$, a line for $n = 2$, a plane for $n = 3$. For any nonzero scalar b, $\{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} = c\} = \{\mathbf{x} : (b\mathbf{z}) \cdot \mathbf{x} = bc\}$. Thus the vector $\mathbf{z}$ and scalar c defining a hyperplane are determined only up to a scalar multiple. If $c \neq 0$, we may, for instance, always take $c = 1$.

A hyperplane $P = \{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} = c\}$ is *parallel* to $P_1 = \{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} = c_1\}$ for any $c_1 \neq c$. If $c_1 = 0$, then P_1 contains 0, and P_1 is a vector subspace of E^n of dimension $n - 1$. This last statement follows from Problem 5.

EXAMPLE 1. Find the hyperplane P in E^4 which contains the four points $\mathbf{e}_1, \mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{e}_2 + 3\mathbf{e}_3, \mathbf{e}_3 + 4\mathbf{e}_4$. Every $\mathbf{x} \in P$ must satisfy the equation $\mathbf{z} \cdot \mathbf{x} = c$, where $\mathbf{z}$ and c must be found. Taking in turn $\mathbf{x} = \mathbf{e}_1$, $\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2, \ldots$, we obtain

$$c = \mathbf{z} \cdot \mathbf{e}_1 = z^1, \qquad c = \mathbf{z} \cdot (\mathbf{e}_1 + 2\mathbf{e}_2) = z^1 + 2z^2,$$
$$c = \mathbf{z} \cdot (\mathbf{e}_2 + 3\mathbf{e}_3) = z^2 + 3z^3, \qquad c = \mathbf{z} \cdot (\mathbf{e}_3 + 4\mathbf{e}_4) = z^3 + 4z^4.$$

From these equations, the components $z^1, \ldots, z^4$ of the vector $\mathbf{z}$ satisfy

$$z^1 = c, \qquad z^2 = 0, \qquad z^3 = \frac{c}{3}, \qquad z^4 = \frac{c}{6}.$$

Taking for convenience $c = 6$, we have

$$P = \{\mathbf{x} : 6x^1 + 2x^3 + x^4 = 6\}.$$

Definition. Let $\mathbf{z} \neq \mathbf{0}$. A *closed half-space* is a set of the form $\{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} \geq c\}$, and an *open half-space* is a set of the form $\{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} > c\}$ (see Figure 1.5).

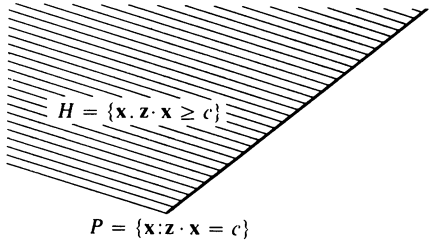


Figure 1.5

A set H of the form $\{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} \leq c\}$, $\mathbf{z} \neq \mathbf{0}$, is a closed half-space, since H is also $\{\mathbf{x} : (-\mathbf{z}) \cdot \mathbf{x} \geq -c\}$. The same remark applies to open half-spaces. A hyperplane $P = \{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} = c\}$ divides E^n into two half-spaces. More precisely, $E^n - P$ is the union of the open half-spaces

$$\{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} > c\} \quad \text{and} \quad \{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} < c\}.$$

Spheres and spherical balls

Given $\mathbf{x}$ and $\delta > 0$, the set $\{\mathbf{y} : |\mathbf{y} - \mathbf{x}| = \delta\}$ is called the $(n-1)$-*sphere* with center $\mathbf{x}$ and radius δ. For $n = 1$, it consists of two points $x \pm \delta$; for $n = 2$, it is a circle, and for $n = 3$ a sphere. The set $\{\mathbf{y} : |\mathbf{y} - \mathbf{x}| < \delta\}$ is the *open spherical n-ball* with center $\mathbf{x}$ and radius δ. The corresponding set $\{\mathbf{y} : |\mathbf{y} - \mathbf{x}| \leq \delta\}$ is called a *closed spherical n-ball.*

The terms "open" and "closed" are justified in Section 1.4. It turns out that an open half-space or spherical n-ball is an open set, and a closed half-space or spherical n-ball is a closed set.

Convex sets

This concept is defined as follows.

Definition. Let $K \subset E^n$. Then K is a *convex* set if the line segment joining any two points of K is contained in K (Figure 1.6).

Convex Not convex

Figure 1.6

E^n itself is a convex set. The empty set and sets with just one point trivially satisfy the definition; hence they are convex. The reader should be able to think of several kinds of geometric objects such as lines, planes, spherical balls, regular solids, and so on, which appear to be convex sets. However, geometric intuition is not always a reliable guide, especially in four or more dimensions. In any case, intuition is no substitute for a proof that the set in question is actually convex. To show that a set K is convex directly from the definition, we must verify that for every $\mathbf{x}_1, \mathbf{x}_2 \in K$ and $t \in [0, 1]$, the point $\mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2$ also belongs to K. In the definition, we assumed that $\mathbf{x}_1 \neq \mathbf{x}_2$. But if $\mathbf{x}_1 = \mathbf{x}_2$, it is trivial that $\mathbf{x} \in K$, since $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2$.

EXAMPLE 2. Any closed half-space is a convex set. Let $H = \{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} \geq c\}$, $\mathbf{z} \neq \mathbf{0}$. Let $\mathbf{x}_1, \mathbf{x}_2 \in H$ and $\mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2$, where $t \in [0, 1]$. Then $\mathbf{z} \cdot \mathbf{x}_1 \geq c$ and $\mathbf{z} \cdot \mathbf{x}_2 \geq c$. Since $t \geq 0$, $t\mathbf{z} \cdot \mathbf{x}_1 \geq tc$; and since $1 - t \geq 0$, $(1 - t)\mathbf{z} \cdot \mathbf{x}_2 \geq (1 - t)c$. Consequently,

$$\mathbf{z} \cdot \mathbf{x} = t\mathbf{z} \cdot \mathbf{x}_1 + (1 - t)\mathbf{z} \cdot \mathbf{x}_2 \geq tc + (1 - t)c = c.$$

This shows that $\mathbf{x} \in H$. Therefore H is a convex set. Similarly, any hyperplane is a convex set (Problem 12) and any open half-space is a convex set.

EXAMPLE 3. Let U be an open spherical n-ball, namely, $U = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < \delta\}$, for some $\mathbf{x}_0$ and $\delta > 0$. To show that U is a convex set, we proceed as in Example 2. Let $\mathbf{x}_1, \mathbf{x}_2 \in U$ and $\mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2$, where $t \in [0, 1]$. Then

$$|\mathbf{x}_1 - \mathbf{x}_0| < \delta, \qquad |\mathbf{x}_2 - \mathbf{x}_0| < \delta,$$
$$\mathbf{x} - \mathbf{x}_0 = t(\mathbf{x}_1 - \mathbf{x}_0) + (1 - t)(\mathbf{x}_2 - \mathbf{x}_0),$$
$$|\mathbf{x} - \mathbf{x}_0| \leq t|\mathbf{x}_1 - \mathbf{x}_0| + (1 - t)|\mathbf{x}_2 - \mathbf{x}_0| < \delta.$$

Hence $\mathbf{x} \in U$.

The convex subsets of E^n have many remarkable properties. There is an extensive mathematical literature devoted to them [6] [9] [13]. In Section 1.5 we give a brief introduction to the theory. The main result (Theorem 1.1) is the characterization of closed convex sets as intersections of closed half-spaces.

PROBLEMS

1. Let $n = 3$. Find the plane that contains the three points $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3 - 3\mathbf{e}_1$. Sketch its intersection with the first octant in E^3.

2. (a) Find the hyperplane in E^4 containing the four points $\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3, 3\mathbf{e}_4 - \mathbf{e}_2$.
   (b) Find the value of t for which $t(\mathbf{e}_1 - \mathbf{e}_2) + (1 - t)\mathbf{e}_4$ is in this hyperplane.

3. Let l denote the line in E^4 through $\mathbf{e}_1 - \mathbf{e}_3$ and $-\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_4$. Find the hyperplane P through $\mathbf{e}_1 - \mathbf{e}_3$ to which l is perpendicular.

4. Let $\mathscr{V} = \{(x, y, z) : 2x + 3y - z = 0\}$. Show that $\mathscr{V}$ is a 2-dimensional vector subspace of E^3, and find a basis for $\mathscr{V}$. ($\mathscr{V}$ is a vector subspace of E^n if $\mathbf{x}, \mathbf{y} \in \mathscr{V}$ imply $\mathbf{x} + \mathbf{y} \in \mathscr{V}$ and $c\mathbf{x} \in \mathscr{V}$ for any scalar c.)

5. Let $\mathscr{V} = \{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} = 0\}$, where $\mathbf{z} \neq \mathbf{0}$ is given. Show that $\mathscr{V}$ is an $(n - 1)$-dimensional vector subspace of E^n, and find a basis for $\mathscr{V}$.

6. Show that $\{\mathbf{x} : |\mathbf{x} - \mathbf{x}_1| = |\mathbf{x} - \mathbf{x}_2|\}$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are given points in E^n, is a hyperplane.

7. Show that $\{\mathbf{x} : |\mathbf{x} - \mathbf{x}_1| = c|\mathbf{x} - \mathbf{x}_2|\}$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are given points in E^n and $0 < c < 1$, is an $(n - 1)$-sphere.

8. Let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$ be such that $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_{n-1} - \mathbf{x}_0$ are linearly independent. Prove that there is exactly one hyperplane containing $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$.

9. A set $P = \{\mathbf{x} : \mathbf{z}_i \cdot \mathbf{x} = c_i \text{ for } i = 1, \ldots, n - k\}$, where $\mathbf{z}_1, \ldots, \mathbf{z}_{n-k}$ are linearly independent vectors, is called a *k-plane* in E^n. Let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k$ be such that $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_k - \mathbf{x}_0$ are linearly independent. Prove that there is exactly one k-plane containing $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k$.

10. Prove that any line in E^n is a convex set.

11. Show that K is a convex set by directly applying the definition. Sketch K in the cases $n = 1, 2, 3$.
   (a) $K = \{\mathbf{x} : |x^1| + \cdots + |x^n| \leq 1\}$.
   (b) $K = \{\mathbf{x} = c^1\mathbf{v}_1 + \cdots + c^n\mathbf{v}_n,\ 0 \leq c^i \leq 1 \text{ for } i = 1, \ldots, n\}$, where $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for E^n. This is the *n-parallelepiped* spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with $\mathbf{0}$ as a vertex.

12. Let P be a hyperplane. Prove that the line through any two points of P is contained in P. Why does this imply that P is a convex set?

1.4 Basic topological notions in E^n

We now introduce some basic concepts that are essential to a careful treatment of several-variable calculus. These concepts are developed further in Chapter 2.

Let us begin by making precise the idea of being "strictly inside" a set A, "strictly outside" A, or neither. Points with these properties will be called, respectively, interior, exterior, or frontier points. We first define the concept of neighborhood.

Definition. A *neighborhood* of a point $\mathbf{x}_0 \in E^n$ is a spherical n-ball $U = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < \delta\}$, where $\delta > 0$ is called the *radius* of U.

We also call U the δ-neighborhood of $\mathbf{x}_0$. Let A be any subset of E^n.

Definition. A point $\mathbf{x}$ is called *interior* to A if there is some neighborhood U of $\mathbf{x}$ such that $U \subset A$. If some neighborhood of $\mathbf{x}$ is contained in the complement $A^c = E^n - A$, then $\mathbf{x}$ is *exterior* to A. If every neighborhood of $\mathbf{x}$ contains at least one point of A and at least one point of A^c, then $\mathbf{x}$ is a *frontier* point of A.

An interior point of A necessarily is a point of A, and an exterior point must be a point of A^c. However, a frontier point may belong either to A or to A^c.

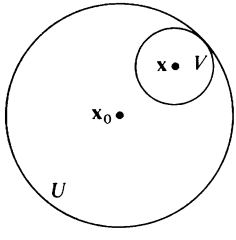


Figure 1.7

EXAMPLE 1. Let U be the δ-neighborhood of $\mathbf{x}_0$ (Figure 1.7). Let us show that every point of U is interior to U. Given $\mathbf{x} \in U$, let $r = \delta - |\mathbf{x} - \mathbf{x}_0|$ and let V be the r-neighborhood of $\mathbf{x}$. If $\mathbf{y} \in V$, then $\mathbf{y} - \mathbf{x}_0 = (\mathbf{y} - \mathbf{x}) + (\mathbf{x} - \mathbf{x}_0)$ and by the triangle inequality

$$|\mathbf{y} - \mathbf{x}_0| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{x}_0|,$$
$$|\mathbf{y} - \mathbf{x}_0| < r + |\mathbf{x} - \mathbf{x}_0| = \delta.$$

Hence $\mathbf{y} \in U$. This shows that $V \subset U$. Similarly, every point $\mathbf{x}$ such that $|\mathbf{x} - \mathbf{x}_0| > \delta$ is exterior to U. If $|\mathbf{x} - \mathbf{x}_0| = \delta$, $\mathbf{x}$ is a frontier point.

EXAMPLE 2. Let $n = 1$. Then neighborhoods are open intervals $(x_0 - \delta, x_0 + \delta)$. Let A be the set of all rational numbers. Any open interval contains both rational and irrational numbers. Hence every point of E^1 is a frontier point of A.

Definition. The *interior* of a set A is the set of all points interior to A. It is denoted by int A. The set of all frontier points of A is the *boundary* (or *frontier*) of A, and is denoted by fr A. The set $A \cup \text{fr } A$ is the *closure* of A and is denoted by cl A.

In Example 1, int $U = U$ and fr U is the $(n-1)$-dimensional sphere of radius δ. In Example 2, int A is the empty set, and fr $A = \text{cl } A = E^1$.

EXAMPLE 3. Let $A = E^n$. Then int $E^n = \text{cl } E^n = E^n$ and fr E^n is the empty set.

EXAMPLE 4. Let A be a subset of E^1, which has an upper bound. Then $\sup A \in \text{fr} A$, where $\sup A$ denotes the least upper bound (Section 1.1). Similarly, $\inf A \in \text{fr } A$ if A has a lower bound.

The closure of a set A consists of those points not exterior to A. Thus

$$(\text{cl } A)^c = \text{int } (A^c).$$

It is always true that int $A \subset A$. If these two sets are the same, then A is called an *open set*.

Definition. A set A is *open* if every point of A is interior to A.

From the above examples any neighborhood is an open set and E^n is an open set. The empty set furnishes another example of an open set. The interior of any set is an open set (Problem 4). We encounter many more open sets, starting with the homework problems at the end of the section. The collection of all open sets define what is called the *topology* of E^n. In Chapter 2 we see that such properties as connectedness of a set and continuity of a function can be expressed in terms of open sets. These are examples of topological properties.

Let us next show that the union of open sets is open, and that the intersection of finitely many open sets is open. We begin by considering two sets A and B.

Proposition 1.1. *If A and B are open sets, then $A \cup B$ and $A \cap B$ are open.*

PROOF. To show that $A \cup B$ is open, we show that every point of $A \cup B$ is interior to $A \cup B$. Let $\mathbf{x} \in A \cup B$. We must find a neighborhood U of $\mathbf{x}$ such that $U \subset A \cup B$. By the definition of the union of two sets, either $\mathbf{x} \in A$ or $\mathbf{x} \in B$. If $\mathbf{x} \in A$, then there is a neighborhood U of $\mathbf{x}$ such that $U \subset A$. Since $A \subset A \cup B$, $U \subset A \cup B$. Similarly, if $\mathbf{x} \in B$ there is a neighborhood of $\mathbf{x}$ contained in $A \cup B$. This proves that $A \cup B$ is open.

If $\mathbf{x} \in A \cap B$, then $\mathbf{x}$ has neighborhoods U_1, U_2 such that $U_1 \subset A$ and $U_2 \subset B$. Let $U_3 = U_1 \cap U_2$. Then U_3 is a neighborhood of $\mathbf{x}$ and $U_3 \subset A \cap B$. Therefore $A \cap B$ is open. □

In order to discuss unions and intersections of more than two sets, let us introduce some set-theoretic notation. By *indexed collection* of sets let us mean a function with domain some nonempty set $\mathcal{I}$ (called an *index set*) whose values are subsets of some set S. In the present instance $S = E^n$.

Let A_μ denote the value of the function at $\mu \in \mathscr{I}$. Moreover, let

$$\bigcup_{\mu \in \mathscr{I}} A_\mu = \{p \in S : p \in A_\mu \text{ for some } \mu \in \mathscr{I}\},$$

$$\bigcap_{\mu \in \mathscr{I}} A_\mu = \{p \in S : p \in A_\mu \text{ for every } \mu \in \mathscr{I}\}.$$

These sets are, respectively, the *union* and *intersection* of the indexed collection. If $\mathscr{I}$ is a finite set, then the indexed collection is called *finite*. For instance, if $\mathscr{I} = \{1, 2, \ldots, m\}$, we write the collection as $A_1, A_2, \ldots, A_m$. If $\mathscr{I} = \{1, 2, \ldots\}$, then the indexed collection is an infinite sequence of sets and is written $A_1, A_2, \ldots$, or $[A_m]$, $m = 1, 2, \ldots$ In that case the union is written $A_1 \cup A_2 \cup \cdots$ or $\bigcup_{m=1}^{\infty} A_m$, with similar notations for the intersection.

Proposition 1.2. *The union of any indexed collection of open sets is open. The intersection of any finite indexed collection of open sets is open.*

The proof is almost the same as for Proposition 1.1.

EXAMPLE 5. Let $n = 1$ and $A = \{x : 0 < x < 1,\ x \neq m^{-1},\ m = 2, 3, \ldots\}$. We can write $A = A_1 \cup A_2 \cup \cdots$, where $A_m = \{x : (m + 1)^{-1} < x < m^{-1}\}$. Each set A_m is open, and hence by Proposition 1.2, A is open.

EXAMPLE 6. Let A_m be the m^{-1}-neighborhood of a point x_0, $m = 1, 2, \ldots$. Then

$$A_1 \cap A_2 \cap \cdots = \{x_0\},$$

which is not an open set.

The concept of closed set is defined by considering set complements.

Definition. A set A is *closed* if its complement A^c is open.

In other words, A is a closed set if A contains all of its frontier points, which is to say $A = \text{cl } A$.

Every door is either open or closed. However, many sets are neither open nor closed.

EXAMPLE 7. Let $A = \{x : a \leq x < b\}$. The points a and b are frontier points of A, with $a \in A$, $b \notin A$. The set A is neither open nor closed. In the notation of Section 1.1, $A = [a, b)$. Its interior and closure are respectively the open interval (a, b) and the closed interval $[a, b]$.

We recall that

$$(A \cap B)^c = A^c \cup B^c, \qquad (A \cup B)^c = A^c \cap B^c.$$

More generally, for any indexed collection of sets

$$\left(\bigcap_{\mu\in\mathscr{I}} A_\mu\right)^c = \bigcup_{\mu\in\mathscr{I}} A_\mu^c, \qquad \left(\bigcup_{\mu\in\mathscr{I}} A_\mu\right)^c = \bigcap_{\mu\in\mathscr{I}} A_\mu^c.$$

We have the following statement from Propositions 1.1 and 1.2.

Proposition 1.3. *The intersection of any indexed collection of closed sets is closed. The union of any finite indexed collection of closed sets is closed.*

Besides indexed collections, we sometimes consider unindexed collections of sets. (We use the term "collection of sets," rather than "set of sets," for a set whose elements are subsets of some given set S.) Let us use German script letters to denote collections of sets. For instance, the elements of a finite collection $\mathfrak{A} = \{A_1, \ldots, A_m\}$ of subsets of E^n are the sets $A_i \subset E^n$, $i = 1, \ldots, m$.

The *union* and *intersection* of a collection $\mathfrak{A}$ of sets are, respectively, the sets

$$\bigcup_{A\in\mathfrak{A}} A = \{p \in S : p \in A \text{ for some } A \in \mathfrak{A}\},$$

$$\bigcap_{A\in\mathfrak{A}} A = \{p \in S : p \in A \text{ for every } A \in \mathfrak{A}\}.$$

If each set of the collection is indexed by itself (taking $\mathscr{I} = \mathfrak{A}$, $A_A = A$), then this definition of union and intersection agrees with the one for indexed collections. Propositions 1.2 and 1.3 remain valid for unindexed collections.

Problems

1. Let U_1 be the δ-neighborhood of $\mathbf{x}_1$ and U_2 the δ-neighborhood of $\mathbf{x}_2$. Show that $U_1 \cap U_2$ is empty if and only if $\delta \leq \frac{1}{2}|\mathbf{x}_1 - \mathbf{x}_2|$.
2. Find int A, fr A, cl A if A is:
   (a) $\{\mathbf{x} : 0 < |\mathbf{x} - \mathbf{x}_0| \leq \delta\}$, $\delta > 0$.
   (b) $\{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| = \delta\}$, $\delta > 0$.
   (c) $\{(x, y) : 0 < y < x + 1, x > -1\}$.
   (d) $\{(r\cos\theta, r\sin\theta) : 0 < r < 1, 0 < \theta < 2\pi\}$.
   (e) $\{(x, y) : x$ or y is irrational$\}$.
   (f) Any finite set.
   (g) $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $n = 1$.
3. In Problem 2 which sets are open? Which are closed?
4. Let A be any set. Show that int A is open, and that both fr A and cl A are closed.
5. (a) Show that the half space $\{\mathbf{x} : \mathbf{z}\cdot\mathbf{x} < c\}$ is an open set.
   [*Hint*: $|\mathbf{z}\cdot\mathbf{y} - \mathbf{z}\cdot\mathbf{x}| \leq |\mathbf{z}||\mathbf{y} - \mathbf{x}|$.]
   (b) Show that $\{\mathbf{x} : \mathbf{z}\cdot\mathbf{x} \geq c\}$ is a closed set, using (a), and that the hyperplane $\{\mathbf{x} : \mathbf{z}\cdot\mathbf{x} = c\}$ is a closed set.

6. Show that:
(a) $\text{fr } A = \text{fr}(A^c)$. (b) $\text{cl } A = \text{cl}(\text{cl } A)$.
(c) $\text{fr } A = \text{cl } A \cap \text{cl}(A^c)$. (d) $\text{int } A = (\text{cl}(A^c))^c$.

7. Show by giving examples that the following are in general *false*:
(a) $\text{int}(\text{cl } A) = \text{int } A$. (b) $\text{fr}(\text{fr } A) = \text{fr } A$.

8. Let A be open and B closed. Show that $A - B$ is open, and that $B - A$ is closed.

9. Show that:
(a) $\text{int}(A \cap B) = (\text{int } A) \cap (\text{int } B)$.
(b) $\text{cl}(A \cup B) = (\text{cl } A) \cup (\text{cl } B)$. [*Hint*: Part (a).]

10. Show that:
(a) $\text{int}(A \cup B) \supset (\text{int } A) \cup (\text{int } B)$.
(b) $\text{cl}(A \cap B) \subset (\text{cl } A) \cap (\text{cl } B)$.
Give examples in which $=$ does not hold.

*1.5 Convex sets

In this section we give a brief introduction to the theory of convex sets. Our treatment of convexity continues in Section 3.6 with a discussion of convex and concave functions.

Proposition 1.4. *If $K_1, \ldots, K_m$ are convex sets, then their intersection $K_1 \cap \cdots \cap K_m$ is convex.*

PROOF. Let $\mathbf{x}_1$, $\mathbf{x}_2$ be any two points of $K_1 \cap \cdots \cap K_m$, $\mathbf{x}_1 \neq \mathbf{x}_2$. Let l denote the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$. For each $j = 1, \ldots, m$, $\mathbf{x}_1, \mathbf{x}_2 \in K_j$. Since K_j is convex $l \subset K_j$ for each $j = 1, \ldots, m$. Thus $l \subset K_1 \cap \cdots \cap K_m$. □

In particular, Proposition 1.4 applies if each K_j is a half-space. A set that is the intersection of a finite number of closed half-spaces is called a *convex polytope*. Since a half-space is a convex set, any convex polytope is a convex set.

EXAMPLE 1. Let T be a triangle in the plane E^2. Then T is the intersection of three half-planes, bounded by the lines through the sides of T.

A convex polytope is the set of all points $\mathbf{x}$ that satisfy a given finite system of linear inequalities of the form $\mathbf{z}^j \cdot \mathbf{x} \geq c^j$, $j = 1, \ldots, m$. The theory of linear programming is concerned with the problem of maximizing or minimizing a linear function subject to such a system of linear inequalities. It has various interesting economic and engineering applications [10, 13]. In Section 3.6 it is shown that the maximum and minimum values of a linear function must occur at "extreme points" of K, at least if K is compact.

In the proof of Proposition 1.4 we did not really use the fact that the number of sets K_j is finite. Therefore we have:

Proposition 1.5. *The intersection of any collection of convex sets is a convex set.* □

In particular, the intersection of any collection of half-spaces is convex. The intersection of any collection of closed sets is a closed set. Hence, if each of the half-spaces is closed, the intersection is a closed, convex set. An important fact about closed convex sets is that, excluding trivial cases, the converse holds. The converse can be stated in a slightly sharper form, in that only half-spaces bounded by supporting hyperplanes need be used (Theorem 1.1). In order to do this we first state the following.

Definition. Let K be a closed convex set. Assume that K is neither the empty set nor E^n. A hyperplane P is called *supporting* for K if $P \cap K$ is not empty and K is contained in one of the two closed half-spaces bounded by P.

If P is supporting for K, the set $P \cap K$ is convex by Proposition 1.4, and contains only boundary points of K (Problem 5). Given any boundary point of K, there is at least one supporting hyperplane containing it. This can be deduced from Theorem 1.1 and results to be proved in Section 2.4 (Problem 7). If K has interior points and the boundary fr K is "sufficiently smooth," then given $\mathbf{y} \in \text{fr } K$, there is just one supporting hyperplane containing $\mathbf{y}$. It is the tangent hyperplane to fr K at $\mathbf{y}$, and can be found by the methods of calculus. This will be explained in Section 4.7.

If, for example, T is a triangle in E^2, then each vertex is contained in an infinite number of supporting lines to T. Each other boundary point is contained in a single supporting line, the line through the edge of T containing it (Figure 1.8).

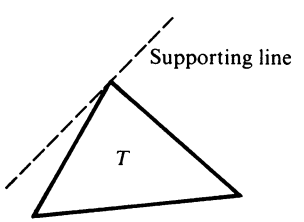


Figure 1.8

EXAMPLE 2. Let $B = \{\mathbf{x} : |\mathbf{x}| \leq 1\}$, the closed unit n-ball. Then

$$\text{fr } B = \{\mathbf{x} : |\mathbf{x}| = 1\}.$$

Given $\mathbf{y} \in \text{fr } B$, let

$$H_{\mathbf{y}} = \{\mathbf{x} : \mathbf{y} \cdot \mathbf{x} \leq 1\},$$
$$P_{\mathbf{y}} = \{\mathbf{x} : \mathbf{y} \cdot \mathbf{x} = 1\},$$

so that $P_{\mathbf{y}}$ is the hyperplane bounding $H_{\mathbf{y}}$. By Cauchy's inequality and the fact that $|\mathbf{y}| = 1$,

$$\mathbf{y} \cdot \mathbf{x} \leq |\mathbf{y}|\,|\mathbf{x}| = |\mathbf{x}|.$$

Equality holds if and only if $\mathbf{x}$ is a positive scalar multiple of $\mathbf{y}$. Hence $B \subset H_{\mathbf{y}}$ and $B \cap P_{\mathbf{y}}$ consists only of $\mathbf{y}$. The supporting hyperplane to B at $\mathbf{y}$ is $P_{\mathbf{y}}$ (Figure 1.9).

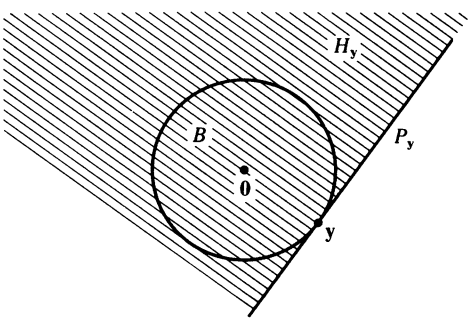


Figure 1.9

Again let K be any nonempty, closed convex set that is a proper subset of $E^n (K \neq E^n)$. Let $\mathscr{H}_K$ denote the collection of all closed half-spaces H such that $K \subset H$ and the hyperplane P bounding H is supporting for K. For instance, the collection $\mathscr{H}_B$ in the above example consists of the various half-spaces $H_{\mathbf{y}}$ for all possible choices of $\mathbf{y} \in \text{fr } B$.

The notation

$$\bigcap_{H \in \mathscr{H}_K} H$$

stands for the intersection of all half-spaces $H \in \mathscr{H}_K$.

Theorem 1.1. $K = \bigcap_{H \in \mathscr{H}_K} H.$

PROOF. For convenience let us set

$$K_1 = \bigcap_{H \in \mathscr{H}_K} H.$$

Since $K \subset H$ for each $H \in \mathscr{H}_K$, $K \subset K_1$. Let us show that $K = K_1$.

Suppose it is not. Then there exists some $\mathbf{x}_1 \in K_1 - K$. Since K is a closed set, there is a point $\mathbf{x}_0 \in K$ nearest $\mathbf{x}_1$, that is, $|\mathbf{x} - \mathbf{x}_1| \geq |\mathbf{x}_0 - \mathbf{x}_1|$ for every $\mathbf{x} \in K$ (Figure 1.10). This fact about closed sets is proved in Section 2.4.

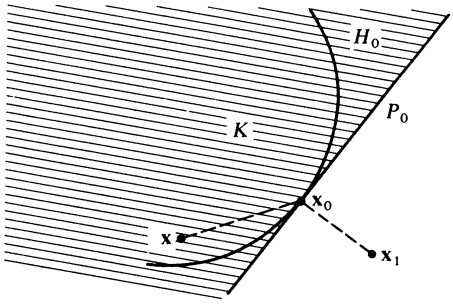


Figure 1.10

Consider the closed half-space

$$H_0 = \{\mathbf{x} : (\mathbf{x}_0 - \mathbf{x}_1) \cdot (\mathbf{x} - \mathbf{x}_0) \geq 0\}.$$

Then $\mathbf{x}_1 \notin H_0$ since

$$(\mathbf{x}_0 - \mathbf{x}_1) \cdot (\mathbf{x}_1 - \mathbf{x}_0) = -|\mathbf{x}_0 - \mathbf{x}_1|^2 < 0.$$

To show that $K \subset H_0$, let $\mathbf{x}$ be any point of K. Since K is convex, $t\mathbf{x} + (1 - t)\mathbf{x}_0 \in K$ for every $t \in [0, 1]$. Then

$$|t\mathbf{x} + (1 - t)\mathbf{x}_0 - \mathbf{x}_1|^2 \geq |\mathbf{x}_0 - \mathbf{x}_1|^2,$$

or

$$|(\mathbf{x}_0 - \mathbf{x}_1) + t(\mathbf{x} - \mathbf{x}_0)|^2 \geq |\mathbf{x}_0 - \mathbf{x}_1|^2,$$
$$|\mathbf{x}_0 - \mathbf{x}_1|^2 + 2t(\mathbf{x}_0 - \mathbf{x}_1) \cdot (\mathbf{x} - \mathbf{x}_0) + t^2|\mathbf{x} - \mathbf{x}_0|^2 \geq |\mathbf{x}_0 - \mathbf{x}_1|^2.$$

Subtracting $|\mathbf{x}_0 - \mathbf{x}_1|^2$ from both sides and dividing by t, we get for $0 < t \leq 1$

$$2(\mathbf{x}_0 - \mathbf{x}_1) \cdot (\mathbf{x} - \mathbf{x}_0) + t|\mathbf{x} - \mathbf{x}_0|^2 \geq 0.$$

Letting $t \to 0^+$ we find that $2(\mathbf{x}_0 - \mathbf{x}_1) \cdot (\mathbf{x} - \mathbf{x}_0) \geq 0$, which shows that $\mathbf{x} \in H_0$. Thus $K \subset H_0$. The boundary of H_0 is

$$P_0 = \{\mathbf{x} : (\mathbf{x}_0 - \mathbf{x}_1) \cdot (\mathbf{x} - \mathbf{x}_0) = 0\},$$

and $\mathbf{x}_0 \in P_0$. Hence $P_0 \cap K$ is not empty, and therefore P_0 is supporting for K. This shows that $H_0 \in \mathscr{H}_K$. Consequently, $K_1 \subset H_0$. But $\mathbf{x}_1 \in K_1$, $\mathbf{x}_1 \notin H_0$, a contradiction. Therefore $K = K_1$. □

Convex combinations

The definition of convex set is expressed in terms of pairs of points. It can also be given in terms of convex combinations of any finite number m of points. Let $\mathbf{x}_1, \ldots, \mathbf{x}_m$ be distinct points ($\mathbf{x}_j \neq \mathbf{x}_k$ if $j \neq k$).

Definition. A point $\mathbf{x}$ is a *convex combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_m$ if there exist scalars $t^1, \ldots, t^m$ such that

$$\mathbf{x} = \sum_{j=1}^{m} t^j \mathbf{x}_j, \qquad 1 = \sum_{j=1}^{m} t^j, \qquad t^j \geq 0 \quad \text{for } j = 1, \ldots, m.$$

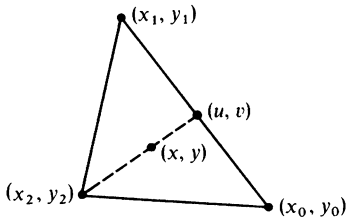


Figure 1.11

To say that $\mathbf{x}$ is a convex combination of two points of S is merely to say that $\mathbf{x}$ lies on some line segment with endpoints in S. For instance, if S is the circle with equation $x^2 + y^2 = a^2$, then every point in the circular disk $\{(x, y) : x^2 + y^2 \leq a^2\}$ bounded by S is a convex combination of two points of S.

On the other hand, if S consists of three noncollinear points (x_0, y_0), (x_1, y_1), (x_2, y_2), then each boundary point of the triangle with these points as vertices is a convex combination of two points of S, but the interior points are not. However, each interior point (x, y) is a convex combination of (x_2, y_2) and some point (u, v) on the edge opposite (x_2, y_2) (Figure 1.11). Since (u, v) is a convex combination of (x_0, y_0) and (x_1, y_1), we can write (x, y) as a convex combination of the three points (x_0, y_0), (x_1, y_1), (x_2, y_2) as follows. Writing $\mathbf{x} = (x, y)$, $\mathbf{x}_j = (x_j, y_j)$, there exist $s, t \in [0, 1]$ such that

$$\mathbf{x} = t[s\mathbf{x}_0 + (1 - s)\mathbf{x}_1] + (1 - t)\mathbf{x}_2 = t^0\mathbf{x}_0 + t^1\mathbf{x}_1 + t^2\mathbf{x}_2,$$

where $t^0 = ts, t^1 = t(1 - s), t^2 = 1 - t$ are nonnegative and $t^0 + t^1 + t^2 = 1$.

Proposition 1.6. *A set K is convex if and only if every convex combination of points of K is a point of K.*

PROOF. Let K be convex. Let us prove by induction on m that if $\mathbf{x}$ is any convex combination of $\mathbf{x}_1, \ldots, \mathbf{x}_m \in K$, then $\mathbf{x} \in K$. The case $m = 2$ is the definition of convexity. Assuming the result true for the integer $m \geq 2$, let $\mathbf{x}$ be a convex combination of points $\mathbf{x}_1, \ldots, \mathbf{x}_{m+1}$ of K,

$$\mathbf{x} = \sum_{j=1}^{m+1} t^j\mathbf{x}_j, \qquad 1 = \sum_{j=1}^{m+1} t^j, \qquad t^j \geq 0 \quad \text{for } j = 1, \ldots, m + 1.$$

If $t^{m+1} = 1$, then $t^j = 0$ for $j \leq m$ and $\mathbf{x} = \mathbf{x}_{m+1}$ is in K. If $t^{m+1} < 1$, let

$$t = 1 - t^{m+1}, \qquad s^j = t^j/t \quad \text{for} \quad j = 1, \ldots, m,$$

$$\mathbf{y} = \sum_{j=1}^{m} s^j\mathbf{x}_j.$$

Then $\mathbf{y}$ is a convex combination of $\mathbf{x}_1, \ldots, \mathbf{x}_m$. By the induction hypothesis, $\mathbf{y} \in K$. But

$$\mathbf{x} = t\mathbf{y} + (1 - t)\mathbf{x}_{m+1},$$

and $t \in [0, 1]$. Hence $\mathbf{x} \in K$.

Conversely, assume that every convex combination of points of K is a point of K. In particular, this is true for convex combinations of any pair $\mathbf{x}_1, \mathbf{x}_2$ of points of K. Hence K is convex. □

Let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r$, $r \leq n$, be distinct points of E^n such that the differences $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_r - \mathbf{x}_0$ form a linearly independent set. The set of all convex combinations of $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r$ is called the *r-simplex with vertices* $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r$. A 1-simplex is a line segment, a 2-simplex a triangle, and a 3-simplex a tetrahedron. According to Proposition 1.6, any simplex whose vertices lie in a convex set K is contained in K (Figure 1.12).

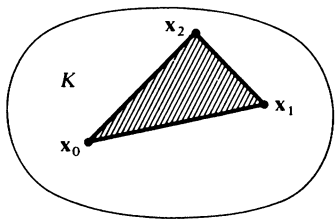


Figure 1.12

A point $\mathbf{x}$ of an r-simplex can be written in a unique way as a convex combination

$$\mathbf{x} = \sum_{j=0}^{r} t^j \mathbf{x}_j$$

of the vertices $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r$ (Problem 4). The numbers $t^0, t^1, \ldots, t^r$ are called the *barycentric coordinates* of $\mathbf{x}$. The $(r-1)$-dimensional *face* opposite the vertex $\mathbf{x}_i$ is the set of points of the r-simplex with $t^i = 0$.

For example, the vertices $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$ of a triangle have barycentric coordinates $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, respectively. The midpoint of the face opposite $\mathbf{x}_0$ has barycentric coordinates $(0, \frac{1}{2}, \frac{1}{2})$. The interior points of the triangle have barycentric coordinates (t^0, t^1, t^2), all of which are strictly positive. In each case $t^0 + t^1 + t^2 = 1$.

The simplex with vertices $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n$ is called the *standard n-simplex*. It will be denoted by Σ, and is of use in Section 5.7 in connection with integration. The barycentric coordinates $(t^0, t^1, \ldots, t^n)$ of a point $\mathbf{x} \in \Sigma$ are given by $t^i = x^i$ for $i = 1, \ldots, n$, $t^0 = 1 - (x^1 + \cdots + x^n)$.

Further results about convex combinations

In the definition of convex combination, no upper bound was put on the positive integer m. However, for most purposes one need consider only $m \leq n + 1$. More precisely:

Proposition 1.7. *If $S \subset E^n$ and $\mathbf{x}$ is a convex combination of points of S, then $\mathbf{x}$ is a convex combination of $n + 1$ or fewer points of S.*

PROOF. Let $\mathbf{x}$ be a convex combination with $m > n + 1$, $t^j > 0$ for $j = 1, \ldots, m$, and $\mathbf{x}_1, \ldots, \mathbf{x}_m \in S$. Let us show that $\mathbf{x}$ is a convex combination of $m - 1$ points of S. Since $m - 1 > n$, there exist $c^1, \ldots, c^{m-1}$ not all 0 such that

$$c^1(\mathbf{x}_1 - \mathbf{x}_m) + \cdots + c^{m-1}(\mathbf{x}_{m-1} - \mathbf{x}_m) = \mathbf{0}.$$

Let $c^m = -(c^1 + \cdots + c^{m-1})$. Then

$$\sum_{j=1}^{m} c^j \mathbf{x}_j = \mathbf{0}, \qquad \sum_{j=1}^{m} c^j = 0.$$

Let

$$s^j = t^j - \alpha c^j \quad \text{for } j = 1, \ldots, m,$$

where α is a positive number chosen so that $s^j \geq 0$ for each $j = 1, \ldots, m$ and $s^k = 0$ for some k. Explicitly,

$$\frac{1}{\alpha} = \max\left\{\frac{c^1}{t^1}, \ldots, \frac{c^m}{t^m}\right\}.$$

Then

$$\mathbf{x} = \sum_{j \neq k} s^j \mathbf{x}_j, \qquad 1 = \sum_{j \neq k} s^j,$$

and consequently $\mathbf{x}$ is a convex combination of the $m - 1$ points $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$, $\mathbf{x}_{k+1}, \ldots, \mathbf{x}_m$.

Either $m - 1 = n + 1$, or else the same argument shows that $\mathbf{x}$ is a convex combination of $m - 2$ points of S. Continuing, we find that $\mathbf{x}$ must be a convex combination of $n + 1$ or fewer points of S. □

If S is the set of vertices of an n-simplex T and each of the barycentric coordinates of $\mathbf{x}$ is positive, then $\mathbf{x}$ is not a convex combination of fewer than $n + 1$ points of S. Hence the number $n + 1$ is the best possible in Proposition 1.7.

A slightly better result is possible if S is a connected set. A set S is called disconnected if there exist open sets D and E such that $S \subset D \cup E$, $D \cap S$ and $E \cap S$ are both nonempty, but $D \cap E \cap S$ is empty. *Connected* means not disconnected. For further discussion of connectedness, see Section 2.7.

Proposition 1.8. *If S is a connected subset of E^n and x is a convex combination of points of S, then x is a convex combination of n or fewer points of S.*

PROOF. Suppose that $\mathbf{x}^*$ is a point which is a convex combination of $n + 1$ points $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ of a connected set S, but not of fewer than $n + 1$ points of S. The differences $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$ form a linearly independent set; if not, the reasoning used to prove the proposition above shows that $\mathbf{x}^*$ is a convex combination of n of the points $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$. Therefore $\mathbf{x}_0$,

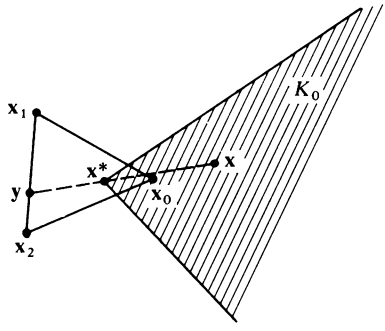


Figure 1.13

$\mathbf{x}_1, \ldots, \mathbf{x}_n$ are the vertices of an n-simplex T, and all the barycentric coordinates of $\mathbf{x}^*$ are positive. Let T_0 be the face of T opposite $\mathbf{x}_0$, and

$$K_0 = \{\mathbf{x} : \mathbf{x}^* = t\mathbf{x} + (1 - t)\mathbf{y}, \text{ where } \mathbf{y} \in T_0 \text{ and } t \in [0, 1]\}.$$

K_0 is a convex polytope, and its boundary fr K_0 consists of portions of the hyperplanes which contain $\mathbf{x}^*$ and the $(n - 2)$-dimensional faces of T_0 (we leave the verification of this to the reader). If fr K_0 intersects S, then $\mathbf{x}^*$ is a convex combination of fewer than $n + 1$ points of S, contrary to hypothesis (Figure 1.13). Hence $S \cap \text{fr } K_0$ is empty. The interior int K_0 and the complement $K_0^c = E^n - K_0$ are open sets, their union contains S, and their intersection is empty. But $\mathbf{x}_0 \in \text{int } K_0$ and $\mathbf{x}_i \in K_0^c$ for $i = 1, \ldots, n$. This implies that S is disconnected, which is a contradiction. □

By slightly refining the proof, an even stronger result is obtained. Suppose that $S = S_1 \cup \cdots \cup S_k$, where $k \leq n$ and $S_1, \ldots, S_k$ are connected sets. For each $i = 1, \ldots, n$ consider the corresponding convex polytope K_i. Then int $K_i \cap$ int K_j is empty whenever $i \neq j$ and $S \cap \text{fr } K_i$ is empty for every i. Moreover, $\mathbf{x}_i \in \text{int } K_i$. Since $k \leq n$, some pair of the points $\mathbf{x}_i, \mathbf{x}_j$ must belong to the same set S_p. Then S_p is not connected, a contradiction. Hence, if S is the union of n or fewer connected sets, every $\mathbf{x}$ which is a convex combination of points of S is a convex combination of n or fewer points of S.

Problems

1. Show that each of the following subsets of E^2 is closed and convex by writing it as the intersection of closed half-planes:
   (a) The regular hexagon with center (0, 0) and $\mathbf{e}_1$ as one vertex.
   (b) $\{(x, y) : y \geq |x|, -1 \leq x \leq 1\}$.
   (c) $\{(x, y) : y \leq \log x, x > 0\}$.
   (d) $\{(x, y) : 0 \leq y \leq \sin x, 0 \leq x \leq \pi\}$.
2. Write the standard n-simplex as the intersection of $n + 1$ closed half-spaces. Illustrate for $n = 2$ and $n = 3$.

3. Write $\frac{1}{4}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2$ as a convex combination of $\mathbf{e}_1, \frac{4}{3}\mathbf{e}_2 - \mathbf{e}_1$. Also write it as a convex combination of $\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2$. Illustrate.

4. Show that if $\mathbf{x}$ can be represented in two ways as a convex combination of $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r$, then $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_r - \mathbf{x}_0$ form a linearly dependent set. [*Hint*: If $\mathbf{x} = t^0\mathbf{x}_0 + \cdots + t^r\mathbf{x}_r$ and $t^0 + \cdots + t^r = 1$, then $\mathbf{x} - \mathbf{x}_0 = t^1(\mathbf{x}_1 - \mathbf{x}_0) + \cdots + t^r(\mathbf{x}_r - \mathbf{x}_0)$.]

5. Prove that a supporting hyperplane for a closed convex set K can contain no interior point of K.

6. Let K be any convex set. Prove that its interior and its closure are also convex sets.

7. The *barycenter* of an r-simplex is the point at which the barycentric coordinates are equal, $t^0 = t^1 = \cdots = t^r$.
   (a) Show that the barycenter of a triangle is at the intersection of the medians.
   (b) State and prove a corresponding result for $r \geq 3$.

8. Let $\mathbf{x}$ be a convex combination of $\mathbf{x}_1, \ldots, \mathbf{x}_m$ and let $\mathbf{x}_j$ be a convex combination of $\mathbf{y}_{j1}, \ldots, \mathbf{y}_{jm_j}, j = 1, \ldots, m$. Show that $\mathbf{x}$ is a convex combination of $\mathbf{z}_1, \ldots, \mathbf{z}_p$, which are the distinct elements of the set $\{\mathbf{y}_{ik} : k = 1, \ldots, m_j, j = 1, \ldots, m\}$.

9. Let S be any subset of E^n. The set $\hat{S}$ of all convex combinations of points of S is the *convex hull* of S.
   (a) Using Problem 8, show that $\hat{S}$ is convex.
   (b) Using Proposition 1.6, show that if K is convex and $S \subset K$, then $\hat{S} \subset K$. Thus the convex hull is the smallest convex set containing S.

10. Given $\mathbf{x}_0$ and $\delta > 0$, let $C = \{\mathbf{x} : |x^i - x_0^i| \leq \delta,\ i = 1, \ldots, n\}$, an *n-cube* with center $\mathbf{x}_0$ and side length 2δ. The *vertices* of C are those $\mathbf{x}$ with $|x^i - x_0^i| = \delta$ for $i = 1, \ldots, n$. Show that C is the convex hull of its set of vertices. [*Hint*: Use induction on n.]

11. Let K be a closed subset of E^n such that both K and its complement $E^n - K$ are nonempty convex sets. Prove that K is a half-space.

12. Let A and B be convex subsets of E^n. The *join* of A and B is the set of all $\mathbf{x}$ such that $\mathbf{x}$ lies on a line segment with one endpoint in A and the other in B. Show that the join of A and B is a convex set.

2 Elementary topology of E^n

In this chapter we explore various ramifications of the topological notions introduced in Section 1.4. We begin with some basic properties of functions, limits, and continuity. Then we turn in Sections 2.3 and 2.4 to some properties of E^n that depend essentially on the least upper bound axiom for real numbers. Among these properties are the convergence of Cauchy sequences (Theorem 2.2) and the Bolzano–Weierstrass theorem. In Section 2.6 a very general concept is introduced, that of topological space. In particular, any set $S \subset E^n$ becomes a topological space with the relative topology which S inherits from E^n. The concepts of connected set and compact set are introduced in Sections 2.7 and 2.8. It turns out that these properties are preserved by continuous functions (Theorems 2.8 and 2.10). At the end of the chapter, the concept of metric space is introduced, and some interesting special cases are considered. Uniform convergence of a sequence of functions is studied as convergence in a certain metric on a space of bounded functions.

2.1 Functions

One should think of a function f as assigning to each element p of some set S an element $f(p)$ of another set T. The element $f(p)$ is called the *value* of f at p. However, this is not a satisfactory definition of "function" because of the ambiguity of the word "assigns." We give a more careful definition, in terms of cartesian product sets.

Cartesian product sets

If S and T are sets, then the *cartesian product* set $S \times T$ is formed by taking all ordered pairs (p, q) where $p \in S$ and $q \in T$. For example, if $S = \{1, 2, \ldots, n\}$ and $T = \{1, 2, \ldots, m\}$, then the elements of $S \times T$ are pairs (i, j) of positive

integers with $1 \leq i \leq n$, $1 \leq j \leq m$. In the same way, the plane E^2 is the cartesian product $E^1 \times E^1$.

If $S_1, \ldots, S_n$ are sets, then the *n-fold cartesian product* $S_1 \times \cdots \times S_n$ is formed by taking all (ordered) n-tuples $(p_1, \ldots, p_n)$, where $p_i \in S_i$ for each $i = 1, \ldots, n$. In particular, $E^n = E^1 \times \cdots \times E^1$. In E^2, the cartesian product $[a, b] \times [c, d]$ is the rectangle $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$.

Relations, functions

Any subset f of the cartesian product $S \times T$ is called a *relation* between S and T. A relation f is called a *function* if for every $p \in S$ there is exactly one $q \in T$ such that $(p, q) \in f$. This element q is denoted by $f(p)$.

The set S is the *domain* of f. We shall sometimes say that f is a function *from S into T*. If for every $q \in T$ there is some $p \in S$ such that $q = f(p)$, then we say that f is *onto T*. A function f is *univalent* (or *one-one*) if $p_1 \neq p_2$ implies that $f(p_1) \neq f(p_2)$.

This book is concerned with the calculus of functions whose domains are subsets of E^n. Such functions are frequently called by the suggestive but imprecise name, "functions of n real variables." We may occasionally use this name in passages intended to motivate a more careful discussion to follow. However, we never try to make precise the phrase "n real variables." It was only after such vague terms as "variable" and "quantity" were abandoned that calculus was put on a foundation acceptable by present-day standards.

A function f from a set S into E^1 is a *real valued* function. When $S \subset E^1$, f is a real valued "function of one real variable." Among such functions are the algebraic functions and elementary transcendental functions (sin, cos, tan, log, etc.), which should be familiar from elementary calculus. The exponential function is denoted in this book by "exp." Thus $\exp x = e^x$, where e is the base for natural logarithms.

Functions with values in some euclidean E^m, $m > 1$, are called *vector valued* and are indicated by boldface letters (say, $\mathbf{f}$ or $\mathbf{g}$). A vector valued function $\mathbf{f}$ from a subset D of some euclidean E^n into E^m is also called a *transformation* from D into E^m. By merely writing "transformation" in place of "vector-valued function," we have of course introduced no new mathematical idea. However, the word "transformation" is supposed to have a geometric flavor which aids intuition. Some authors say "mapping" instead of "transformation." The differential calculus of transformations is developed in Chapter 4.

If f and g are functions with the same domain S and values in a vector space $\mathscr{V}$, then the sum $f + g$ is defined by

$$(f + g)(p) = f(p) + g(p)$$

for every $p \in S$. In particular, it makes sense to speak of sums of real valued functions or of transformations. If f has values in $\mathscr{V}$ and ϕ is real valued, then ϕf is the $\mathscr{V}$-valued function given by

$$(\phi f)(p) = \phi(p) f(p)$$

for every $p \in S$. If ϕ is a constant function, $\phi(p) = c$ for every $p \in S$, then we write cf instead of ϕf.

Restriction of a function

Often one is interested only in the values of a function f for elements of some subset A of its domain. The *restriction* of f to A is the function with domain A and the same values as f there. It is denoted by $f \mid A$. Thus

$$f \mid A = \{(p, f(p)) : p \in A\}.$$

For instance, if a real-valued function f is integrated over an interval $I \subset E^1$, then it is only $f \mid I$ which is important. The values of f outside I do not affect the integral.

Images, inverse images

Let f be a function from a set S into a set T. The *image* under f of a set $A \subset S$ is the set $f(A) = \{f(p) : p \in A\}$. It is a subset of T, and in fact the restriction $f \mid A$ is a function from A onto $f(A)$. The *inverse image* of a set $B \subset T$ is the set $f^{-1}(B) = \{p : f(p) \in B\}$. It is a subset of S.

EXAMPLE 1. Let $f(x) = x^2$. Then $f([-2, 2]) = [0, 4]$, $f(E^1) = [0, \infty)$, $f^{-1}([1, 3]) = [-\sqrt{3}, -1] \cup [1, \sqrt{3}]$. The function f is not univalent since $f(-x) = f(x)$.

EXAMPLE 2. Let f be a function from a set S into a set T. Show that

$$(*) \qquad A \subset f^{-1}(f(A))$$

for any $A \subset S$. Consider any $p \in A$. Then $f(p) \in f(A)$ by definition of $f(A)$. Take $B = f(A)$ in the definition of inverse image set above. Then $p \in f^{-1}(f(A))$. Since this is true for each $p \in A$, we get (*).

EXAMPLE 3. Show that if f is univalent in Example 2, then

$$(**) \qquad A = f^{-1}(f(A)).$$

It suffices to show that $A \supset f^{-1}(f(A))$, since the opposite inclusion is (*). Consider any $p \in f^{-1}(f(A))$. Then $f(p) \in f(A)$. Therefore $f(p) = f(p')$ for some $p' \in A$. Since f is univalent, $p = p'$. Thus $p \in A$ as required.

PROBLEMS

1. (a) Let $f(x) = \cos x$. Find $f(E^1)$, $f([-\pi/4, \pi/2])$, $f^{-1}([0, 1])$.
   (b) Let $g = f \mid [0, \pi]$. Find $g([0, \pi])$, $g^{-1}([0, 1])$. Is g univalent?

2. The equations $s = (x^2 + y^2)^{1/2}, t = x - y$ define a transformation $\mathbf{f}$ from E^2 into E^2, such that $\mathbf{f}(x, y) = (s, t)$. Let $A = \{(x, y) : x^2 + y^2 \leq a^2\}$, where $a > 0$ is given.
   (a) Find $\mathbf{f}(A)$.
   (b) Find $\mathbf{f}^{-1}(A)$.

3. Let f be a function from S into T. Show that, for any $B \subset T$:
(a) $B \supset f(f^{-1}(B))$.
(b) $B = f(f^{-1}(B))$, if f is onto B.

4. Let f be a function from S into T. Show that, for any subsets A and B of S:
(a) $f(A \cup B) = f(A) \cup f(B)$.
(b) $f(A \cap B) \subset f(A) \cap f(B)$.
(c) If f is univalent, then $f(A \cap B) = f(A) \cap f(B)$.

5. Let f be a function from S into T. Show that, for any subsets C and D of T:
(a) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
(b) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
(c) $f^{-1}(D^c) = [f^{-1}(D)]^c$.

6. Let S and T be sets, and let $\pi(p, q) = p$ for all $p \in S$, $q \in T$. The function π *projects* $S \times T$ onto S. Let $R \subset S \times T$ be a relation. Show that R is a function if and only if $\pi | R$ is univalent and onto S.

7. Let $1 \leq s \leq n - 1$. Let us regard E^n as the cartesian product $E^s \times E^{n-s}$, and write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, where $\mathbf{x}' = (x^1, \ldots, x^s)$, $\mathbf{x}'' = (x^{s+1}, \ldots, x^n)$. Let $\pi(\mathbf{x}) = \mathbf{x}'$ be the projection of E^n onto E^s. Show that $\pi(A)$ is an open subset of E^s if A is an open subset of E^n.

8. Let $A \subset E^s$, $B \subset E^{n-s}$, and regard the cartesian product $A \times B$ as a subset of E^n, as in Problem 7.
(a) Show that $A \times B$ is open if both A and B are open.
(b) Show that $A \times B$ is closed if both A and B are closed.

2.2 Limits and continuity of transformations

Let us now suppose that $\mathbf{f}$ is a function from a set $D \subset E^n$ into E^m, where n and m are positive integers. As already mentioned, such functions are called *transformations* in this book.

The definition of "limit" for transformations is patterned after the one encountered in elementary calculus for real-valued functions of one variable. A *punctured* neighborhood of $\mathbf{x}_0$ is a neighborhood with the center $\mathbf{x}_0$ removed. Let us assume that D contains some punctured neighborhood of $\mathbf{x}_0$. For the definition of "limit," $\mathbf{x}_0$ itself need not be in D. If $\mathbf{x}_0 \in D$, the value of f at $\mathbf{x}_0$ is irrelevant.

Definition. If for every neighborhood V of $\mathbf{y}_0$ there is a punctured neighborhood U of $\mathbf{x}_0$ such that $\mathbf{f}(U) \subset V$, then $\mathbf{y}_0$ is the *limit* of the transformation $\mathbf{f}$ at $\mathbf{x}_0$ (Figure 2.1).

In the definition it is understood that the radius of U is small enough so that $U \subset D$. The notations $\mathbf{y}_0 = \lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x}) \to \mathbf{y}_0$ as $\mathbf{x} \to \mathbf{x}_0$ are used to mean that $\mathbf{y}_0$ is the limit of $\mathbf{f}$ at $\mathbf{x}_0$.

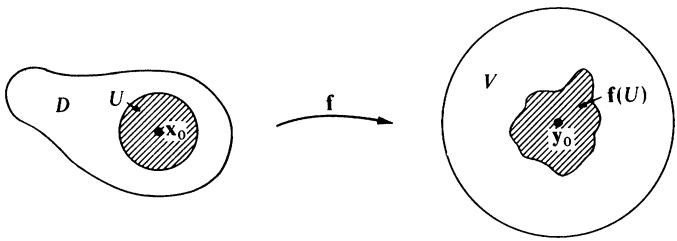


Figure 2.1

If we let ε and δ denote the radii of V and U respectively, then the definition may be rephrased as follows: $\mathbf{f}(\mathbf{x}) \to \mathbf{y}_0$ as $\mathbf{x} \to \mathbf{x}_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{f}(\mathbf{x}) - \mathbf{y}_0| < \varepsilon$ whenever $0 < |\mathbf{x} - \mathbf{x}_0| < \delta$. The number δ depends, of course, on ε and may also depend on $\mathbf{x}_0$. Given ε and $\mathbf{x}_0$, there is a largest possible δ. However, there is ordinarily no reason to try to calculate it.

Let us first show that limits behave properly with respect to sums and products. Let $\mathbf{f}$ and $\mathbf{g}$ have the same domain and values in the same euclidean E^m.

Proposition 2.1. *If* $\mathbf{y}_0 = \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})$ *and* $\mathbf{z}_0 = \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x})$, *then*

(1) $\mathbf{y}_0 + \mathbf{z}_0 = \lim_{\mathbf{x}\to\mathbf{x}_0}[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})]$.

(2) $c\mathbf{y}_0 = \lim_{\mathbf{x}\to\mathbf{x}_0} c\,\mathbf{f}(\mathbf{x})$, *for any scalar* c.

(3) $\mathbf{y}_0 \cdot \mathbf{z}_0 = \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})$.

PROOF. Let V be any neighborhood of $\mathbf{y}_0 + \mathbf{z}_0$ and let ε be its radius. Let V_1, V_2 be the neighborhoods of radius $\varepsilon/2$ of $\mathbf{y}_0$, $\mathbf{z}_0$, respectively. If $\mathbf{y} \in V_1$ and $\mathbf{z} \in V_2$, then

$$(\mathbf{y} + \mathbf{z}) - (\mathbf{y}_0 + \mathbf{z}_0) = (\mathbf{y} - \mathbf{y}_0) + (\mathbf{z} - \mathbf{z}_0).$$

By the triangle inequality

$$|(\mathbf{y} + \mathbf{z}) - (\mathbf{y}_0 + \mathbf{z}_0)| \leq |\mathbf{y} - \mathbf{y}_0| + |\mathbf{z} - \mathbf{z}_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\mathbf{y} + \mathbf{z} \in V$. By hypothesis there exist punctured neighborhoods U_1, U_2 of $\mathbf{x}_0$ such that $\mathbf{f}(U_1) \subset V_1$ and $\mathbf{g}(U_2) \subset V_2$. Let $U = U_1 \cap U_2$, which is also a punctured neighborhood of $\mathbf{x}_0$. If $\mathbf{x} \in U$, then $\mathbf{f}(\mathbf{x}) \in V_1$, $\mathbf{g}(\mathbf{x}) \in V_2$. Consequently, $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \in V$, which shows that $(\mathbf{f} + \mathbf{g})(U) \subset V$. This proves (1). The proof of (2) is left to the reader (Problem 2).

To prove (3) let V_0 be the neighborhood of $\mathbf{y}_0$ of radius 1, and U_0 be a punctured neighborhood of $\mathbf{x}_0$ such that $\mathbf{f}(U_0) \subset V_0$. Let

$$C = \max\{|\mathbf{y}_0| + 1, |\mathbf{z}_0|\}.$$

If $\mathbf{y} \in V_0$, then $\mathbf{y} = \mathbf{y}_0 + (\mathbf{y} - \mathbf{y}_0)$. By the triangle inequality

$$|\mathbf{y}| \leq |\mathbf{y}_0| + |\mathbf{y} - \mathbf{y}_0| < |\mathbf{y}_0| + 1,$$

and hence $|\mathbf{y}| < C$. Now

$$\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) - \mathbf{y}_0 \cdot \mathbf{z}_0 = \mathbf{f}(\mathbf{x}) \cdot [\mathbf{g}(\mathbf{x}) - \mathbf{z}_0] + \mathbf{z}_0 \cdot [\mathbf{f}(\mathbf{x}) - \mathbf{y}_0].$$

From the triangle inequality and Cauchy's inequality,

$$(*) \qquad |\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) - \mathbf{y}_0 \cdot \mathbf{z}_0| \leq |\mathbf{f}(\mathbf{x})||\mathbf{g}(\mathbf{x}) - \mathbf{z}_0| + |\mathbf{z}_0||\mathbf{f}(\mathbf{x}) - \mathbf{y}_0|.$$

Given $\varepsilon > 0$, let V_1, V_2 be the neighborhoods of radius $\varepsilon/2C$ of $\mathbf{y}_0$, $\mathbf{z}_0$, respectively, and let $V_1' = V_0 \cap V_1$. If $\varepsilon \leq 2C$, then $V_1' = V_1$. By hypothesis there are punctured neighborhoods U_1, U_2 of $\mathbf{x}_0$ such that $\mathbf{f}(U_1) \subset V_1', \mathbf{g}(U_2) \subset V_2$. Let $U = U_1 \cap U_2$. For every $\mathbf{x} \in U$, $\mathbf{f}(\mathbf{x}) \in V_0$ and hence $|\mathbf{f}(\mathbf{x})| < C$. From (*),

$$|\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) - \mathbf{y}_0 \cdot \mathbf{z}_0| < C\frac{\varepsilon}{2C} + C\frac{\varepsilon}{2C} = \varepsilon$$

for every $\mathbf{x} \in U$. This proves (3). □

Definition. A transformation $\mathbf{f}$ is called *bounded* on a set A if there exists C such that $|\mathbf{f}(\mathbf{x})| \leq C$ for every $\mathbf{x} \in A$.

In the course of the proof we showed that if $\mathbf{f}$ has a limit at $\mathbf{x}_0$, then $\mathbf{f}$ is bounded on some punctured neighborhood U_0 of $\mathbf{x}_0$.

Limits of components

Since $\mathbf{f}$ has values in E^m,

$$\mathbf{f}(\mathbf{x}) = (f^1(\mathbf{x}), \ldots, f^m(\mathbf{x}))$$

where f^i is a real valued function called the ith component of $\mathbf{f}$ (with respect to the standard basis of E^m). For instance, in Problem 2, Section 2.1, $f^1(x, y) = (x^2 + y^2)^{1/2}$ and $f^2(x, y) = x - y$. The next proposition shows that convergence of $\mathbf{f}(\mathbf{x})$ to $\mathbf{y}_0$ is equivalent to convergence of each component $f^i(\mathbf{x})$ to the corresponding component of $\mathbf{y}_0$.

Proposition 2.2. $\mathbf{y}_0 = \lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x})$ *if and only if* $y_0^i = \lim_{\mathbf{x} \to \mathbf{x}_0} f^i(\mathbf{x})$ *for each* $i = 1, \ldots, m$.

The proof is left to the reader (Problem 10).

Limits along lines

Proposition 2.3. *If* $\mathbf{y}_0 = \lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x})$, *then for any* $\mathbf{v} \neq \mathbf{0}$,

$$\mathbf{y}_0 = \lim_{t \to 0} \mathbf{f}(\mathbf{x}_0 + t\mathbf{v}).$$

PROOF. Let V be any neighborhood of $\mathbf{y}_0$. There exists $\delta > 0$ such that $\mathbf{f}(\mathbf{x}) \in V$ whenever $0 < |\mathbf{x} - \mathbf{x}_0| < \delta$. If $0 < |t| < \delta/|\mathbf{v}|$, then

$$|(\mathbf{x}_0 + t\mathbf{v}) - \mathbf{x}_0| = |t||\mathbf{v}| < \delta.$$

Hence $\mathbf{f}(\mathbf{x}_0 + t\mathbf{v}) \in V$ for every t in the punctured $\delta/|\mathbf{v}|$-neighborhood of 0. □

The points $\mathbf{x}_0 + t\mathbf{v}$ lie on the line through $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{v}$. Roughly speaking, Proposition 2.3 states that if $\mathbf{f}$ has a limit $\mathbf{y}_0$ at $\mathbf{x}_0$, then $\mathbf{y}_0$ is also the limit as $\mathbf{x}_0$ is approached along any line containing $\mathbf{x}_0$. When $\mathbf{f}$ fails to have a limit at $\mathbf{x}_0$ this fact can often be discovered by testing $\mathbf{f}$ along various lines.

EXAMPLE 1. Let $f(x, y) = x^2/(x^2 + y^2)$, $(x, y) \neq (0, 0)$, and let $\mathbf{x}_0 = (0, 0)$. Taking $\mathbf{v} = \mathbf{e}_1 = (1, 0), f(t, 0) = 1$ for every $t \neq 0$. Hence $f(t, 0) \to 1$ as $t \to 0$. Similarly, taking $\mathbf{v} = \mathbf{e}_2 = (0, 1), f(0, t) = 0$ for every $t \neq 0$ and $f(0, t) \to 0$ as $t \to 0$. Since these limits are different, f has no limit at $(0, 0)$.

As we show below, if $\mathbf{x}_0 = (x_0, y_0) \neq (0, 0)$, then $f(x, y) \to f(x_0, y_0)$ as $(x, y) \to (x_0, y_0)$. Thus $(0, 0)$ is the only point where there is no limit.

The converse to Proposition 2.3 is false, as shown in Example 2.

EXAMPLE 2. Let $f(x, y) = (y^2 - x)^2/(y^4 + x^2)$, $(x, y) \neq (0, 0)$, and again let $\mathbf{x}_0 = (0, 0)$. Consider any $\mathbf{v} = (h, k) \neq (0, 0)$. Then

$$f(th, tk) = \frac{(tk^2 - h)^2}{t^2k^4 + h^2}$$

which tends to 1 as $t \to 0$. However, every punctured neighborhood of $(0, 0)$ contains part of the parabola $y^2 = x$, and $f(y^2, y) = 0$. Hence f does not have a limit at $(0, 0)$.

Continuity

Let us now suppose that $\mathbf{x}_0$ is an interior point of the domain D of the transformation $\mathbf{f}$.

Definition. A transformation $\mathbf{f}$ is *continuous* at $\mathbf{x}_0$ if $\mathbf{f}(\mathbf{x}_0) = \lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x})$. When $\mathbf{f}$ is continuous at $\mathbf{x}_0$, punctured neighborhoods may be replaced by neighborhoods. The definition may be restated: $\mathbf{f}$ is continuous at $\mathbf{x}_0$ if for every neighborhood V of $\mathbf{f}(\mathbf{x}_0)$ there is a neighborhood U of $\mathbf{x}_0$ such that $\mathbf{f}(U) \subset V$.

EXAMPLE 3. Let $f(x, y) = (xy)^{1/3}$. Show directly from the definition that f is continuous at $(0, 0)$. Denote the radii of the neighborhoods U and V by δ and ε, respectively. We must show that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x, y) - f(0, 0)| < \varepsilon$ whenever $x^2 + y^2 < \delta^2$. Note that $f(0, 0) = 0$.

From the inequality $x^2 \pm 2xy + y^2 \geq 0$, we get

$$|xy| \leq \tfrac{1}{2}(x^2 + y^2)$$

$$|f(x, y)| = |xy|^{1/3} \leq 2^{-1/3}(x^2 + y^2)^{1/3}.$$

Given $\varepsilon > 0$, we choose $\delta = \sqrt{2\varepsilon^3}$. Then $x^2 + y^2 < \delta^2$ implies $|f(x, y)| < \varepsilon$ as required.

One does not ordinarily have to verify continuity of a transformation directly from the definition (as in Example 3). Instead, one builds up a stock of transformations known to be continuous and shows that other transformations can be expressed in terms of these by such operations as addition, multiplication and composition of functions.

Let us consider for the moment real valued functions f, g.

Proposition 2.4. *Let f and g be continuous at $\mathbf{x}_0$. Then $f + g$ and fg are continuous at x_0. If $g(x_0) \neq 0$, then fg^{-1} is continuous at x_0.*

This is a consequence of Proposition 2.1 and Problem 8.

The continuity of a transformation $\mathbf{f} = (f^1, \ldots, f^m)$ can be verified from continuity of its components f^i:

Proposition 2.5. $\mathbf{f}$ *is continuous at $\mathbf{x}_0$ if and only if f^i is continuous at $\mathbf{x}_0$ for each $i = 1, \ldots, m$.*

This is a consequence of Proposition 2.2.

EXAMPLE 4. A constant function, $f(\mathbf{x}) = c$ for all $\mathbf{x}$, is continuous. In the definition, the neighborhood U can be chosen arbitrarily.

EXAMPLE 5. Let $\mathbf{I}(\mathbf{x}) = \mathbf{x}$, the identity transformation. Then $\mathbf{I}$ is everywhere continuous (take $U = V$ above). Therefore the components of $\mathbf{I}$ are everywhere continuous. In this book these components are called the *standard cartesian coordinate functions*, and are denoted by $X^1, \ldots, X^n$. For each $\mathbf{x} = (x^1, \ldots, x^n)$, $X^i(\mathbf{x}) = x^i$.

EXAMPLE 6. Any polynomial in n variables is everywhere continuous. This is proved by induction on the degree of the polynomial using the continuity of the coordinate functions X^i and of constant functions.

EXAMPLE 7. A rational function $f(\mathbf{x}) = P(\mathbf{x})/Q(\mathbf{x})$, where P and Q are polynomials, is continuous at each point where $Q(\mathbf{x}) \neq 0$ by Proposition 2.4. For instance, in Examples 1 and 2, f is continuous at each $(x, y) \neq (0, 0)$.

It is shown in Theorem 2.7 that the composite of two continuous transformations is continuous. In Section 4–4 it is shown that any differentiable transformation is continuous. For a real valued function f of one variable, differentiability of f at x_0 is equivalent to the existence of the derivative $f'(x_0)$.

Limits at ∞

Let us call a set of the form $\{\mathbf{x} : |\mathbf{x}| > b\}$ a *punctured neighborhood* of ∞. The definition of "limit at ∞" then reads: $\mathbf{y}_0 = \lim_{|\mathbf{x}|\to\infty} \mathbf{f}(\mathbf{x})$ if for every neighborhood V of $\mathbf{y}_0$ there exists a punctured neighborhood U of ∞ such that $\mathbf{f}(U) \subset V$.

When f is real valued we say that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = +\infty$ if for every $C > 0$ there is a punctured neighborhood U of $\mathbf{x}_0$ such that $f(\mathbf{x}) > C$ whenever $\mathbf{x} \in U$. The definition of "$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = -\infty$" is similar.

Problems

1. Find the limit at $\mathbf{x}_0$ if it exists.
(a) $f(x, y) = xy/(x^2 + y^2)$, $\mathbf{x}_0 = \mathbf{e}_1 + \mathbf{e}_2$.
(b) $f(x, y) = xy/(x^2 + y^2)$, $\mathbf{x}_0 = (0, 0)$.
(c) $f(x) = (1 - \cos x)/x^2$, $x_0 = 0$. [*Hint*: $\lim_{\lambda\to 0} (\sin x)/x = 1$.]
(d) $\mathbf{f}(x) = |x - 2|\mathbf{e}_1 + |x + 2|\mathbf{e}_2$, $x_0 = 3$.
(e) $\mathbf{f}(x, y) = y\mathbf{e}_1 + (xy)^2/[(xy)^2 + (x - y)^2]\mathbf{e}_2$, $\mathbf{x}_0 = (0, 0)$.
At which points is each of these functions continuous?

2. Prove (2) of Proposition 2.1.

3. Show that if $\mathbf{y}_0 = \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})$, then $|\mathbf{y}_0| = \lim_{\mathbf{x}\to\mathbf{x}_0} |\mathbf{f}(\mathbf{x})|$. Prove that the converse holds if $\mathbf{y}_0 = \mathbf{0}$.

4. Let $f(x) = |x|^a$, where $a > 0$. Show that f is continuous at $x_0 = 0$ directly from the definition of continuous function.

5. Let $f(x, y) = x \cos(y^{-1})$ if $y \neq 0$, and $f(x, 0) = 0$. At which points is f continuous?

6. Find the limit if it exists.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^2 + y^2}$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$.

(c) $\displaystyle\lim_{|\mathbf{x}|\to\infty} \frac{(\mathbf{x}\cdot\mathbf{x}_1)(\mathbf{x}\cdot\mathbf{x}_2)}{\mathbf{x}\cdot\mathbf{x}}$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are given vectors not $\mathbf{0}$.

(d) $\displaystyle\lim_{|\mathbf{x}|\to\infty} \frac{|\mathbf{x} - \mathbf{x}_1|}{|\mathbf{x} - \mathbf{x}_2|}$.

7. (a) Let $g(\mathbf{x}) = |\mathbf{f}(\mathbf{x})|^a$ where $a > 0$. Suppose that $\mathbf{f}$ is continuous at $\mathbf{x}_0$ and $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$. Show that g is continuous at $\mathbf{x}_0$.
(b) Use (a) and Problem 3 to give another proof that the function in Example 3 is continuous at (0, 0).

8. Let $y_0 = \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$, $z_0 = \lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})$. Show that if $z_0 \neq 0$ then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{y_0}{z_0}.$$

9. Show that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = +\infty$ if and only if $\lim_{\mathbf{x}\to\mathbf{x}_0}[f(\mathbf{x})]^{-1} = 0$ and $f(\mathbf{x}) > 0$ for every $\mathbf{x}$ in some punctured neighborhood of $\mathbf{x}_0$.

10. Prove Proposition 2.2 in two different ways. [*Hints*: For one proof use the definition of limit directly and the inequality, for any vector $\mathbf{h} = (h^1, \ldots, h^n)$,

$$|h^i| \leq |\mathbf{h}| \leq \sqrt{n}(|h^1| + \cdots + |h^n|).$$

Take $\mathbf{h} = \mathbf{f}(\mathbf{x}) - \mathbf{y}_0$. For the other proof, write $\mathbf{f} = f^1\mathbf{e}_1 + \cdots + f^n\mathbf{e}_n$ and note that $\mathbf{f}(\mathbf{x}) \cdot \mathbf{e}_i = f^i(\mathbf{x})$.]

2.3 Sequences in E^n

An *infinite sequence* is a function whose domain is the set of positive integers. For brevity, we use the term "sequence" to mean infinite sequence. In this section let us consider sequences with values in E^n. It is customary to denote by $\mathbf{x}_m$ the value of the function at the integer $m = 1, 2, \ldots$, and to call $\mathbf{x}_m$ the mth *term* of the sequence. The sequence itself is denoted by $\mathbf{x}_1, \mathbf{x}_2, \ldots$, or for brevity by $[\mathbf{x}_m]$. It must not be confused with the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ whose elements are the terms of the sequence. This set may be finite or infinite. For instance if $x_m = (-1)^m$ then the sequence is $-1, 1, -1, \ldots$, and the set $\{x_1, x_2, \ldots\}$ has only two elements -1 and 1.

Definition. Suppose that for every $\varepsilon > 0$ there exists a positive integer N such that $|\mathbf{x}_m - \mathbf{x}_0| < \varepsilon$ for every $m \geq N$. Then $\mathbf{x}_0$ is the *limit* of the sequence $[\mathbf{x}_m]$.

The notations "$\mathbf{x}_0 = \lim_{m\to\infty} \mathbf{x}_m$" and "$\mathbf{x}_m \to \mathbf{x}_0$ as $m \to \infty$" are used to mean that $\mathbf{x}_0$ is the limit of the sequence $[\mathbf{x}_m]$. A sequence is called *convergent* if it has a limit, otherwise *divergent*. The integer N in the definition depends of course on ε. Given ε there is a smallest possible choice for N. However, for purposes of the theory of limits it is of no interest to calculate it. What matters is the fact that some N exists.

Proposition 2.6. *Let* $\mathbf{x}_0 = \lim_{m\to\infty} \mathbf{x}_m$, $\mathbf{y}_0 = \lim_{m\to\infty} \mathbf{y}_m$. *Then*:

(a) $\mathbf{x}_0 + \mathbf{y}_0 = \lim_{m\to\infty}(\mathbf{x}_m + \mathbf{y}_m)$.

(b) $c\mathbf{x}_0 = \lim_{m\to\infty} c\mathbf{x}_m$ *for any scalar* c.

(c) $\mathbf{x}_0 \cdot \mathbf{y}_0 = \lim_{m\to\infty} \mathbf{x}_m \cdot \mathbf{y}_m$.

Let x_m^i denote the ith component of the vector $\mathbf{x}_m$.

Proposition 2.7. $\mathbf{x}_0 = \lim_{m\to\infty} \mathbf{x}_m$ *if and only if* $x_0^i = \lim_{m\to\infty} x_m^i$ *for each* $i = 1, \ldots, n$.

We leave the proof of Propositions 2.6 and 2.7 to the reader (Problem 6). The proofs of the corresponding Propositions 2.1 and 2.2 in Section 2.2 can be adapted for this purpose.

In this section we prove three important results (Theorems 2.1–2.3) that depend on the least upper bound property of the real numbers (Axiom III, Section 1.1).

The first of these results is about convergence of monotone sequences of real numbers.

A sequence $[x_m]$ of real numbers is called *monotone* if either $x_1 \le x_2 \le x_3 \le \cdots$ or $x_1 \ge x_2 \ge x_3 \ge \cdots$. In the first instance the sequence is *nondecreasing*, in the second *nonincreasing*. A sequence $[\mathbf{x}_m]$ is *bounded* if there is a number C such that $|\mathbf{x}_m| \le C$ for every $m = 1, 2, \ldots$.

Theorem 2.1. *Every bounded monotone sequence of real numbers has a limit.*

PROOF. Let $[x_m]$ be nondecreasing and bounded. Let $x_0 = \sup\{x_1, x_2 \ldots\}$. Given $\varepsilon > 0$ there exists an N such that $x_0 - x_N < \varepsilon$. Otherwise $x_m \le x_0 - \varepsilon$ for every $m = 1, 2, \ldots$, and $x_0 - \varepsilon$ would be a smaller upper bound than the least upper bound x_0. Since the sequence is nondecreasing, $x_N \le x_m \le x_0$ for every $m \ge N$. Hence $|x_m - x_0| = x_0 - x_m < \varepsilon$ for every $m \ge N$. This shows that $x_m \to x_0$ as $m \to \infty$.

If the sequence is nonincreasing and bounded, let $x_0 = \inf\{x_1, x_2, \ldots\}$. In the same way $x_m \to x_0$ as $m \to \infty$. □

EXAMPLE 1. Let $0 < a < 1$. The sequence $a, a^2, a^3, \ldots$ of its powers is decreasing, and is bounded below by 0. The limit of the sequence is $b = \inf\{a, a^2, a^3, \ldots\}$. Since $b \le a^{m+1}$, $a^{-1}b \le a^m$ for $m = 1, 2, \ldots$. Therefore $a^{-1}b$ is a lower bound for $\{a, a^2, a^3, \ldots\}$. Since b is the greatest lower bound, $a^{-1}b \le b$. However, $a^{-1}b \ge b$ since $0 < a < 1$; and hence $a^{-1}b = b$. This implies that $b = 0$.

EXAMPLE 2. Let $y_m = \sum_{k=0}^{m}(1/k!)$. The sequence $[y_m]$ is nondecreasing. Since $(k!)^{-1} \le 2^{-k+1}$ for $k \ge 1$

$$y_m \le 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{m-1}} < 3.$$

Thus $[y_m]$ is bounded. This sequence has a limit y_0.

EXAMPLE 3. Let $x_m = (1 + 1/m)^m$. From the binomial formula

$$x_m = 1 + m\frac{1}{m} + \frac{m(m-1)}{2!}\frac{1}{m^2} + \cdots + \frac{m!}{m!}\frac{1}{m^m},$$

which can be rewritten as

$$x_m = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{m}\right) + \frac{1}{3!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)$$
$$+ \cdots + \frac{1}{m!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\cdots\left(1 - \frac{m-1}{m}\right).$$

When x_m is replaced by x_{m+1}, the product multiplying $(k!)^{-1}$ becomes larger for $k = 2, 3, \ldots, m$, and an additional term appears. Thus $x_m \leq x_{m+1}$. Moreover, $x_m \leq y_m \leq y_0 \leq 3$, with y_m as in Example 2. Thus the sequence $[x_m]$ has a limit x_0, and $x_0 \leq y_0$. In fact, $x_0 = y_0$ (Problem 11). This number is the base for natural logarithms $x_0 = e = 2.718\cdots$.

Let us turn to the second main result of the present section. The definition of limit has the sometimes inconvenient feature that it refers not only to the sequence $[\mathbf{x}_m]$, but to the limit $\mathbf{x}_0$. This limit $\mathbf{x}_0$ is not known in advance, although one can sometimes guess what $\mathbf{x}_0$ must be. The following is a related concept due to Cauchy. It is stated purely in terms of the sequence, without reference to a limit.

Definition. If for every $\varepsilon > 0$ there exists a positive integer N such that $|\mathbf{x}_l - \mathbf{x}_m| < \varepsilon$ for every $l, m \geq N$, then $[\mathbf{x}_m]$ is a *Cauchy sequence.*

We may agree that $l \geq m$ in this definition without loss of generality.

EXAMPLE 4. Let $x_m = \sum_{k=1}^{m} (\cos k)/k^2$. Let us show that $[x_m]$ is a Cauchy sequence. If $l \geq m$, then

$$x_l - x_m = \sum_{k=m+1}^{l} \frac{\cos k}{k^2}$$

$$|x_l - x_m| \leq \sum_{k=m+1}^{l} \frac{1}{k^2}.$$

Since $k^{-2} \leq t^{-2}$ for $k - 1 \leq t \leq k$, we get (as in the integral test for convergence of an infinite series)

$$|x_l - x_m| \leq \int_m^l \frac{dt}{t^2} = m^{-1} - l^{-1} < m^{-1}.$$

Given $\varepsilon > 0$, choose $N > \varepsilon^{-1}$. Then

$$|x_l - x_m| < \varepsilon \quad \text{for } N \leq m \leq l.$$

Thus $[x_m]$ is a Cauchy sequence.

Fortunately, it turns out that the concepts of Cauchy sequence and convergent sequence are equivalent, for sequences in E^n.

Theorem 2.2 (Cauchy convergence criterion). *A sequence* $[\mathbf{x}_m]$ *is convergent if and only if it is a Cauchy sequence.*

PROOF. Let $[\mathbf{x}_m]$ be convergent, and $\mathbf{x}_0$ be its limit. Then given $\varepsilon > 0$ there exists N such that $|\mathbf{x}_m - \mathbf{x}_0| < \varepsilon/2$ for every $m \geq N$. Now

$$\mathbf{x}_l - \mathbf{x}_m = (\mathbf{x}_l - \mathbf{x}_0) + (\mathbf{x}_0 - \mathbf{x}_m).$$

If $l, m \geq N$, then by the triangle inequality

$$|\mathbf{x}_l - \mathbf{x}_m| \leq |\mathbf{x}_l - \mathbf{x}_0| + |\mathbf{x}_0 - \mathbf{x}_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $[\mathbf{x}_m]$ is a Cauchy sequence.

The proof of the converse is more difficult. Let us first show that every Cauchy sequence is bounded. If $[\mathbf{x}_m]$ is Cauchy, then taking $\varepsilon = 1$ in the definition there is an N such that $|\mathbf{x}_l - \mathbf{x}_m| < 1$ for every $l, m \geq N$. In particular, let $l = N$ and let $C = \max\{|\mathbf{x}_1|, \ldots, |\mathbf{x}_{N-1}|, |\mathbf{x}_N| + 1\}$. Then by the triangle inequality and the fact that $\mathbf{x}_m = \mathbf{x}_N + (\mathbf{x}_m - \mathbf{x}_N)$,

$$|\mathbf{x}_m| \leq |\mathbf{x}_N| + |\mathbf{x}_m - \mathbf{x}_N| < |\mathbf{x}_N| + 1$$

if $m \geq N$. Therefore $|\mathbf{x}_m| \leq C$ for every $m = 1, 2, \ldots$.

Next, let $[x_m]$ be a Cauchy sequence of real numbers, and C be a number such that $|x_m| \leq C$ for every m. For each $m = 1, 2, \ldots$ let

$$y_m = \inf\{x_m, x_{m+1}, \ldots\}.$$

Since $\{x_{m+1}, x_{m+2}, \ldots\} \subset \{x_m, x_{m+1}, \ldots\}$, we must have $y_m \leq y_{m+1}$. Since $x_m \leq C$ for every m, $y_m \leq C$ for every m. The sequence $[y_m]$ is nondecreasing and bounded. By Theorem 2.1 the sequence $[y_m]$ has a limit y_0. Let us show that $x_m \to y_0$ as $m \to \infty$. Given $\varepsilon > 0$ there exists N such that for every $m \geq N$, $|x_N - x_m| < \varepsilon/2$, and hence

$$x_N - \frac{\varepsilon}{2} < x_m < x_N + \frac{\varepsilon}{2}.$$

If $m \geq N$, then $x_N - \varepsilon/2$ is a lower bound and $x_N + \varepsilon/2$ an upper bound for $\{x_m, x_{m+1}, \ldots\}$. Therefore when $m \geq N$

$$x_N - \frac{\varepsilon}{2} \leq y_m \leq y_0 \leq x_N + \frac{\varepsilon}{2},$$

$$|x_m - y_0| \leq |x_m - x_N| + |x_N - y_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that $x_m \to y_0$ as $m \to \infty$.

Finally, if $[\mathbf{x}_m]$ is a Cauchy sequence in E^n, then for each $i = 1, \ldots, n$ the components form a Cauchy sequence $[x_m^i]$ of real numbers. This follows from the definition of Cauchy sequence and the inequality $|x_l^i - x_m^i| \leq |\mathbf{x}_l - \mathbf{x}_m|$. Each of the sequences $[x_m^i]$ has a limit y_0^i, $i = 1, \ldots, n$. By Proposition 2.7, $\mathbf{x}_m \to \mathbf{y}_0$ as $m \to \infty$. □

For the next theorem we need the concept of diameter of a set. A nonempty set A is called *bounded* if there exists $C > 0$ such that $|\mathbf{x}| \leq C$ for all $\mathbf{x} \in A$. The *diameter* of a bounded set A is

$$\operatorname{diam} A = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in A\}.$$

Theorem 2.3 (Cantor). *Let $[A_m]$ be a sequence of closed sets such that $A_1 \supset A_2 \supset \cdots$ and $0 = \lim_{m\to\infty} \operatorname{diam} A_m$. Then $A_1 \cap A_2 \cap \cdots$ contains a single point.*

PROOF. For each $m = 1, 2, \ldots$ let $\mathbf{x}_m$ be a point of A_m. Let us show that $[\mathbf{x}_m]$ is a Cauchy sequence. Given $\varepsilon > 0$, there exists N such that $\operatorname{diam} A_N < \varepsilon$. If $l, m \geq N$, then $\mathbf{x}_l, \mathbf{x}_m \in A_N$ since $A_l \subset A_N$, $A_m \subset A_N$. Therefore

$$|\mathbf{x}_l - \mathbf{x}_m| \leq \operatorname{diam} A_N < \varepsilon.$$

By Theorem 2.2 the sequence $[\mathbf{x}_m]$ has a limit $\mathbf{x}_0$. For each $l = 1, 2, \ldots$, $\mathbf{x}_m \in A_l$ for every $m \geq l$ since $A_m \subset A_l$. Since A_l is closed, $\mathbf{x}_0 \in A_l$ (Problem 5). Since this is true for each l, $\mathbf{x}_0 \in A_1 \cap A_2 \cap \cdots$.

If $\mathbf{x} \in A_1 \cap A_2 \cap \cdots$, then $\mathbf{x} \in A_m$ and

$$0 \leq |\mathbf{x} - \mathbf{x}_0| \leq \operatorname{diam} A_m, \qquad m = 1, 2, \ldots$$

Since $\operatorname{diam} A_m \to 0$ as $m \to \infty$, $|\mathbf{x} - \mathbf{x}_0| = 0$ and $\mathbf{x} = \mathbf{x}_0$. □

EXAMPLE 5. Let $n = 1$, $A_m = \{m, m + 1, m + 2, \ldots\}$. Then each A_m is closed, unbounded, and $A_1 \supset A_2 \supset \cdots$. The intersection $A_1 \cap A_2 \cap \cdots$ is empty.

If in Theorem 2.3 it is not assumed that the diameter of A_m tends to 0, then it is still true that $A_1 \cap A_2 \cap \cdots$ is not empty, provided some set A_m is bounded. This is proved in Section 2.4.

Note. Theorems 2.1–2.3 depend in an essential way on the least upper bound Axiom III. Let us suppose, on the other hand, that instead of Axiom III we took the following.

Axiom III_1

(a) *E^1 has the archimedean property.*
(b) *Every Cauchy sequence in E^1 has a limit.*

Then Axioms III and III_1 are equivalent in the presence of Axioms I and II, Section 1.1. This means that if either III or III_1 is taken as an axiom, then the other can be proved as a theorem. We have shown that III implies III_1. To show that III_1 implies III, we first note that if III_1 is taken as an axiom, then Theorem 2.3 remains valid (with the same proof). It then suffices to deduce the least upper bound property from the archimedean property and Theorem 2.3. To prove this, let S be any subset of E^1 that is bounded above, and let c be some upper bound for S. Let us define a sequence of closed intervals

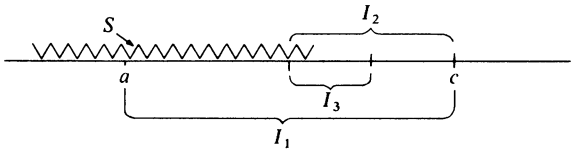


Figure 2.2

$I_1, I_2, \ldots$ as follows: Let a be some point of S, and $I_1 = [a, c]$. Divide I_1 at the midpoint $(a + c)/2$ into two congruent closed intervals. If $(a + c)/2$ is an upper bound for S, let I_2 be the left-hand interval, otherwise let I_2 be the right-hand interval. In general, suppose $m \geq 1$ and I_m has been defined. If the midpoint of I_m is an upper bound for S, let I_{m+1} be the left half of I_m, otherwise the right half. The archimedean property implies that for any $x \geq 0$, the sequence $[m^{-1}x]$ tends to 0 as $m \to \infty$. Since $0 \leq 2^{-m} \leq m^{-1}$, the sequence $[2^{-m}x]$ also tends to 0. Let $x = 2(c - a)$. Now $I_1 \supset I_2 \supset \cdots$ and the length of I_m is $2^{-m}x$. By Theorem 2.3, $I_1 \cap I_2 \cap \cdots$ contains a single point x_0. By the construction, $x_0 = \sup S$ (see Figure 2.2).

Infinite series

Formally, an infinite series is an expression written $\sum_{k=1}^{\infty} \mathbf{x}_k$ or $\mathbf{x}_1 + \mathbf{x}_2 + \cdots$. To be more precise, with any sequence $[\mathbf{x}_k]$ is associated another sequence $[\mathbf{s}_m]$, where $\mathbf{s}_m = \mathbf{x}_1 + \cdots + \mathbf{x}_m$ is called the mth *partial sum*. This pair of sequences defines an *infinite series*. If the sequence of partial sums has a limit $\mathbf{s}$, then the series is *convergent* and $\mathbf{s}$ is its *sum*. This is denoted by $\mathbf{s} = \mathbf{x}_1 + \mathbf{x}_2 + \cdots$. If the sequence of partial sums has no limit, then the series is *divergent*.

If $\mathbf{s} = \mathbf{x}_1 + \mathbf{x}_2 + \cdots$, $\mathbf{t} = \mathbf{y}_1 + \mathbf{y}_2 + \cdots$, then $\mathbf{s} + \mathbf{t} = (\mathbf{x}_1 + \mathbf{y}_1) + (\mathbf{x}_2 + \mathbf{y}_2) + \cdots$ and $c\mathbf{s} = (c\mathbf{x}_1) + (c\mathbf{x}_2) + \cdots$ for any scalar c. This follows from the definition and Proposition 2.6. Some further elementary properties are given in Problems 7(c) and 8.

Problems

In Problems 1 and 2 you may use the results of Problems 9 and 10.

1. Find the limit if it exists.
(a) $x_m = (2^m - 2^{-m})/(3^m + 3^{-m})$.
(b) $x_m = \sin(m\pi/2)$.
(c) $x_m = \sin m\pi$.
(d) $x_m = ((m + 1)/(m - 1))^m$. [*Hint*: $(1 + 1/m)^m \to e$ as $m \to \infty$.]
(e) $x_m = ((m^2 + 1)/(m^2 - 1))^m$.

2. Find the limit if it exists, using Proposition 2.
(a) $(x_m, y_m) = ((1 + m)/(1 - 2m), 1/(1 + m))$.
(b) $(x_m, y_m) = (2^{-m}, 1 + m)$.
(c) $(x_m, y_m) = (1 - 2^{-m}, (m^2 + 3^m)/m!)$.

3. Show that a sequence $[\mathbf{x}_m]$ has at most one limit $\mathbf{x}_0$. [*Hint*: If $\mathbf{y}_0$ were another limit, let $\varepsilon = |\mathbf{x}_0 - \mathbf{y}_0|/2$.]

4. (a) Let A be a closed set. Show that if $\mathbf{x}_m \in A$ for $m = 1, 2, \ldots$ and $\mathbf{x}_0 = \lim_{m\to\infty} \mathbf{x}_m$, then $\mathbf{x}_0 \in A$.
   (b) If A is not closed, show that there exists a sequence $[\mathbf{x}_m]$, with $\mathbf{x}_m \in A$ for $m = 1, 2, \ldots$, converging to a limit $\mathbf{x}_0 \notin A$.

5. Show that if $\mathbf{x}_m \in A$ for every $m \geq l$ and $\mathbf{x}_0 = \lim_{m\to\infty} \mathbf{x}_m$, then $\mathbf{x}_0 \in \text{cl } A$.

6. (a) Prove Proposition 2.6.
   (b) Prove Proposition 2.7.

7. (*Comparison tests.*) Show that:
   (a) If $0 \leq x_m \leq y_m$ for every $m \geq l$ and $y_m \to 0$ as $m \to \infty$, then $x_m \to 0$ as $m \to \infty$.
   (b) If $[x_m]$, $[y_m]$ are nondecreasing sequences such that $x_m \leq y_m$ for each $m = 1, 2, \ldots$ and $y_m \to y$ as $m \to \infty$, then $[x_m]$ has a limit $x \leq y$.
   (c) If $0 \leq x_m \leq y_m$ for every $m = 1, 2, \ldots$ and $t = y_1 + y_2 + \cdots$, then the series $x_1 + x_2 + \cdots$ converges with sum $s \leq t$.

8. An infinite series $\mathbf{x}_1 + \mathbf{x}_2 + \cdots$ converges *absolutely* if the series of nonnegative numbers $|\mathbf{x}_1| + |\mathbf{x}_2| + \cdots$ converges. Prove that any absolutely convergent infinite series is convergent. [*Hint*: Show that the sequence $[\mathbf{s}_m]$ of partial sums is Cauchy.]

9. Show that if $a > 0$, then
   (a) $\lim_{m\to\infty} a^{1/m} = 1$. (b) $\lim_{m\to\infty} a^m/m! = 0$.
   (c) $\lim_{m\to\infty}(x_m)^{1/m} = 1$ provided $\lim_{m\to\infty} x_m = a$.
   [*Hints*: For part (a) reduce to the case $0 < a < 1$. By Example 1, if $b < 1$ then $a \leq b^m$ for only finitely many m. For part (b), compare with the sequence $[c/m]$ for suitable c and suitable l in Problem 7(a).]

10. Let $x_0 = \lim_{m\to\infty} x_m$, $y_0 = \lim_{m\to\infty} y_m$, and assume that $y_m \neq 0$ for $m = 0, 1, 2, \ldots$. Show that $x_0/y_0 = \lim_{m\to\infty} x_m/y_m$. [*Hint*: By (c) of Proposition 2.6 it suffices to show that $y_0^{-1} = \lim_{m\to\infty} y_m^{-1}$.]

11. Show that $x_0 = y_0$ in Example 3.

2.4 Bolzano–Weierstrass theorem

Suppose that an infinite number of points lie in a box. It is intuitively reasonable that they cannot remain scattered but must accumulate at some points of the box. The purpose of the present section is to put this idea on a precise basis. We begin with definitions of the concepts of isolated point and accumulation point of a set $A \subset E^n$.

Definition. A point $\mathbf{x}_0$ is an *isolated point* of A if there exists a neighborhood U of $\mathbf{x}_0$ such that $A \cap U = \{\mathbf{x}_0\}$.

Definition. A point $\mathbf{x}_0$ is an *accumulation point* of A if every neighborhood of $\mathbf{x}_0$ contains an infinite number of points of A.

In the definition of accumulation point we do not require that $\mathbf{x}_0 \in A$. However, the next proposition shows that accumulation points are in the closure of A.

Proposition 2.8. *A point $\mathbf{x}_0$ is an accumulation point of A if and only if $\mathbf{x}_0 \in \operatorname{cl} A$ and $\mathbf{x}_0$ is not an isolated point of A.*

PROOF. Suppose that $\mathbf{x}_0 \in \operatorname{cl} A$ and $\mathbf{x}_0$ is not an isolated point of A. Let U_1 be any neighborhood of $\mathbf{x}_0$. From the definitions of closure and isolated point there exists $\mathbf{x}_1 \in A \cap U_1$, $\mathbf{x}_1 \neq \mathbf{x}_0$. Let us show that $A \cap U_1$ contains infinitely many points. Suppose not. Then $A \cap U_1$ contains finitely many points $\mathbf{x}_1, \ldots, \mathbf{x}_l$ different from $\mathbf{x}_0$. Choose a smaller neighborhood U of radius $\delta < \min\{|\mathbf{x}_m - \mathbf{x}_0|, m = 1, \ldots, l\}$. Either $A \cap U$ is empty or $A \cap U = \{\mathbf{x}_0\}$. The first possibility contradicts the assumption that $\mathbf{x}_0 \in \operatorname{cl} A$, and the second the assumption that $\mathbf{x}_0$ is not isolated. Thus $\mathbf{x}_0$ is an accumulation point of A. The converse follows immediately from the definitions. □

EXAMPLE 1. Every point of the closed interval $[a, b]$ is an accumulation point of the open interval (a, b).

EXAMPLE 2. Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Every point of A is an isolated point of A. The set A has the single accumulation point 0.

EXAMPLE 3. Let $A = \{\text{all rational numbers}\}$. Then every point of E^1 is an accumulation point of A.

The next proposition characterizes accumulation points in terms of convergent sequences.

Proposition 2.9. *A point $\mathbf{x}_0$ is an accumulation point of A if and only if there exists a sequence $[\mathbf{x}_m]$ with limit $\mathbf{x}_0$, such that $\mathbf{x}_m \in A$ and $\mathbf{x}_m \neq \mathbf{x}_0$ for $m = 1, 2, \ldots$.*

The proof is left to the reader (Problem 2).

We come now to the main results of the present section. A set A is called infinite if the number of its elements is not finite. A set $I \subset E^n$ of the form

$$I = \left\{\mathbf{x}: |x^i - x_0^i| \leq \frac{a}{2}, i = 1, \ldots, n\right\},$$

where

$$\mathbf{x} = (x^1, \ldots, x^n), \qquad \mathbf{x}_0 = (x_0^1, \ldots, x_0^n),$$

is called an *n-cube* with center $\mathbf{x}_0$ and side length a. If A is any bounded set, then $A \subset I$ for some n-cube I.

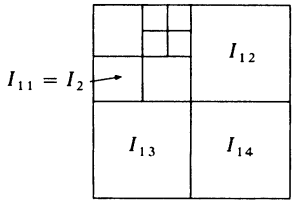


Figure 2.3

Theorem (Bolzano–Weierstrass). *Every bounded infinite set $A \subset E^n$ has at least one accumulation point.*

PROOF. Let A be a bounded infinite set and I_1 be some closed n-cube containing A. Divide I_1 into $m = 2^n$ closed congruent n-cubes $I_{11}, \ldots, I_{1m}$ as indicated in Figure 2.3. Since A is an infinite set, $A \cap I_{1k}$ must be infinite for at least one $k = 1, \ldots, m$. Choose some such k and let $I_{1k} = I_2$. In the same way divide I_2 into 2^n closed congruent n-cubes $I_{21}, \ldots, I_{2m}$. As before, $A \cap I_{2k}$ is infinite for at least one k. Choose such a k and let $I_{2k} = I_3$. Continuing, we obtain closed n-cubes $I_1 \supset I_2 \supset I_3 \supset \cdots$ such that $A \cap I_l$ is infinite for each $l = 1, 2, \ldots$ and diam $I_l \to 0$ as $l \to \infty$. By Theorem 2.3, $I_1 \cap I_2 \cap \cdots$ has a single point $\mathbf{x}_0$. If U is any neighborhood of $\mathbf{x}_0$, then $I_l \subset U$ for large enough l. Since $A \cap I_l \subset A \cap U$, $A \cap U$ is an infinite set. Therefore $\mathbf{x}_0$ is an accumulation point of A. □

The Bolzano–Weierstrass theorem has the following useful consequences.

Corollary 1. *Let $S \subset E^n$ be closed and bounded. Then every infinite set $A \subset S$ has at least one accumulation point $\mathbf{x}_0 \in S$.*

PROOF. Since S is a bounded set and $A \subset S$, A is a bounded set. By the Bolzano–Weierstrass theorem, A has an accumulation point $\mathbf{x}_0$. By Proposition 2.8, $\mathbf{x}_0 \in \text{cl } A$. Since S is closed, $\text{cl } A \subset S$. Thus $\mathbf{x}_0 \in S$. □

Note. The *converse* to Corollary 1 is true. To prove it consider first an unbounded set S. For each $m = 1, 2, \ldots$ there exists $\mathbf{x}_m \in S$ such that $|\mathbf{x}_m| \geq m$. The set $A = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ is infinite and has no accumulation point. If S is not closed, then there exists a point $\mathbf{x}_0 \in \text{fr } S - S$. For $m = 1, 2, \ldots$ there exists $\mathbf{x}_m \in S$ such that $|\mathbf{x}_m - \mathbf{x}_0| < 1/m$. The set $A = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ is infinite and has the single accumulation point $\mathbf{x}_0$. But $\mathbf{x}_0 \notin S$.

Corollary 2. *Let $A_1, A_2, \ldots$ be nonempty bounded, closed subsets of E^n such that $A_1 \supset A_2 \supset \cdots$. Then $\bigcap_{m=1}^{\infty} A_m$ is not empty.*

PROOF. For $m = 1, 2, \ldots$ choose some point $\mathbf{x}_m \in A_m$. Let $A = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$. If A is a finite set, then some $\mathbf{x} = \mathbf{x}_m$ is repeated infinitely often in the sequence $[\mathbf{x}_m]$. Since $A_1 \supset A_2 \supset \cdots$, $\mathbf{x} \in A_m$ for every $m = 1, 2, \ldots$. Thus $\mathbf{x} \in \bigcap_{m=1}^{\infty} A_m$.

Let us consider the case when A is an infinite set. Then $A \subset A_1$ since each $A_m \subset A_1$. Since A_1 is bounded, A is bounded. Let $\mathbf{x}_0$ be an accumulation point of A. As in the proof of Corollary 1, we have $\mathbf{x}_0 \in A_1$ since A_1 is a closed set and $A \subset A_1$. For each $m = 1, 2, \ldots, \mathbf{x}_0$ is also an accumulation point of the set $\{\mathbf{x}_m, \mathbf{x}_{m+1}, \ldots\}$. Since this set is contained in A_m and A_m is closed, we get in the same way $\mathbf{x}_0 \in A_m$. Since this is true for each m, $\mathbf{x}_0 \in \bigcap_{m=1}^{\infty} A_m$. □

Corollary 3 shows the existence of a point in any closed set A nearest a given point $\mathbf{x}_0 \notin A$.

Corollary 3. *Let A be a closed, nonempty subset of E^n and $\mathbf{x}_0 \notin A$. Then there exists $\mathbf{x}_1 \in A$ such that $|\mathbf{x} - \mathbf{x}_0| \geq |\mathbf{x}_1 - \mathbf{x}_0|$ for all $\mathbf{x} \in A$.*

PROOF. Let $S_r = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ denote the closed spherical n-ball with center $\mathbf{x}_0$ and radius r. Let

$$d = \inf\{|\mathbf{x} - \mathbf{x}_0| : \mathbf{x} \in A\}$$

$$d_m = d + \frac{1}{m}, \quad m = 1, 2, \ldots$$

$$A_m = A \cap S_{d_m}.$$

The sets $A_1, A_2, \ldots$ satisfy the hypotheses of Corollary 2. Let $\mathbf{x}_1 \in \bigcap_{m=1}^{\infty} A_m$. Then $\mathbf{x}_1 \in A$ since each $A_m \subset A$. By definition of d, $|\mathbf{x}_1 - \mathbf{x}_0| \geq d$. Since $\mathbf{x}_1 \in A_m$, and $A_m \subset S_{d_m}$, $|\mathbf{x}_1 - \mathbf{x}_0| \leq d + 1/m$ for each $m = 1, 2, \ldots$. Thus $|\mathbf{x}_1 - \mathbf{x}_0| = d$. □

Note. The point $\mathbf{x}_1$ in Corollary 3 need not be unique. However, $\mathbf{x}_1$ is unique if A is also convex (Problem 3).

PROBLEMS

1. Find all accumulation points of A:
   (a) $A = \{(-1)^m m(1 + m)^{-1} : m = 1, 2, \ldots\}$.
   (b) $A = \left\{\left(\cos\frac{2m\pi}{5}, \sin\frac{2m\pi}{5}\right) : m = 1, 2, \ldots\right\}$.
   (c) $A = \left\{\left(\left(1 - \frac{1}{m}\right)\cos\frac{2m\pi}{5}, \left(1 - \frac{1}{m}\right)\sin\frac{2m\pi}{5}\right) : m = 1, 2, \ldots\right\}$.
   (d) $A = \{(x, y) : (x^2 + y^2)(y^2 - x^2 + 1) \leq 0\}$.
   (e) $A = \{\cos m : m = 1, 2, \ldots\}$.
2. Prove Proposition 2.9.
3. Let A be a closed, convex, nonempty set, and $\mathbf{x}_0 \notin A$. Show that there is exactly one point $\mathbf{x}_1 \in A$ nearest $\mathbf{x}_0$.

4. A set A is called *dense in* B if every point of B is an accumulation point of A.
   (a) Suppose that A has no isolated points. Show that A is dense in B if and only if $B \subset \text{cl } A$.
   (b) Suppose that A is dense in B and B is dense in C. Show that A is dense in C.

5. Let $C = [0, 1] - (A_1 \cup A_2 \cup \cdots)$, where $A_1 = (\frac{1}{3}, \frac{2}{3})$, $A_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, $A_3 = (\frac{1}{27}, \frac{2}{27}) \cup \cdots \cup (\frac{25}{27}, \frac{26}{27})$, and A_j is the union of 2^{j-1} open intervals of length 3^{-j} chosen similarly (see Figure 2.4). [*Note*: C is called the *Cantor set*.]
   (a) Show that C is a closed set.
   (b) Show that C is dense in no open set.

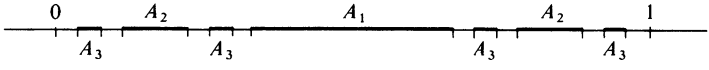


Figure 2.4

6. (*Subsequences.*) Let $[\mathbf{x}_m]$ be a sequence, and $\mathbf{y}_l = \mathbf{x}_{m_l}$ for $l = 1, 2, \ldots$, where $m_1 < m_2 < \ldots$. Then $[\mathbf{y}_l]$ is called a *subsequence* of $[\mathbf{x}_m]$.
   (a) Show that any bounded sequence in E^n has a convergent subsequence.
   (b) A set S is called *sequentially compact* if: any bounded sequence $[\mathbf{x}_m]$, with $\mathbf{x}_m \in S$ for $m = 1, 2, \ldots$, has a subsequence $[\mathbf{y}_l]$ such that $\mathbf{y}_l \to \mathbf{y}_0$ as $l \to \infty$, $\mathbf{y}_0 \in S$. Show that a nonempty set $S \subset E^n$ is sequentially compact if and only if S is closed and bounded.

7. Let $\mathbf{y}$ be any frontier point of a closed convex set K. Show that K has a supporting hyperplane P that contains $\mathbf{y}$. [*Hint*: Let $\{\mathbf{y}_m\}$ be a sequence of points exterior to K such that $\mathbf{y}_m$ tends to $\mathbf{y}$ as $m \to \infty$. Let $\mathbf{x}_m$ be a point of K nearest to $\mathbf{y}_m$ and

$$\mathbf{u}_m = \frac{\mathbf{y}_m - \mathbf{x}_m}{|\mathbf{y}_m - \mathbf{x}_m|}.$$

   Then $|\mathbf{u}_m| = 1$ and $\mathbf{x}_m$ tends to $\mathbf{y}$ as $m \to \infty$. By the proof of Theorem 1.1 there is a supporting hyperplane of the form $\{\mathbf{x} : \mathbf{u}_m \cdot (\mathbf{x} - \mathbf{x}_m) = 0\}$. Let $\mathbf{u}$ be an accumulation point of the bounded set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots\}$ and $P = \{\mathbf{x} : \mathbf{u} \cdot (\mathbf{x} - \mathbf{y}) = 0\}$.]

2.5 Relative neighborhoods, continuous transformations

For the definition in Section 2.2 of a transformation continuous at a point $\mathbf{x}_0$, it is assumed that $\mathbf{x}_0$ is interior to the domain D. However, we often wish to discuss continuity at points which are not interior to the domain. Moreover, even if the domain D is an open subset of E^n, we may be interested only in the restriction of the transformation to some set $S \subset D$.

We easily circumvent this apparent difficulty by introducing the idea of relative neighborhood.

Definition. Let S be a nonempty subset of E^n. A *relative neighborhood* of a point $\mathbf{x} \in S$ is any set U such that $U = S \cap W$, where W is a neighborhood of $\mathbf{x}$ in E^n (Figure 2.5).

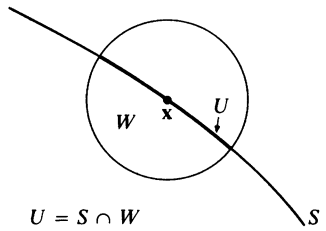


Figure 2.5

In considering the relative neighborhood $U = S \cap W$ one simply ignores all points of W not in S.

Let us now consider a transformation $\mathbf{f}$, whose domain is a set $S \subset E^n$.

Definition. The transformation $\mathbf{f}$ is *continuous* at $\mathbf{x}_0$ if, for every neighborhood V of $\mathbf{f}(\mathbf{x}_0)$, there exists a relative neighborhood U of $\mathbf{x}_0$ such that $\mathbf{f}(U) \subset V$. If $\mathbf{f}$ is continuous at every point $\mathbf{x}_0 \in S$, then we call $\mathbf{f}$ *continuous on* S.

An equivalent form of the definition of continuity at $\mathbf{x}_0$ is: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| < \varepsilon$ whenever $\mathbf{x} \in S$ and $|\mathbf{x} - \mathbf{x}_0| < \delta$ (cf. Section 2.2.)

EXAMPLE 1. Let $S = [a, b]$. The relative neighborhoods are described in Problem 3. Let f be real valued. For $a < x_0 < b$, continuity at x_0 means the same as that in Section 2.2. Continuity at a means the following: given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $a \leq x < a + \delta$. This is expressed by writing $f(a) = \lim_{x \to a^+} f(x)$, where the notation $^+$ denotes right-hand limit. Similarly, continuity at b is written $f(b) = \lim_{x \to b^-} f(x)$, where superscript $^-$ denotes left-hand limit.

Theorem 2.4. *Let* $\mathbf{f}$ *be continuous on a closed, bounded set* S. *Then* $\mathbf{f}(S)$ *is also a closed, bounded set.*

In this section we give a proof of Theorem 2.4 using Corollary 1 to the Bolzano–Weierstrass theorem. A different proof is given in Section 2.8, using the concept of compactness.

PROOF. The proof is by contradiction. Suppose that $\mathbf{f}(S)$ is unbounded. Then for each $m = 1, 2, \ldots$ there exists $\mathbf{x}_m \in S$ such that $|\mathbf{f}(\mathbf{x}_m)| \geq m$. The set $A = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ is infinite, and $A \subset S$. By Corollary 1, Section 2.4, A has an accumulation point $\mathbf{x}_0 \in S$. Let V be the neighborhood of $\mathbf{x}_0$ of radius 1. Since $\mathbf{f}$ is continuous at $\mathbf{x}_0$, there exists a relative neighborhood U of $\mathbf{x}_0$ such that $\mathbf{f}(U) \subset V$. In particular, $\mathbf{f}(\mathbf{x}_m) \in V$ for infinitely many m such that $\mathbf{x}_m \in U$. Since $|\mathbf{f}(\mathbf{x}_m)| \geq m$, this is a contradiction. Thus $\mathbf{f}(S)$ is bounded.

Next suppose that $\mathbf{f}(S)$ is not closed. Then there exists $\mathbf{y}_0 \in \mathrm{cl}[\mathbf{f}(S)] - \mathbf{f}(S)$. For $m = 1, 2, \ldots$ there exists $\mathbf{y}_m \in \mathbf{f}(S)$ such that $\mathbf{y}_m \to \mathbf{y}_0$ as $m \to \infty$. Choose $\mathbf{z}_m \in S$ with $\mathbf{y}_m = \mathbf{f}(\mathbf{z}_m)$. As before, the set $B = \{\mathbf{z}_1, \mathbf{z}_2, \ldots\}$ has an accumulation point $\mathbf{z}_0 \in S$. Since $\mathbf{y}_0 \notin \mathbf{f}(S)$, $\mathbf{y}_0 \neq \mathbf{f}(\mathbf{z}_0)$. Let V be a neighborhood of $\mathbf{f}(\mathbf{z}_0)$ of radius $\frac{1}{2}|\mathbf{f}(\mathbf{z}_0) - \mathbf{y}_0|$. Since $\mathbf{f}$ is continuous at $\mathbf{z}_0$, there exists a relative neighborhood U of $\mathbf{z}_0$ with $\mathbf{f}(U) \subset V$. In particular, $\mathbf{y}_m = \mathbf{f}(\mathbf{z}_m)$ is in V for infinitely many m for which $\mathbf{z}_m \in U$. This contradicts the fact that $\mathbf{y}_m \to \mathbf{y}_0$ as $m \to \infty$. Thus $\mathbf{f}(S)$ is closed. □

If we specialize Theorem 2.4 to real valued functions, we get a theorem about the existence of maxima and minima.

Theorem 2.5. *Let f be a real valued function continuous on a closed, bounded set S. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in S$ such that*

$$f(\mathbf{x}_1) \leq f(\mathbf{x}) \leq f(\mathbf{x}_2)$$

for all $\mathbf{x} \in S$.

PROOF. We recall that if $T \subset E^1$ is bounded and closed, then $y_1 = \inf T$ and $y_2 = \sup T$ are points of T (Example 4, Section 1.4). Let $T = f(S)$. By Theorem 2.4, T is closed and bounded. Take $\mathbf{x}_i$ such that $y_i = f(\mathbf{x}_i)$, $i = 1, 2$. □

The function f is said to have a *minimum* on S at $\mathbf{x}_1$ and a *maximum* on S at $\mathbf{x}_2$.

PROBLEMS

1. In each case show that f has a minimum on S, but no maximum on S. Which assumption in Theorem 2.5 is violated?
   (a) $S = (0, 1]$, $f(x) = x^{-1}$.
   (b) $S = E^n$, $f(\mathbf{x}) = \dfrac{|\mathbf{x}|}{1 + |\mathbf{x}|}$.
2. Given $\mathbf{x}_0$, let $f(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_0|$. Show that f has a minimum on any closed, nonempty set $A \subset E^n$. (This gives another proof of Corollary 3, Section 2.4.)
3. Let $S = [a, b]$.
   (a) Show that the relative neighborhoods of a are the half open intervals $[a, c)$ with $a < c < b$, and S.
   (b) Let $a < x_0 < \frac{1}{2}(a + b)$. Show that the relative neighborhoods of x_0 are as follows: $(x_0 - \delta, x_0 + \delta)$ if $0 < \delta < x_0 - a$; $[a, x_0 + \delta)$ if $x_0 - a \leq \delta < b - x_0$; S.
   (c) Describe the relative neighborhoods in the remaining cases $\frac{1}{2}(a + b) \leq x_0 \leq b$.
4. Define the projection π from E^n onto E^s as in Problem 7, Section 2.1.
   (a) Show that $\pi(A)$ is closed and bounded if A is closed and bounded.
   (b) Give an example of a closed set A such that $\pi(A)$ is not closed.

5. (a) Let $\mathbf{f}$ be continuous on S, and let $S_1 \subset S$. Show that the restriction $\mathbf{f}|S_1$ is continuous on S_1.

(b) Let $f(x) = 1 - x$ if $x \geq 0$ and $f(x) = 0$ if $x < 0$. Let $S_1 = [0, \infty)$, $S = E^1$. Show that $f|S_1$ is continuous, but f is not continuous at each point of S_1.

6. Let $\mathbf{f}$ be a transformation with domain S. Show that $\mathbf{f}$ is continuous at $\mathbf{x}_0$ if and only if $\mathbf{f}(\mathbf{x}_0) = \lim_{m \to \infty} \mathbf{f}(\mathbf{x}_m)$ for every sequence $[\mathbf{x}_m]$ such that $\mathbf{x}_m \in S$ for $m = 1, 2, \ldots$ and $\mathbf{x}_m \to \mathbf{x}_0$ as $m \to \infty$.

7. Let f be continuous on E^n. Suppose, moreover, that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and that $f(c\mathbf{x}) = cf(\mathbf{x})$ for any $\mathbf{x}$ and $c > 0$. Show that there exist $a > 0$ and $b > 0$ such that $a|\mathbf{x}| \leq f(\mathbf{x}) \leq b|\mathbf{x}|$. [*Hint*: First consider $\{\mathbf{x} : |\mathbf{x}| = 1\}$.]

8. (*Uniform continuity.*) A transformation $\mathbf{f}$ is *uniformly continuous* on $S \subset E^n$ if given $\varepsilon > 0$ there exists $\delta > 0$ (depending only on ε) such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$ for every $\mathbf{x}, \mathbf{y} \in S$ with $|\mathbf{x} - \mathbf{y}| < \delta$. Show that if S is closed and bounded then every $\mathbf{f}$ continuous on S is uniformly continuous on S. [*Hint*: If not, then there exists $\varepsilon > 0$ and for $m = 1, 2, \ldots, \mathbf{x}_m, \mathbf{y}_m \in S$ such that $|\mathbf{f}(\mathbf{x}_m) - \mathbf{f}(\mathbf{y}_m)| \geq \varepsilon$ and $|\mathbf{x}_m - \mathbf{y}_m| \leq 1/m$. Let $\mathbf{x}_0$ be an accumulation point of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$. Show that the continuity of $\mathbf{f}$ at $\mathbf{x}_0$ is contradicted.]

2.6 Topological spaces

In order to proceed further with the study of subsets of E^n and continuous transformations, it is convenient to introduce a very general concept—that of topological space. In this section S denotes a set, not necessarily a subset of E^n, and p denotes a point of S.

The notion of topological space occurs in practically all branches of mathematics. There are several equivalent definitions; of these, we give the one in terms of neighborhoods.

Definition. Let S be a nonempty set. For every $p \in S$ let $\mathscr{U}_p$ be a collection of subsets of S called *neighborhoods* of p such that:

(1) Every point p has at least one neighborhood.
(2) Every neighborhood of p contains p.
(3) If U_1 and U_2 are neighborhoods of p, then there is a neighborhood U_3 of p such that $U_3 \subset U_1 \cap U_2$.
(4) If U is a neighborhood of p and $q \in U$, then there is a neighborhood V of q such that $V \subset U$.

Then S is a *topological space.*

More precisely, the topological space is S together with the collections $\mathscr{U}_p$ of neighborhoods. However, it is common practice to omit explicit reference to the collections of neighborhoods when no ambiguity can arise.

For our purposes, the following two examples of topological spaces are of primary importance.

EXAMPLE 1. Let $S = E^n$, and as in Section 1.4, let $\mathscr{U}_{\mathbf{x}}$ be the collection of all open spherical n-balls with center $\mathbf{x}$. Clearly, Axioms (1) and (2) of the definition above are satisfied, and in (3) we may take $U_3 = U_1 \cap U_2$. Axiom (4) is verified in Example 1, Section 1.4. Thus E^n is a topological space.

EXAMPLE 2. Let $S \subset E^n$. Let neighborhoods of $\mathbf{x} \in S$ be all relative neighborhoods $U = S \cap W$, where W is a neighborhood of $\mathbf{x}$ in E^n, as in Section 2.5. The topology on S defined by the collections of relative neighborhoods is the *relative topology*. It is discussed further later in the section. Roughly speaking, the relative topology is the one obtained by simply ignoring the complementary set $S^c = E^n - S$.

Further examples of topological spaces appear in the homework problems. The metric spaces (Section 2.9) furnish an important class of topological spaces.

In any topological space S, the basic notions of interior, frontier, and closure are defined just as in Section 1.4 for the topological space E^n. For instance, p is interior to a set $A \subset S$ if some neighborhood of p is contained in A.

Definition. Let S be a topological space. A set $A \subset S$ is *open* if every point $p \in A$ is interior to A. A set $A \subset S$ is *closed* if $S - A$ is open.

Axiom (4) (in the preceding definition) guarantees that any neighborhood is an open set. Propositions 1.1–1.3 in Section 1.4 about unions and intersections of open sets (or closed sets) remain true in any topological space. The proofs are almost the same as before.

It often happens that two different collections of neighborhoods $\mathscr{U}_p$, $\mathscr{U}'_p$ lead to the same collection of open subsets of S. In E^n we need not have started with spherical neighborhoods. For instance, the neighborhoods obtained from any noneuclidean norm on E^n lead to the same open sets as in Section 1.4. (cf. Section 2.11).

The open sets, and not the particular kinds of neighborhoods from which they were obtained, determine all of the topological properties of S. Thus we say that the collections $\mathscr{U}_p$, $\mathscr{U}'_p$ *define the same topology on* S if they lead to the same collection of open sets.

EXAMPLE 3. Let $S = E^n$. Let $\mathscr{U}'_{\mathbf{x}}$ consist of all n-cubes

$$U' = \{\mathbf{y} : |y^i - x^i| < \delta, i = 1, \ldots n\}$$

for any $\delta > 0$. The reader should verify that the collections $\mathscr{U}'_{\mathbf{x}}$ of "neighborhoods" in this sense define the same topology on E^n as the usual topology of E^n in Example 1 (Problem 1). The key fact is that each $U' \in \mathscr{U}'_{\mathbf{x}}$ is contained in some $U \in \mathscr{U}_{\mathbf{x}}$, and vice versa.

Continuous functions

The concept of continuous function from one topological space into another is a natural extension of that already considered for transformations between euclidean spaces.

Definition. Let f denote a function from a topological space S into a topological space T. The function f is *continuous at* p_0 if for every neighborhood V of $f(p_0)$ there exists a neighborhood U of p_0 such that $f(U) \subset V$.

If $S \subset E^n$, $T = E^m$, and S is given the relative topology, then this definition of continuity agrees with that in Section 2.5.

Proposition 2.4'. *Let f and g be real valued functions, continuous at p_0. Then $f + g$ and fg are continuous at p_0. If $g(p_0) \neq 0$, then fg^{-1} is continuous at p_0.*

Proposition 2.5'. *A function* $\mathbf{f}$*, from a topological space S into E^m, is continuous at p_0 and only if each component f^i is continuous at p_0, $i = 1, \ldots, m$.*

We leave the proof of Propositions 2.4' and 2.5' to the reader (Problems 11 and 12).

Definition. If f is continuous at every $p \in S$, we call f *continuous on* S.

The following is a convenient characterization of continuous functions, in terms of open subsets of S and T.

Theorem 2.6. *f is continuous on S if and only if the inverse image $f^{-1}(B)$ of any open set B is open.*

PROOF. Let f be continuous on S and $B \subset T$ be open. Let p be any point of $f^{-1}(B)$ and V be a neighborhood of $f(p)$ such that $V \subset B$. Since f is continuous, there is a neighborhood U of p such that $f(U) \subset V$. Then $U \subset f^{-1}(B)$, which shows that $f^{-1}(B)$ is open.

Conversely, let $f^{-1}(B)$ be open for each open set B. Let p be any point of S, and V be any neighborhood of $f(p)$. Since V is open, $f^{-1}(V)$ is open and contains p. Let U be a neighborhood of p such that $U \subset f^{-1}(V)$. Then $f(U) \subset V$, which shows that f is continuous at p. Since this is true for every $p \in S$, f is continuous on S. □

Since the complement of an open set is closed and $f^{-1}(B^c) = [f^{-1}(B)]^c$, we have the following.

Corollary 1. *f is continuous on S if and only if the inverse image $f^{-1}(B)$ of any closed set B is closed.*

Let us next specialize to real valued functions f and special choices for B.

Corollary 2. *If f is real valued and continuous on S, then $\{p : f(p) > c\}$ is an open set, and $\{p : f(p) \geq c\}$, $\{p : f(p) = c\}$ are closed sets.*

PROOF. In this case $T = E^1$. The semiinfinite interval $B = (c, \infty)$ is open, and $\{p : f(p) > c\} = f^{-1}(B)$. The other statements are obtained by taking $B = [c, \infty)$, $B = \{c\}$, which are closed sets. □

By applying Corollary 2 to $-f$ and $-c$, the set $\{p : f(p) < c\}$ is open and $\{p : f(p) \leq c\}$ is closed. Corollary 2 often provides a convenient way to show that a given set is open or closed.

EXAMPLE 4. Let $A = \{(x, y) : y^2 < x^2 + 1\}$. Take $S = E^2$ and $f(x, y) = x^2 - y^2 + 1$. Then $A = \{(x, y) : f(x, y) > 0\}$; hence A is open. Similarly, the set $\{(x, y) : y^2 \leq x^2 + 1\}$ and the hyperbola $\{(x, y) : y^2 = x^2 + 1\}$ are closed sets.

Composites

Let f be a function from S into T, and g from R into S. The composite $f \circ g$ is defined by

$$(f \circ g)(r) = f[g(r)] \quad \text{for every } r \in R.$$

Theorem 2.7. *If g is continuous at r_0 and f is continuous at $p_0 = g(r_0)$, then $f \circ g$ is continuous at r_0.*

PROOF. Let V be any neighborhood of $f(p_0)$ (see Figure 2.6). There is a neighborhood U of p_0 such that $f(U) \subset V$. Moreover, there is a neighborhood W of r_0 such that $g(W) \subset U$. Then $(f \circ g)(W) = f[g(W)] \subset f(U) \subset V$. This shows that $f \circ g$ is continuous at r_0. □

EXAMPLE 5. Let $D \subset E^n$ be open. Given $\mathbf{x}_0$ and $\mathbf{v}$, let $\mathbf{g}(t) = \mathbf{x}_0 + t\mathbf{v}$ for every scalar t. Then $\mathbf{g}$ is continuous from E^1 into E^n. By Proposition 2.6, $\Delta = \{t : \mathbf{x}_0 + t\mathbf{v} \in D\}$ is open. Let f be continuous on D and let $\phi(t) = f(\mathbf{x}_0 + t\mathbf{v})$ for $t \in \Delta$. Then $\phi = f \circ (\mathbf{g} | \Delta)$, and by Theorem 2.7, ϕ is continuous on Δ. This result is similar to Proposition 2.3.

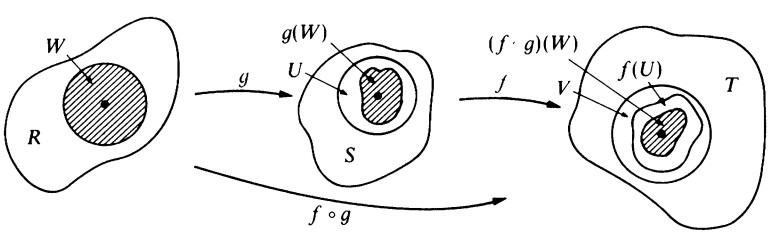


Figure 2.6

The following consequence of Theorem 2.7 is used in Chapter 5.

EXAMPLE 6. Let $1 \leq s \leq n-1$. Let us regard E^n as the cartesian product $E^s \times E^{n-s}$ and write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, where $\mathbf{x}' = (x^1, \ldots, x^s)$, $\mathbf{x}'' = (x^{s+1}, \ldots, x^n)$. Given $\mathbf{x}_0' \in E^s$, let $\mathbf{g}$ be the function from E^{n-s} into E^n such that $\mathbf{g}(\mathbf{x}'') = (\mathbf{x}_0', \mathbf{x}'')$ for every $\mathbf{x}'' \in E^{n-s}$. Such a function $\mathbf{g}$ is called an *injection*. Since $|\mathbf{g}(\mathbf{x}'') - \mathbf{g}(\mathbf{y}'')| = |\mathbf{x}'' - \mathbf{y}''|$, $\mathbf{g}$ is continuous. Let $D \subset E^n$ be open, and let $D(\mathbf{x}_0') = \{\mathbf{x}'' : (\mathbf{x}_0', \mathbf{x}'') \in D\}$. Since $D(\mathbf{x}_0') = \mathbf{g}^{-1}(D)$, by Theorem 2.6 $D(\mathbf{x}_0')$ is an open subset of E^{n-s}. Let f be continuous on D. The function $f(\mathbf{x}_0', \ )$ whose value at each $\mathbf{x}'' \in D(\mathbf{x}_0')$ is $f(\mathbf{x}_0', \mathbf{x}'')$ is the composite of f and $\mathbf{g}|D(\mathbf{x}_0')$. By Theorem 2.7, $f(\mathbf{x}_0', \ )$ is continuous. Similarly, given $\mathbf{x}_0''$, the set $\{\mathbf{x}' : (\mathbf{x}', \mathbf{x}_0'') \in D\}$ is open and the function $f(\ , \mathbf{x}_0'')$ is continuous.

Subspaces

Let S_0 be a topological space and S a nonempty subset of S_0. By disregarding the complement $S_0 - S$, S becomes a topological space in the following way. If $p \in S$, then a *neighborhood of p relative to S* is a set $U = S \cap W$, where W is a neighborhood in S_0 of p. Axioms (1) through (4) (in the first definition in this section) are satisfied. For instance, to prove (3) let U_1, U_2 be relative neighborhoods of p. Then $U_1 = S \cap W_1$, $U_2 = S \cap W_2$, where W_1 and W_2 are neighborhoods in S_0 of p. Since S_0 is a topological space, there is a neighborhood W_3 in S_0 of p with $W_3 \subset W_1 \cap W_2$. Then $U_3 = S \cap W_3$ is a relative neighborhood of p and $U_3 \subset U_1 \cap U_2$. This topology is called the *relative topology* induced on S by the topology of S_0, and S is a topological subspace of S_0.

In particular, if $S_0 = E^n$, this is the relative topology on S mentioned in Section 2.5 and Example 2 (above).

The following proposition gives a convenient way to describe the sets that are open relative to S, or closed relative to S.

Proposition 2.10. *Let S be a topological subspace of S_0. A set $A \subset S$ is open in the relative topology if and only if $A = S \cap D$, where D is an open subset of S_0. Similarly, A is closed in the relative topology if and only if $A = S \cap D$ where D is a closed subset of S_0.*

PROOF. Suppose that $A = S \cap D$, where D is an open subset of S_0. Any $p \in A$ has a neighborhood W in S_0 such that $W \subset D$. Then $U = S \cap W$ is a relative neighborhood of p such that $U \subset A$. Hence A is relatively open.

Conversely, let A be relatively open. For each $p \in A$, let U_p be some relative neighborhood of p with $U_p \subset A$. Then $U_p = S \cap W_p$, where W_p is a neighborhood of p in S_0. Let $D = \bigcup_{p \in A} W_p$. Then D is an open subset of S_0, and $A = S \cap D$.

The statements about closed sets are then obtained by considering the sets $S - A$ and $S_0 - D$. □

EXAMPLE 7. Let $S = [a, b]$, $S_0 = E^1$, as in Example 1, Section 2.5. For $a < c < b$, the interval $A = [a, c)$ is open relative to S, since $A = S \cap D$ where $D = (-\infty, c)$ is an open subset of E^1.

EXAMPLE 8. Let $S = \{\text{all rational numbers}\}$, $S_0 = E^1$, $A = \{\text{rational } x : x^2 < 2\}$. Then

$$A = S \cap (-\infty, \sqrt{2}) = S \cap (-\infty, \sqrt{2}].$$

Since $(-\infty, \sqrt{2})$ is an open subset of E^1 and $(-\infty, \sqrt{2}]$ is a closed subset of E^1, A is both open and closed relative to S.

Homeomorphisms

Let f be a univalent function from a topological space S onto a topological space T. Then f has an inverse f^{-1}, whose domain is T. For each $q \in T$ its value $f^{-1}(q)$ is the unique $p \in S$ such that $f(p) = q$. If both f and its inverse f^{-1} are continuous functions, then f is a *homeomorphism*. If there is a homeomorphism from S onto T, then S and T are *homeomorphic* topological spaces.

In topology, homeomorphic spaces S and T may be regarded as indistinguishable. Every topological property enjoyed by S is also enjoyed by T.

For a deeper introduction to the literature on the general theory of topological spaces, we refer to the work by Kelley [14].

PROBLEMS

1. Consider E^n with $\mathscr{U}_{\mathbf{x}}$ the collection of neighborhoods of $\mathbf{x}$ (open spherical n-balls with center $\mathbf{x}$), and $\mathscr{U}'_{\mathbf{x}}$ the collection of "neighborhoods" in Example 3.
   (a) Verify that E^n with the collections $\mathscr{U}'_{\mathbf{x}}$ of "neighborhoods" satisfies Axioms (1) through (4) for a topological space.
   (b) Show that each $U' \in \mathscr{U}'_{\mathbf{x}}$ contains some $U \in \mathscr{U}_{\mathbf{x}}$; and each $U \in \mathscr{U}_{\mathbf{x}}$ contains some $U' \in \mathscr{U}'_{\mathbf{x}}$.
   (c) Using (*b*), show that the collections $\mathscr{U}_{\mathbf{x}}$, $\mathscr{U}'_{\mathbf{x}}$ lead to the same collection of open sets, and hence define the same topology on E^n.

2. Use Corollary 2 to show that each of the following sets is closed:
   (a) $\{x : -2 \leq x \leq 2, x^3 - x \geq 0\}$.
   (b) $\{(x, y) : x^4 + y^4 = 1\}$.
   (c) $\{\mathbf{x} : \mathbf{y}_0 \cdot \mathbf{x} \leq |\mathbf{x}|\}$, $\mathbf{y}_0$ a given vector.

3. Let $S_0 = E^1$. Find whether A is open relative to S, closed relative to S, or neither.
   (a) $S = \{x : a \leq x \leq b, x \neq c\}$, $A = [a, c)$ where $a < c < b$.
   (b) $S = (0, 1]$, $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.
   (c) $S = [0, 1]$, $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

4. Let $S_0 = E^2$. Find whether A is open relative to S, closed relative to S, or neither.
   (a) $S = \{(x, y) : 0 < x^2 + y^2 < 1\}$, $A = \{(x, y) \in S : x^2 \geq y^2\}$.
   (b) $S = \{(x, y) : x^2 + y^2 = 4\}$, $A = \{(x, y) \in S : x^2 < y^2\}$.

5. Let S be an open subset of S_0. Show that the relatively open sets are just those open subsets of S_0 contained in S.

Problems 6 through 9 give some examples of topological spaces.

6. (*Indiscrete spaces.*) Let S be any set, and let every $p \in S$ have exactly one "neighborhood," namely, S itself; that is, each $\mathscr{U}_p$ consists of the set S only.
(a) Verify Axioms (1) through (4).
(b) Show that the only open sets are S and the empty set.
(c) Show that any real valued function continuous on S is constant.

7. (*Discrete spaces.*) Let S be a topological space such that the set $\{p\}$ with the one element p is a "neighborhood" of p; that is, $\{p\} \in \mathscr{U}_p$ for each $p \in S$.
(a) Show that every subset of S is open.
(b) Show that every function with domain S is continuous.
(c) Suppose that $S \subset E^n$ with the relative topology. Show that S is a discrete space if and only if every $\mathbf{x} \in S$ is isolated.

8. Let $S = E^1$ with the following nonstandard topology. In this topology the "δ-neighborhood" of x_0 is the interval $[x_0, x_0 + \delta)$.
(a) Verify Axioms (1) through (4).
(b) Show that $[0, \infty)$ is both open and closed in this topology.
(c) Let f be a real valued function from E^1 with this topology into E^1 with the standard topology. Show that continuity of f at x_0 is equivalent to $f(x_0) = \lim_{x \to x_0^+} f(x)$, where superscript $^+$ denotes righthand limit.

9. Consider the following nonstandard topology on the plane E^2: In this topology the "δ-neighborhood" of (x_0, y_0) is the set $\{(x, y) : x_0 - \delta < x < x_0 + \delta, y = y_0\}$.
(a) Verify Axioms (1) through (4).
(b) Show that if $D \subset E^2$ is open in the usual sense, then D is open in this topology, but not conversely.
(c) Let $f(x, y) = g(x)h(y)$, where g and h have domain E^1 and g is continuous in the usual topology of E^1. Show that f is continuous in this topology.

10. Let $S = \{(x, y) : x^2 + y^2 = 1\}$, $T = \{(x, y) : |x| + |y| = 1\}$, both with the relative topology. Show that S and T are homeomorphic topological spaces, by finding a homeomorphism $\mathbf{f}$ from S onto T.

11. Prove Proposition 2.4′.

12. Prove Proposition 2.5′.

2.7 Connectedness

From the intuitive point of view, a set should be regarded as connected if it consists of one piece. Thus an interval on the real line E^1 is connected, while the set $[0, 1] \cup [2, 3]$ is disconnected. For more complicated sets, intuition is not a reliable guide.

Definition. Let S_0 be a topological space. A set $S \subset S_0$ is *disconnected* if there exist nonempty sets A and B such that $S = A \cup B$, $A \cap B$ is empty, and A, B are both open relative to S. If S is not disconnected, then S is a *connected* set.

If S_0, considered as a subset of itself, is connected, then S_0 is a *connected topological space*.

EXAMPLE 1. Let $S = [0, 1] \cup [2, 3], S_0 = E^1$. To show that S is disconnected, let $A = [0, 1]$, $B = [2, 3]$. Then $A = S \cap (-1, \frac{3}{2})$, which shows that A is open relative to S by Proposition 2.10. Similarly, B is open relative to S.

EXAMPLE 2. Let $S = \{(x, y) : y^2 \leq x^2, x \neq 0\}$, $S_0 = E^2$. To show that S is disconnected, let $A = S \cap H_+$, $B = S \cap H_-$, where H_+, H_- are the open half-planes $H_+ = \{(x, y) : x > 0\}$, $H_- = \{(x, y) : x < 0\}$.

Let us next characterize completely the connected subsets of E^1.

Definition. A nonempty set $J \subset E^1$ is an *interval* if for every $x, y \in J$, $x < y$, the set $[x, y]$ is contained in J (Figure 2.7).

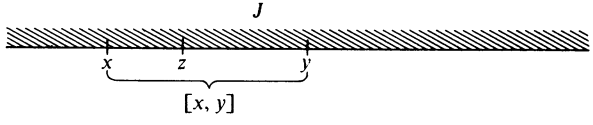


Figure 2.7

In Example 2, Section 1.1, four types of finite interval and four types of semi-infinite interval were considered. Each of these is an interval according to the definition just given. Moreover, E^1 is an interval, and any set $\{\mathbf{x}\}$ with a single point is an interval. These 10 types of intervals are the only possibilities (Problem 5).

Proposition 2.11. *A set $S \subset E^1$ is connected if and only if S is an interval.*

PROOF. If S is not an interval, then there exist $x, y \in S$, $x < y$, and $z \notin S$ such that $x < z < y$. Let $A = S \cap (-\infty, z)$, $B = S \cap (z, \infty)$. Then A and B are nonempty and relatively open, $A \cup B = S$, and $A \cap B$ is empty. Therefore S is disconnected.

Conversely, suppose that some interval J is disconnected. Then $J = A \cup B$ where A and B are not empty and open relative to J, and $A \cap B$ is empty. Let $x_1 \in A$ and $x_2 \in B$. The notation A, B may be chosen so that $x_1 < x_2$. Since J is an interval, $[x_1, x_2] \subset J$; hence the fact that A is relatively open implies that there exists $\delta_1 > 0$ such that $[x_1, x_1 + \delta_1) \subset A$. Similarly, there exists $\delta_2 > 0$ such that $(x_2 - \delta_2, x_2] \subset B$. Letting $B_1 = \{x \in B; x > x_1\}$ and $y = \inf B_1$, we have $x_1 < y < x_2$. Since J is an interval, $y \in J$. If $y \in A$, then some interval $(y - \delta, y + \delta)$ is contained in A and $y + \delta$ is a lower bound for B_1, contrary to the fact that y is the greatest lower bound. Similarly, if $y \in B$, then some interval $(y - \delta, y + \delta)$ is contained in B_1, and y is not a lower bound. This is a contradiction. □

Let f be a function from S_0 into a topological space T. For $S \subset S_0$, we call f continuous on S if the restriction $f|S$ is continuous at each $p \in S$ when S is given the relative topology. This agrees with the terminology "continuous on S" in Section 2.5 when $S_0 = E^n$ and in Section 2.6.

Theorem 2.8. *If S is a connected set and f is continuous on S, then $f(S)$ is a connected set.*

PROOF. Consider first the case $S = S_0$ and $f(S) = T$. We argue by contradiction. Suppose that T is disconnected. Then $T = P \cup Q$ where P, Q are open, nonempty, and $P \cap Q$ is empty. Let $A = f^{-1}(P)$, $B = f^{-1}(Q)$. By Theorem 2.6, A and B are open. Moreover, $S = A \cup B$, $A \cap B$ is empty, and neither A nor B is empty. Thus S is disconnected, contrary to hypothesis.

To reduce the general case to this one, consider S as a topological space with the relative topology. The space S is connected. Replace f by $f \mid S$, and T by $f(S)$, regarded as a topological subspace of T with the relative topology. Then $f \mid S$ is continuous, regarded as a function from S onto $f(S)$. By what has already been proved, $f(S)$ is connected. □

Corollary (intermediate value theorem). *If S is a connected set and f is real-valued and continuous on S, then $f(S)$ is an interval.*

PROOF. By Proposition 2.11 every connected subset of E^1 is an interval. □

As a particular instance of the intermediate value theorem, suppose that $S = [a, b]$, a closed, bounded interval. If f is real valued and continuous on S, then Theorem 2.5 implies that f has a maximum and minimum on S. Since S is connected, the intermediate value theorem implies that $f(x)$ assumes on S every value between its maximum and minimum.

**Pathwise connectedness*

Let p and q be points of a topological space S. A *path in S from p to q* is a continuous function g from $[0, 1]$ into S with $g(0) = p$, $g(1) = q$. If every such pair of points can be joined by a path in S, then S is called *pathwise connected.*

Proposition 2.12. *If S is pathwise connected, then S is connected.*

PROOF. If S is disconnected, then $S = A \cup B$ as in the definition of disconnected space. Let $p \in A$, $q \in B$, and g be a path in S joining p and q. Since g is continuous, $g^{-1}(A)$ and $g^{-1}(B)$ are open relative to $[0, 1]$, their union is $[0, 1]$, and their intersection is empty. This contradicts the fact that $[0, 1]$ is connected. □

EXAMPLE 3. Let $S = S_1 \cup S_2$, where

$$S_1 = \{(0, y) : -1 \leq y \leq 1\},$$
$$S_2 = \{(x, \sin 1/x) : x > 0\}$$

Then it can be shown that S is connected but not pathwise connected (Problem 12).

On the other hand, any *open* connected subset of E^n is pathwise connected. In fact, any two points of D can be connected by a polygonal path (Problem 10).

Problems

1. Show from the definition that the following are disconnected subsets of the plane E^2:
   (a) The hyperbola $x^2 - y^2 = 1$.
   (b) Any finite subset of E^2 with at least two elements.
   (c) $\{(x, y) : x^2 + y^2 \leq 4, y \neq x + 1\}$.

2. Show that a topological space S is disconnected if and only if there exists a continuous function f on S, with $f(p) = 0$ or $f(p) = 1$ for all $p \in S$, but $f(p)$ not constant on S.

3. Let S_1 and S_2 be connected subsets of E^n, such that $S_1 \cap S_2$ is not empty. Show that $S_1 \cup S_2$ is connected.

4. Let S be an open subset of E^n. Show that S is disconnected if and only if $S = A \cup B$ where A and B are nonempty open subsets of E^n, such that $A \cap B$ is empty. Prove the corresponding statement in which "open" is replaced everywhere by "closed."

5. Let J be an interval. Show that either $J = E^1$, J has a single point, or J is of one of the eight types listed in Example 2, Section 1.1.

6. Instead of Axiom III (least upper bound property) about the real numbers, take as an axiom the property that E^1 is connected. Prove Axiom III as a theorem. [*Hint*: Let $A = \{\text{all upper bounds of } S\}$ and $B = A^c$. Show that B is open; and if S has no least upper bound, then A is open.]

7. Show that each of the following sets is pathwise connected.
   (a) Any convex set.
   (b) The unit circle $x^2 + y^2 = 1$ in E^2.
   (c) The unit sphere $x^2 + y^2 + z^2 = 1$ in E^3.

8. Let g_1 be a path from p_1 to p_2 and g_2 a path from p_2 to p_3. Let $h(t) = g_1(2t)$ if $0 \leq t \leq \frac{1}{2}$, and $h(t) = g_2(2t - 1)$ if $\frac{1}{2} \leq t \leq 1$. Show that h is a path from p_1 to p_3.

9. Let $D \subset E^n$ be open. By *polygonal path* in D from $\mathbf{x}$ to $\mathbf{y}$ let us mean a path $\mathbf{g}$ in D from $\mathbf{x}$ to $\mathbf{y}$ with the following property: There exist $t_0, t_1, \ldots, t_m$ such that $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ and $\mathbf{g}(t) = \mathbf{g}(t_k) + (t - t_k)\mathbf{v}_k$ if $t_k \leq t \leq t_{k+1}$, where $\mathbf{v}_k = (t_{k+1} - t_k)^{-1}[\mathbf{g}(t_{k+1}) - \mathbf{g}(t_k)]$, $k = 0, 1, \ldots, m - 1$.
   Let $\mathbf{g}$ be a polygonal path in D from $\mathbf{x}$ to $\mathbf{y}$, and U be any convex set such that $\mathbf{y} \in U$ and $U \subset D$. Using Problem 8, find a polygonal path in D from $\mathbf{x}$ to any point $\mathbf{z} \in U$.

10. Let $D \subset E^n$ be open. Given $\mathbf{x} \in D$, let $A = \{\mathbf{y} : \text{there is a polygonal path in } D \text{ from } \mathbf{x} \text{ to } \mathbf{y}\}$ and let $B = D - A$. Using Problem 9, show that A and B are open sets and A is not empty. [*Hint*: Any neighborhood is convex.] If B is not empty, then D is disconnected.

11. A set S is *totally disconnected* if no connected subset of S contains more than one point. Show that the following are totally disconnected subsets of E^1:
   (a) $S = \{\text{rational numbers}\}$.
   (b) $S = C$, the Cantor set (Section 2.4, Problem 5).
   (c) Show that E^1 would be totally disconnected if we gave E^1 the nonstandard topology in Problem 8, Section 2.6.

12. Let S be as in Example 3. Show that:
(a) S is a closed set.
(b) There is no path in S joining $(0, 0)$ and any point of S_2.
(c) S is a connected set.

2.8 Compactness

Let us next introduce another property, called *compactness*, which a subset of a topological space may possess. For a set $S \subset E^n$, compactness of S turns out to be equivalent to S being closed and bounded. Therefore, Theorem 2.4 can be rederived in a more elegant way.

Compactness is defined in terms of open coverings. Let S be a subset of a topological space S_0. A collection $\mathfrak{A}$ of subsets of S_0 is a *covering* of S if every point of S belongs to some set $A \in \mathfrak{A}$, that is, if $S \subset \bigcup_{A \in \mathfrak{A}} A$. If $\mathfrak{A}' \subset \mathfrak{A}$ and $\mathfrak{A}'$ is also a covering of S, then $\mathfrak{A}'$ is a *subcovering*. If every $A \in \mathfrak{A}$ is an open subset of S_0, then $\mathfrak{A}$ is an *open covering* of S.

We are interested in whether every open covering $\mathfrak{A}$ of S has a finite subcovering; i.e. whether finitely many sets $A_1, \ldots, A_m \in \mathfrak{A}$ exist such that $S \subset A_1 \cup \cdots \cup A_m$.

EXAMPLE 1. Let $S \subset E^n$, and fix $a > 0$. Let $U_{\mathbf{x}}$ be the neighborhood of $\mathbf{x}$ of radius a, and $\mathfrak{A}$ the collection of such $U_{\mathbf{x}}$ for all $\mathbf{x} \in S$. Then $\mathfrak{A}$ is an open covering of S. Generally, one will still have a covering if some of the sets $U_{\mathbf{x}}$ are discarded from the collection. In fact, if S is a bounded set, there will be a subcovering $\mathfrak{A}'$ consisting of finitely many $U_{\mathbf{x}}$.

EXAMPLE 2. In Example 1, let $U_{\mathbf{x}}$ be some neighborhood of $\mathbf{x}$ whose radius is no longer fixed, and $\mathfrak{A}$ the collection of such $U_{\mathbf{x}}$ for all $\mathbf{x} \in S$. It may turn out that there is no finite subcovering, even though S is bounded. For instance, let $S = (0, 1]$, $U_x = (\frac{1}{2}x, \frac{3}{2}x)$, and $\mathfrak{A}$ the collection of these open intervals U_x for $0 < x \leq 1$. Then $\mathfrak{A}$ is an open covering of S. However, there do not exist finitely many points $x_1, \ldots, x_m$ in S such that $S \subset U_{x_1} \cup \cdots \cup U_{x_m}$.

Definition. A subset S of a topological space S_0 is *compact* if every open covering of S contains a finite subcovering.

If S_0, considered as a subset of itself, is compact, then S_0 is a *compact topological space.*

Let us consider the particular case $S_0 = E^n$. The compact subsets of E^n are characterized by the following theorem, and its converse.

Theorem (Heine–Borel). *If S is a closed, bounded subset of E^n, then every open covering of S contains a finite subcovering.*

PROOF. Let $\mathfrak{A}$ be an open covering of S. Suppose that no finite collection $\mathfrak{A}' \subset \mathfrak{A}$ covers S. Let us define a sequence of closed, bounded sets $S_1 \supset S_2 \supset \cdots$ such that diam $S_k \to 0$ as $k \to \infty$ and no finite subcollection of $\mathfrak{A}$ covers any S_k. Let $S_1 = S$. Since S_1 is bounded, some closed n-cube I_1 contains S_1. Divide I_1 into n-cubes $I_{11}, \ldots, I_{1m}$ as in the proof of the Bolzano–Weierstrass theorem, and let $S_{1k} = S_1 \cap I_{1k}$. Since S_1 and I_{1k} are closed, so is S_{1k}. If for every $k = 1, \ldots, m$ some finite collection $\mathfrak{A}_k \subset \mathfrak{A}$ covered S_{1k}, then $\mathfrak{A}_1 \cup \cdots \cup \mathfrak{A}_m$ would be a finite subcovering of S, contrary to assumption. Choose some k for which no finite subcollection of $\mathfrak{A}$ covers S_{1k}, and let $S_{1k} = S_2$. Repeating this process, we obtain the desired sequence of closed, bounded sets.

By Theorem 2.3, $S_1 \cap S_2 \cap \cdots$ contains a single point $\mathbf{x}_0$. Since $\mathfrak{A}$ covers S, $\mathbf{x}_0$ belongs to some set $A \in \mathfrak{A}$; and since $\mathfrak{A}$ is an open covering, A is an open set. Therefore, there is a neighborhood U of $\mathbf{x}_0$ such that $U \subset A$. Since diam $S_k \to 0$ as $k \to \infty$, $S_k \subset U$ for large enough k. For such k, S_k is covered by the subcollection of $\mathfrak{A}$ consisting of the single set A. Since by construction no finite subcollection of $\mathfrak{A}$ covers any S_k, this is a contradiction. □

EXAMPLE 3. Let $S = [0, 1]$. For $0 < x \leq 1$, define U_x as in Example 2; and let $U_0 = (-\frac{1}{3}, \frac{1}{3})$. The collection $\mathfrak{A}$ of open intervals U_x for $0 \leq x \leq 1$ is an open covering of $[0, 1]$. By the Heine–Borel theorem, there is finite subcovering. In fact, the three intervals U_0, $U_{1/2}$, U_1 give a covering of $[0, 1]$.

The *converse* to the Heine–Borel theorem is true. To prove it, suppose first that S is unbounded. Let A_m be the neighborhood of $\mathbf{0}$ of radius $m = 1, 2, \ldots$. Then $\{A_1, A_2, \ldots\}$ is an open covering of S with no finite subcovering. If S is not closed, let $\mathbf{x}_0 \in \text{fr } S - S$ and $A_m = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| > 1/m\}$. Then $\{A_1, A_2, \ldots\}$ is an open covering of S with no finite subcovering. This proves the converse.

We summarize these results as follows.

Theorem 2.9. *A set $S \subset E^n$ is compact if and only if S is closed and bounded.*

The reader is cautioned that Theorem 2.9 depends heavily on the finite dimensionality of E^n. The concepts in this theorem all make sense in any normed vector space (Section 2.9). However, closed, bounded subsets of an infinite dimensional normed vector space need not be compact. For an example, see Problem 4, Section 2.9.

Let us next show that the continuous image of a compact set S is compact. Consider a function f from S_0 into some topological space T, and $S \subset S_0$.

Theorem 2.10. *If S is a compact set and f is continuous on S, then $f(S)$ is a compact set.*

PROOF. Consider first the case $S = S_0$. Let $\mathscr{B}$ be any open covering of $f(S)$. Since f is continuous, $f^{-1}(B)$ is open for every $B \in \mathscr{B}$. The collection of sets $f^{-1}(B)$ is an open covering of S. Since S is compact, a finite subcollection $\{f^{-1}(B_1), \ldots, f^{-1}(B_m)\}$ covers S. Then $\{B_1, \ldots, B_m\}$ is a finite subcollection of $\mathscr{B}$ which covers $f(S)$. Hence $f(S)$ is compact. The case $S \subset S_0$ reduces to this one by considering S as a topological space with the relative topology and by replacing f with the restriction $f \mid S$. □

Note that Theorem 2.4 is an immediate consequence of Theorems 2.9 and 2.10. This gives another proof of Theorem 2.4, based on the Heine–Borel theorem rather than the Bolzano–Weierstrass theorem.

PROBLEMS

1. Show that any closed subset of a compact topological space is compact.
2. Let A and B be compact subsets of a topological space S_0. Show that $A \cup B$ is compact.
3. Let $A \subset E^s$ and $B \subset E^{n-s}$ be compact. Show that $A \times B$ is a compact subset of E^n. [*Hint*: Problem 8(b), Section 2.1.]
4. Let A be a nonempty subset of E^n, and let $f(\mathbf{x}) = \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in A\}$. This is the *distance* from $\mathbf{x}$ to A. Show that:
   (a) $f(\mathbf{x}) = 0$ if and only if $\mathbf{x} \in \operatorname{cl} A$.
   (b) $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| \leq |\mathbf{x}_1 - \mathbf{x}_2|$, for every $\mathbf{x}_1, \mathbf{x}_2 \in E^n$; consequently f is continuous on E^n.
   (c) $\{\mathbf{x} : f(\mathbf{x}) \leq c\}$ is compact if A is compact, for any $c \geq 0$.
5. Let A, B be nonempty subsets of E^n, and let $d = \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{x} \in B, \mathbf{y} \in A\}$.
   (a) Show that $d > 0$ if A is closed, B is compact, and $A \cap B$ is empty. [*Hint*: Problem 4.]
   (b) Give an example of closed sets A, B such that $A \cap B$ is empty but $d = 0$.
6. A *dyadic rational* number is a real number of the form $x = j2^{-k}$, where j, k are integers and $k \geq 0$. Let $S = \{\text{all dyadic rational } x : 0 < x < 1\}$. If $x = j2^{-k}$ with j odd, $1 \leq j \leq 2^k - 1$, and $k > 0$, let $U_x = (x - 4^{-k}, x + 4^{-k})$. Show that there do not exist finitely many points $x_1, \ldots, x_m \in S$ such that $S \subset U_{x_1} \cup \cdots \cup U_{x_m}$.
7. Let S be a compact topological space.
   (a) Let A_m be a closed, nonempty subset of S for $m = 1, 2, \ldots$ such that $A_1 \supset A_2 \supset \cdots$. Show that $\bigcap_{m=1}^{\infty} A_m$ is not empty. [*Hint*: Consider the collection $\mathfrak{A}$ of open sets $S - A_m$, $m = 1, 2, \ldots$.]
   (b) Let A_m be a closed, nonempty subset of S for $m = 1, 2, \ldots$, such that $A_1 \cap \cdots \cap A_l$ is not empty for each $l = 1, 2, \ldots$. Show that $\bigcap_{m=1}^{\infty} A_m$ is not empty. [*Hint*: Part (a).]

2.9 Metric spaces

In Section 2.6 a very general notion was introduced, that of topological space, by axiomatizing the idea of neighborhood. In the present section we proceed in a different way, by axiomatizing the idea of distance between any two points

of a set S. We denote the "distance" between $p, q \in S$ by $d(p, q)$, and require d to satisfy Properties (i) through (iii) below. A set provided with such a distance is called a *metric space*. We formalize this as follows.

Definition. A *metric space* is a nonempty set S together with a real valued function d with domain the cartesian product $S \times S$, such that:

(i) $d(p, q) \geq 0$ for every $p, q \in S$, $d(p, q) = 0$ if and only if $p = q$.
(ii) $d(p, q) = d(q, p)$ for every $p, q \in S$.
(iii) $d(p, r) \leq d(p, q) + d(q, r)$ for every $p, q, r \in S$. The function d is called a *metric* on S.

Property (iii) is called the *triangle inequality* in the metric space S.

EXAMPLE 1. Let $S = E^n$, and $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ be the usual euclidean distance. Properties (i) and (ii) clearly hold, and (iii) is just Formula (1.4). If $S \subset E^n$, the euclidean distance also defines a metric on S. Thus any nonempty subset of E^n, with the euclidean distance, is a metric space.

EXAMPLE 2. Let S be a sphere in E^3. As distance $\tilde{d}(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}, \mathbf{y} \in S$ take the length of the shorter great circle arc on S joining $\mathbf{x}$ and $\mathbf{y}$. This gives a different metric on S than the euclidean distance $d(\mathbf{x}, \mathbf{y})$. They satisfy the inequalities

$$d(\mathbf{x}, \mathbf{y}) \leq \tilde{d}(\mathbf{x}, \mathbf{y}) \leq \frac{\pi}{2} d(\mathbf{x}, \mathbf{y}).$$

Although the metrics d and $\tilde{d}$ are different, they will give rise to the same topology on the sphere (the relative topology).

EXAMPLE 3. Let S be any nonempty set. Let $d(p, q) = 1$ if $p \neq q$; and $d(p, p) = 0$. This may seem a rather artificial sort of "distance," but it does have the properties (i), (ii), and (iii) required for a metric space.

Any metric defines a topology of S, in the same way that the euclidean distance defines the usual topology of E^n. By *δ-neighborhood* of a point p in a metric space let us mean the set

$$U = \{q \in S : d(p, q) < \delta\}, \delta > 0.$$

Axioms (1) and (2) for a topological space (in the first definition in Section 2.6) are satisfied. In Axiom (3) we can take $U_3 = U_1 \cap U_2$. To verify (4), let U be the δ-neighborhood of p, and $q \in U$. Let V be the $\delta - d(p, q)$ neighborhood of q. If $r \in V$, then by (iii)

$$d(p, r) \leq d(p, q) + d(q, r) < d(p, q) + \delta - d(p, q).$$

Thus, $d(p, r) < \delta$ for all $r \in V$; that is, $V \subset U$.

In Example 1, neighborhoods of $\mathbf{x}$ are just the usual ones when $S = E^n$, and are the relative neighborhoods when $S \subset E^n$. In Example 3, the δ-neighborhood of p is the one point set $\{p\}$ if $0 < \delta < 1$. With this metric, S is a discrete topological space (Problem 7, Section 2.6).

Complete metric spaces

The concepts of convergent sequence and of Cauchy sequence in a metric space S are defined almost exactly as in Section 2.3. One needs merely to replace euclidean distance by d-distance in S.

Definition. Let $[p_m]$, $m = 1, 2, \ldots$, be a sequence in a metric space S. If for every $\varepsilon > 0$ there exists a positive integer N such that $d(p_l, p_m) < \varepsilon$ for every $l, m \geq N$, then $[p_m]$ is a *Cauchy sequence.*

If for every $\varepsilon > 0$ there exists a positive integer N such that $d(p_m, p_0) < \varepsilon$ for every $m \geq N$, then $[p_m]$ is a *convergent sequence* and p_0 is its limit.

It can be easily shown that every convergent sequence in a metric space is Cauchy, just as for sequences in E^n (Section 2.3). Those metric spaces for which the converse is valid are called *complete.*

Definition. Let S be a metric space, with metric d. Then S is *complete* if every Cauchy sequence $[p_m]$ in S converges to a limit $p_0 \in S$.

EXAMPLE 1 (*continued*). Theorem 2.2 states that E^n, with the euclidean distance is a complete metric space. If $S \subset E^n$ with the euclidean distance, then S is a complete metric space if and only if S is a closed subset of E^n. To see this, suppose that S is closed. Let $[\mathbf{x}_m]$ be any Cauchy sequence, with $\mathbf{x}_m \in S$ for $m = 1, 2, \ldots$. According to Theorem 2.2, the sequence $[\mathbf{x}_m]$ has a limit $\mathbf{x}_0$. Since S is closed, $\mathbf{x}_0 \in S$ (Problem 4a, Section 2.3). Thus S is a complete metric space. On the other hand, if S is not closed, then there exists a sequence $[\mathbf{x}_m]$, with $\mathbf{x}_m \in S$ for $m = 1, 2, \ldots$, converging to a limit $\mathbf{x}_0 \notin S$ (Problem 4b, Section 2.3). The sequence $[\mathbf{x}_m]$ is Cauchy, but has no limit in S. Thus S is not a complete metric space.

In Sections 2.10 and 5.13 we see that certain metric spaces whose elements are functions are complete.

Normed vector spaces

We recall from Appendix A.1 the definition of vector space. Let us axiomatize the idea of norm on a vector space $\mathscr{V}$ over the real numbers. A vector space with a norm then becomes a metric space, by the same procedure used to define the euclidean metric from the euclidean norm in Section 1.2.

Definition. A *normed vector space* is a vector space $\mathcal{V}$ together with a real valued function $\| \ \|$ with domain $\mathcal{V}$, such that:

(1) $\|u\| > 0$ for all $u \in \mathcal{V}$, $u \neq \theta$.
(2) $\|cu\| = |c|\|u\|$ for every real c and $u \in \mathcal{V}$.
(3) $\|u + v\| \leq \|u\| + \|v\|$ for every $u, v \in \mathcal{V}$.

From Property (2) with $c = 0$, we have $\|\theta\| = 0$, where θ is the zero element of $\mathcal{V}$. From (2), (3), and induction on m, one can show that

$$\left\| \sum_{j=1}^{m} c^j u_j \right\| \leq \sum_{j=1}^{m} |c^j| \|u_j\|$$

for every choice of real numbers $c^1, \ldots, c^m$ and $u_1, \ldots, u_m \in \mathcal{V}$.

We define the *distance* between $u, v \in \mathcal{V}$ by

$$d(u, v) = \|u - v\|.$$

Then $\mathcal{V}$ is a metric space. Property (3) implies the triangle inequality (iii).

EXAMPLE 4. Let $\mathcal{V} = E^n$, with the norm of a vector $\mathbf{x} = (x^1, \ldots, x^n)$ equal to

$$\|\mathbf{x}\| = \max\{|x^1|, \ldots, |x^n|\}.$$

Neighborhoods of $\mathbf{x}$ in the metric $d(x, y) = \|x - y\|$ are n-cubes (see Example 3, Section 2.6). This is just one of many nonstandard norms which can be placed on E^n. It is seen in Section 2.11 that all norms on E^n lead to the usual topology of E^n.

EXAMPLE 5. The following space $\mathcal{V}$ is an infinite dimensional version of E^n. Consider infinite sequences of real numbers, which will be written as

$$\mathbf{x} = (x^1, x^2, \ldots, x^m, \ldots).$$

If $\mathbf{y} = (y^1, \ldots, y^m, \ldots)$ is another such sequence, then the sum is defined by

$$\mathbf{x} + \mathbf{y} = (x^1 + y^1, \ldots, x^m + y^m, \ldots).$$

Similarly, $c\mathbf{x} = (cx^1, \ldots, cx^m, \ldots)$. Let $\mathcal{V}$ be the space of such sequences for which the sum of squares of the components x^m is finite. As norm we take

$$\|\mathbf{x}\| = \left[\sum_{m=1}^{\infty} (x^m)^2 \right]^{1/2}.$$

The reader should check that $\mathcal{V}$ is a normed vector space. Just as for E^n, a set $A \subset \mathcal{V}$ is called *bounded* if there is a positive number C such that $\|\mathbf{x}\| \leq C$ for all $\mathbf{x} \in A$. Unlike E^n, there exist closed bounded subsets of $\mathcal{V}$ which are not compact (Problem 4).

A normed vector space $\mathcal{V}$ which is complete (in the metric $d(u, v) = \|u - v\|$) is called a *Banach space*. The space E^n, with any norm, is a finite dimensional Banach space. Examples of infinite dimensional Banach spaces

are the sequence space in Example 5, the space $\mathscr{C}(S)$ in Section 2.10, and the L^p spaces in Section 5.13. We refer to the work by Taylor [22] for an introduction to the theory of Banach spaces and their key role in current mathematical analysis.

Problems

1. Let S be a sphere in E^3, with $\tilde{d}(\mathbf{x}, \mathbf{y})$ as in Example 2. Show that $\tilde{d}$ satisfies properties (i), (ii), and (iii) in the definition of metric space.

2. Let S be a metric space, with metric d.
(a) Show that, for fixed p_0, the function $d(p_0, \quad)$ is continuous on S. (By definition, the value of $d(p_0, \quad)$ at q is $d(p_0, q)$.)
(b) Let $\delta > 0$. Show that $\{q : d(p_0, q) \leq \delta\}$ is closed and the δ-neighborhood $\{q : d(p_0, q) < \delta\}$ is open. Give an example in which the first of these two sets is not the closure of the second.

3. Let S be a metric space with metric d. Let

$$\tilde{d}(p, q) = \frac{d(p, q)}{1 + d(p, q)}.$$

Show that $\tilde{d}$ satisfies properties (i), (ii), and (iii) for a metric space.

4. Let $\mathscr{Y}$ be as in Example 5.
(a) Verify Properties (1) through (3) for the norm $\| \quad \|$ in this example.
(b) Show that $\mathscr{Y}$ is a complete metric space.
(c) For $l = 1, 2, \ldots$ let $\mathbf{e}_l = (0, \ldots, 0, 1, 0, \ldots)$ where $l - 1$ zeros precede the 1. Let $A = \{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$. Show that A is bounded and closed.
(d) Show that the set A in part (c) is not compact. [*Hint*: Let $U_l = \{\mathbf{x} : \|\mathbf{x} - \mathbf{e}_l\| < 1\}$. The collection $\{U_1, U_2, \ldots\}$ covers A.]

5. For any compact subsets A, B of E^n, let $d(A, B)$ be the smallest number a with the following property: for every $\mathbf{x} \in B$ there exists $\mathbf{y} \in A$ such that $|\mathbf{x} - \mathbf{y}| \leq a$, and for every $\mathbf{y} \in A$ there exists $\mathbf{x} \in B$ such that $|\mathbf{x} - \mathbf{y}| \leq a$. Show d is a metric on the space whose elements are all compact subsets of E^n.

6. Let S be a compact metric space, with metric d.
(a) Show that for any $\delta > 0$ there is a finite set $S_\delta \subset S$ such that any $p \in S$ is distant less than δ from some $q \in S_\delta$ [i.e., $d(p, q) < \delta$].
(b) A set $A \subset S$ is *countable* if either A is a finite set or $A = \{p_1, p_2, \ldots\}$ for some infinite sequence $[p_m]$ in S, $p_m \neq p_l$ for $l \neq m$. Show that there is a countable set A with cl $A = S$.

7. A topological space S_0 is called a *Hausdorff* space if S_0 has the property that for every $p, q \in S_0$ $(p \neq q)$ there exist a neighborhood U of p and a neighborhood V of q such that $U \cap V$ is empty.
(a) Show that any metric space is a Hausdorff space.
(b) Show that any compact set $S \subset S_0$ is closed, if S_0 is a Hausdorff space.
(c) Let f be continuous and univalent from a compact space S onto a Hausdorff space T. Show that f^{-1} is continuous from T onto S. [*Hint*: Show that $(f^{-1})^{-1}(B)$ is closed if B is closed.]

2.10 Spaces of continuous functions

To motivate the discussion that follows, let us for the moment consider real valued, continuous functions on a closed, bounded interval $[0, 1]$. There are many possible ways to define a concept of distance between two functions f and g. One such is the integral $\int_0^1 |f(x) - g(x)|dx$. Another is $[\int_0^1 |f(x) - g(x)|^2 dx]^{1/2}$. Such distances are considered in Section 5.13. In the present section we consider another distance:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}.$$

In Figure 2.8 we picture $\{g : d(f, g) \leq \varepsilon\}$, where f is given. For g to be in this set, we must have $|f(x) - g(x)| \leq \varepsilon$ for all $x \in [0, 1]$. This imposes a uniform bound on $|f(x) - g(x)|$; for that reason $d(f, g)$ is called the *uniform distance.*

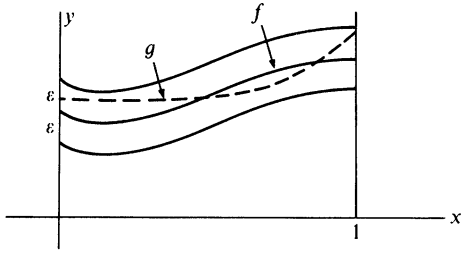


Figure 2.8

Let us now formulate the idea of uniform distance in a more general setting. Let S be a set. A real valued function f, with domain S, is called *bounded* if $f(S)$ is a bounded set. For each such f, let

$$\|f\| = \sup\{|f(p)| : p \in S\}. \tag{2.1}$$

Let us denote by $\mathscr{B}(S)$ the set of all bounded, real valued functions with domain S. If f and g are bounded functions, then the sum $f + g$ is a bounded function. Moreover, any scalar multiple cf is a bounded function. Thus, $\mathscr{B}(S)$ is a vector space, under the usual notions of addition of functions and multiplication by scalars. Moreover, $\| \ \|$ is a norm on $\mathscr{B}(S)$. It is called the *uniform norm.* Properties (1) and (2) for a norm (Section 2.9) are immediate from (2.1). To prove (3), we have for all $p \in S$

$$|(f + g)(p)| = |f(p) + g(p)| \leq |f(p)| + |g(p)|.$$

Since $|f(p)| \leq \|f\|$, $|g(p)| \leq \|g\|$, we have

$$|(f + g)(p)| \leq \|f\| + \|g\|.$$

Since this is true for each $p \in S$,

$$\|f + g\| \leq \|f\| + \|g\|,$$

which is Property (3).

Uniform convergence of sequences

Let $f_1, f_2, \ldots$ be a sequence of real valued functions. One concept of convergence of the sequence to a limiting function f is that $f(p) = \lim_{m \to \infty} f_m(p)$, for each $p \in S$. This concept is called *pointwise convergence* of the sequence $[f_m]$ to f. It is not the concept of convergence with which we are principally concerned here.

Definition. Let $f_1, f_2, \ldots$ be a sequence of functions, such that $f_m \in \mathscr{B}(S)$ for $m = 1, 2, \ldots$. Then f_m tends to f *uniformly* on S as $m \to \infty$ if $\|f_m - f\| \to 0$ as $m \to \infty$.

Thus, uniform convergence of a sequence $[f_m]$ is the same as convergence in the metric d such that $d(f, g) = \|f - g\|$.

EXAMPLE 1. Let $S = [0, 1]$, and let $f_m(x) = x^m$ (mth power of x). Then $f_m(x) \to 0$ as $m \to \infty$ for $0 \leq x < 1$. Since $f_m(1) = 1$, $f_m(1) \to 1$ as $m \to \infty$. The sequence $[f_m]$ converges pointwise to f, where $f(x) = 0$ for $0 \leq x < 1$, $f(1) = 1$. However, the convergence is not uniform. This follows from Theorem 2.11, since f is not a continuous function. A direct proof can also be given (Problem 1).

Proposition 2.13. *The space $\mathscr{B}(S)$, with the metric $d(f, g) = \|f - g\|$, is a complete metric space.*

PROOF. Let $f_1, f_2, \ldots$ be any Cauchy sequence in $\mathscr{B}(S)$. Given $\varepsilon > 0$ there exists N such that $\|f_l - f_m\| < \varepsilon$ for every $l, m \geq N$. Since $|f_l(p) - f_m(p)| \leq \|f_l - f_m\|$ for each $p \in S$,

$$|f_l(p) - f_m(p)| < \varepsilon \text{ for every } l, m \geq N.$$

This shows that, for fixed p, the sequence of real numbers $f_1(p), f_2(p), \ldots$ is Cauchy. By Theorem 2.2, this sequence has a limit, which we denote by $f(p)$. This defines the function f. Let us show that $\|f_m - f\| \to 0$ as $m \to \infty$.

Given $\varepsilon > 0$ there exists N_1 such that $\|f_m - f_l\| < \varepsilon/2$ for every $l, m \geq N_1$. Since $f_m(p) \to f(p)$ as $m \to \infty$, there exists $N_2 > N_1$ such that $|f_l(p) - f(p)| < \varepsilon/2$ for every $l \geq N_2$. Note that N_2 may depend on both p and ε, but N_1 depends only on ε. Then

$$f_m(p) - f(p) = [f_m(p) - f_l(p)] + [f_l(p) - f(p)],$$

$$|f_m(p) - f(p)| \leq |f_m(p) - f_l(p)| + |f_l(p) - f(p)|.$$

Therefore, if $m \geq N_1$ and $l \geq N_2$,

$$|f_m(p) - f(p)| \leq \|f_m - f_l\| + |f_l(p) - f(p)| < \varepsilon.$$

Since this is true for all $p \in S$,

$$\|f_m - f\| = \sup\{|f_m(p) - f(p)| : p \in S\} \leq \varepsilon,$$

for every $m \geq N_1$. This proves that $\|f_m - f\| \to 0$ as $m \to \infty$.

It remains only to verify that f is a bounded function. Take $\varepsilon = 1$ and N_1 as above. Fix $m \geq N_1$. Then $f = (f - f_m) + f_m$, and both $f - f_m$, f_m are bounded functions. Hence f is bounded. □

Let us now suppose that S is a topological space, and consider sequences of continuous functions on S.

Theorem 2.11. *Let $f_1, f_2, \ldots$ be a sequence of functions, such that f_m is continuous on S for each $m = 1, 2, \ldots$ and f_m tends to f uniformly on S as $m \to \infty$. Then f is continuous on S.*

PROOF. Consider any $p_0 \in S$. Given $\varepsilon > 0$, there exists N such that $\|f_m - f\| < \varepsilon/3$ for every $m \geq N$ since f_m tends to f uniformly. Since f_N is continuous, there exists a neighborhood U of p_0 such that

$$|f_N(p) - f_N(p_0)| < \varepsilon/3 \quad \text{for all } p \in U.$$

Recall that $|f_N(p) - f(p)| \leq \|f_N - f\|$ by definition of the norm $\| \ \|$. Then

$$f(p) - f(p_0) = [f(p) - f_N(p)] + [f_N(p) - f_N(p_0)] + [f_N(p_0) - f(p_0)],$$

$$|f(p) - f(p_0)| \leq |f(p) - f_N(p)| + |f_N(p) - f_N(p_0)| + |f_N(p_0) - f(p_0)|,$$

$$|f(p) - f(p_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $p \in U$. Since such a neighborhood U exists corresponding to every $\varepsilon > 0$, f is continuous at p_0. Since p_0 is any point of S, f is continuous on S. □

The space $\mathscr{C}(S)$

Let $\mathscr{C}(S)$ denote the set of all real valued functions f, such that f is bounded and continuous on S. The sum $f + g$ is continuous if f and g are continuous. Moreover, any scalar multiple cf of a continuous f is continuous. Thus $\mathscr{C}(S)$ is a vector subspace of $\mathscr{B}(S)$.

Theorem 2.12. *The space $\mathscr{C}(S)$, with the metric $d(f, g) = \|f - g\|$, is a complete metric space.*

PROOF. Let $[f_m]$, $m = 1, 2, \ldots$, be any Cauchy sequence in $\mathscr{C}(S)$. Since $\mathscr{C}(S) \subset \mathscr{B}(S)$, the sequence is Cauchy in $\mathscr{B}(S)$. By Proposition 2.13 $\mathscr{B}(S)$ is complete. Thus there exists a bounded function f such that $\|f_m - f\| \to 0$

as $m \to \infty$. By Theorem 2.11 f is continuous. Hence $f \in \mathscr{C}(S)$. Thus, any Cauchy sequence $[f_m]$ in $\mathscr{C}(S)$ has a limit f in $\mathscr{C}(S)$, which proves that $\mathscr{C}(S)$ is complete. □

PROBLEMS

1. Show directly from the definition of uniform convergence that the sequence $[f_m]$ in Example 1 does not converge uniformly.
2. Let $f_m(x) = m^2x$ if $0 \le x \le m^{-1}$, $f_m(x) = m(2 - mx)$ if $m^{-1} \le x \le 2m^{-1}$, and $f_m(x) = 0$ for all other x.
   (a) Find $\| f_m \|$.
   (b) Show that f_m tends to 0 pointwise for each x, but that the convergence is not uniform.
3. Let S be a finite set with n elements. Explain how $\mathscr{C}(S)$ can be identified with the space in Example 4, Section 2.9.
4. Let $f(x) = \sum_{k=1}^{\infty} (\sin kx)/k^2$. Use Theorem 2.11 to show that f is continuous on E^1.
5. Let $S \subset E^n$ be compact, and let $\mathscr{S}$ be a compact subset of $\mathscr{C}(S)$. Show that:
   (a) There exists a number C such that $\| f \| \le C$ for all $f \in \mathscr{S}$.
   (b) Given $\varepsilon > 0$ there exists $\delta > 0$ depending on ε (but not on f), such that $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$ for all $f \in \mathscr{S}$ and all $\mathbf{x}, \mathbf{y} \in S$ satisfying $|\mathbf{x} - \mathbf{y}| < \delta$. [*Hint*: Problem 8, Section 2.5 and Problem 6a, Section 2.9.]

*2.11 Noneuclidean norms on E^n

It is sometimes advantageous to consider norms on E^n other than the standard euclidean norm. The distance between two points $\mathbf{x}$ and $\mathbf{y}$ defined by such a norm need not agree with the euclidean distance. As a result, such geometric notions as length, area, and spherical ball are changed when considered with respect to a noneuclidean norm. However, we see that any noneuclidean norm leads to the same collection of open sets as the euclidean norm. Since the collection of open sets determines all of the topological properties of E^n, these properties are therefore independent of the particular norm chosen.

We recall from Section 2.9 that a norm is a real valued function $\| \ \|$ with domain E^n such that:

(1) $\|\mathbf{x}\| > 0$ for every $\mathbf{x} \neq \mathbf{0}$,
(2) $\|c\mathbf{x}\| = |c| \, \|\mathbf{x}\|$ for every c and $\mathbf{x}$, and
(3) $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ for every $\mathbf{x}$ and $\mathbf{y}$.

The norm defines a metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$. The *$\delta$-neighborhood* of $\mathbf{x}_0$ in this metric is $\{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| < \delta\}$. The *closed n-ball* with center $\mathbf{x}_0$ and radius δ, with respect to the norm $\| \ \|$, is $\{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| \le \delta\}$. The main result of this section is a characterization of the closed n-ball with center $\mathbf{0}$ and radius 1. By translations and scalar multiplications, this then characterizes all closed n-balls with respect to the norm $\| \ \|$.

We have already given two examples, namely, the standard euclidean norm ($\|\mathbf{x}\| = |\mathbf{x}|$) and Example 4, Section 2.9. Two further examples are:

EXAMPLE 1. Let

$$\|\mathbf{x}\| = \sum_{i=1}^{n} |x^i|.$$

The n-balls with respect to this norm are convex polytopes. For example, if $n = 2$, the closed unit 2-ball $\{\mathbf{x} : \|\mathbf{x}\| \leq 1\}$ is the square with vertices $\mathbf{e}_1$, $\mathbf{e}_2$, $-\mathbf{e}_1$, $-\mathbf{e}_2$. Compare with Problem 11a, Section 1.3.

EXAMPLE 2. Let (c_{ij}) be a $n \times n$ matrix that is symmetric, $c_{ij} = c_{ji}$ for $i, j = 1, \ldots, n$. Define the function $B(\mathbf{x}, \mathbf{y})$, for each $\mathbf{x} = (x^1, \ldots, x^n)$, $\mathbf{y} = (y^1, \ldots, y^n)$ by

$$B(\mathbf{x}, \mathbf{y}) = \sum_{i,j=1}^{n} c_{ij} x^i y^j. \tag{2.2}$$

The function B is called *bilinear*. Let us assume that $B(\mathbf{x}, \mathbf{x}) > 0$ for every $\mathbf{x} \neq \mathbf{0}$. This means that the matrix (c_{ij}) is positive definite. With B we associate a quadratic norm, given by

$$\|\mathbf{x}\| = \sqrt{B(\mathbf{x}, \mathbf{x})}.$$

Property (1) holds by assumption, while (2) follows from $B(c\mathbf{x}, c\mathbf{x}) = c^2 B(\mathbf{x}, \mathbf{x})$. To establish (3), let us think of $B(\mathbf{x}, \mathbf{y})$ as an inner product on E^n. This is the standard euclidean inner product if (c_{ij}) is the identity matrix, in which case $B(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$. Any such inner product satisfies

$$B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x}), \qquad B(\mathbf{x} + \mathbf{y}, \mathbf{z}) = B(\mathbf{x}, \mathbf{z}) + B(\mathbf{y}, \mathbf{z}),$$
$$B(c\mathbf{x}, \mathbf{y}) = cB(\mathbf{x}, \mathbf{y}), \qquad B(\mathbf{x}, \mathbf{x}) > 0 \text{ if } \mathbf{x} \neq \mathbf{0}.$$

The proof of Cauchy's inequality (1.1) was based on the corresponding properties of the euclidean inner product. The same proof, with each inner product $\mathbf{u} \cdot \mathbf{v}$ in the proof of (1.1) replaced by the corresponding $B(\mathbf{u}, \mathbf{v})$, shows that

$$|B(\mathbf{x}, \mathbf{y})| \leq \sqrt{B(\mathbf{x}, \mathbf{x})}\sqrt{B(\mathbf{y}, \mathbf{y})}.$$

This inequality is then used to show that

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|,$$

exactly in the same way (1.2) was derived from (1.1). This verifies property (3) of the quadratic norm $\| \quad \|$.

The n-balls with respect to any quadratic norm are n-dimensional ellipsoids. This is proved in Section 4.8 by finding a new orthonormal basis for E^n for which the matrix associated with the bilinear function B is diagonal.

Proposition 2.14. *Corresponding to any norm* $\| \ \|$ *on* E^n *there exist positive numbers* m *and* M *such that for every* $\mathbf{x} \in E^n$,

$$m|\mathbf{x}| \leq \|\mathbf{x}\| \leq M|\mathbf{x}|. \tag{2.3}$$

PROOF. Let

$$M = n \max\{\|\mathbf{e}_1\|, \ldots, \|\mathbf{e}_n\|\},$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis for E^n. Then

$$\mathbf{x} = x^1\mathbf{e}_1 + \cdots + x^n\mathbf{e}_n,$$

and therefore

$$\|\mathbf{x}\| \leq \sum_{i=1}^{n} |x^i| \|\mathbf{e}_i\| \leq \frac{M}{n} \sum_{i=1}^{n} |x^i|.$$

But $|x^i| \leq |\mathbf{x}|$ for each $i = 1, \ldots, n$. Hence

$$\|\mathbf{x}\| \leq M|\mathbf{x}|.$$

From this $\|\mathbf{x} - \mathbf{y}\| \leq M|\mathbf{x} - \mathbf{y}|$ for every $\mathbf{x}$ and $\mathbf{y}$, which implies that $\| \ \|$ is a continuous function. Therefore it has a minimum value m on the compact set $\{\mathbf{x} : |\mathbf{x}| = 1\}$,

$$m = \min\{\|\mathbf{x}\| : |\mathbf{x}| = 1\}.$$

By Axiom (1) for a norm $m > 0$. If $\mathbf{x} = \mathbf{0}$, all terms in (2.3) are 0. Given any $\mathbf{x} \neq \mathbf{0}$, let $c = |\mathbf{x}|^{-1}$. Then $|c\mathbf{x}| = c|\mathbf{x}| = 1$, and hence $\|c\mathbf{x}\| \geq m$. By Axiom (2) $\|c\mathbf{x}\| = |c| \|\mathbf{x}\|$, from which

$$\|\mathbf{x}\| \geq m|\mathbf{x}|. \qquad \square$$

From Proposition 2.14,

$$m|\mathbf{x} - \mathbf{y}| \leq \|\mathbf{x} - \mathbf{y}\| \leq M|\mathbf{x} - \mathbf{y}|$$

for every $\mathbf{x}$ and $\mathbf{y}$. This says that the ratio of the $\| \ \|$-distance to the euclidean distance is bounded between m and M. Let us call a set D $\| \ \|$-*open* if every $\mathbf{x}_0 \in D$ has some neighborhood with respect to this norm which is contained in D. If $|\mathbf{x} - \mathbf{x}_0| < \delta/M$, then $\|\mathbf{x} - \mathbf{x}_0\| < \delta$. Therefore the δ-neighborhood of $\mathbf{x}_0$ with respect to the norm $\| \ \|$ contains the ordinary euclidean (δ/M)-neighborhood of $\mathbf{x}_0$. If D is $\| \ \|$-open, then every $\mathbf{x}_0 \in D$ has a euclidean neighborhood contained in D. Hence D is open in the ordinary sense. Similarly, the euclidean δ-neighborhood of $\mathbf{x}_0$ contains the $(m\delta)$-neighborhood of $\mathbf{x}_0$ with respect to the norm $\| \ \|$. It follows that every set which is open in the ordinary sense is also $\| \ \|$-open. Thus a set is $\| \ \|$-open if and only if it is open in the usual sense. Since the open subsets of a topological space determine the topology (Section 2.6), all norms on E^n lead to the same topology of E^n.

The closed unit n-ball

$$K = \{\mathbf{x} : \|\mathbf{x}\| \leq 1\} \tag{2.4}$$

with respect to any norm has the following four properties:

(i) K is compact;
(ii) K is convex;
(iii) K is symmetric about $\mathbf{0}$; and
(iv) K contains a euclidean neighborhood of $\mathbf{0}$.

Symmetry about $\mathbf{0}$ means that $-\mathbf{x} \in K$ for every $\mathbf{x} \in K$. By Axiom (2) with $c = -1$, $\|-\mathbf{x}\| = \|\mathbf{x}\|$, and hence K has Property (iii). From Proposition 2.14, K has Property (iv) and is bounded. For any continuous function f, $\{\mathbf{x} : f(\mathbf{x}) \leq 1\}$ is a closed set. Since $\| \ \|$ is continuous, K is closed. Since K is also bounded, property (i) follows from Theorem 2.9. To prove (ii), let $\mathbf{x}_1, \mathbf{x}_2 \in K$ and $t \in [0, 1]$. Then

$$\|t\mathbf{x}_1 + (1 - t)\mathbf{x}_2\| \leq t\|\mathbf{x}_1\| + (1 - t)\|\mathbf{x}_2\| \leq t + 1 - t.$$

Thus

$$\|t\mathbf{x}_1 + (1 - t)\mathbf{x}_2\| \leq 1; \text{ i.e., } t\mathbf{x}_1 + (1 - t)\mathbf{x}_2 \in K.$$

This proves (ii).

Let us show that, conversely, any set K with these four properties gives rise to a norm with respect to which K is the closed unit n-ball.

Theorem 2.13. *Let K be any set with Properties* (i) *through* (iv). *Let* $\|\mathbf{0}\| = \mathbf{0}$, *and for every* $\mathbf{x} \neq \mathbf{0}$, *let*

$$\|\mathbf{x}\| = \frac{1}{\max\{t : t\mathbf{x} \in K\}}. \tag{2.5}$$

Then $\| \ \|$ *is a norm, and* (2.4) *holds.*

PROOF. By (i) and (iv) there exist $r_1 > 0$ and $r_2 > 0$ such that $\mathbf{y} \in K$ if $|\mathbf{y}| \leq r_1$ and $\mathbf{y} \notin K$ if $|\mathbf{y}| > r_2$. Hence given $\mathbf{x} \neq \mathbf{0}$, $t\mathbf{x} \in K$ if $|t| \leq r_1/|\mathbf{x}|$, and $t\mathbf{x} \notin K$ if $|t| > r_2/|\mathbf{x}|$ (see Figure 2.9). Let

$$S_{\mathbf{x}} = \{t : t\mathbf{x} \in K\}.$$

Then $S_{\mathbf{x}}$ contains the $(r_1/|\mathbf{x}|)$-neighborhood of 0 and is bounded above by $r_2/|\mathbf{x}|$. Since K is a closed set, $S_{\mathbf{x}}$ is also closed. Hence $S_{\mathbf{x}}$ has a largest element $\max S_{\mathbf{x}}$, which is positive. This shows that $\|\mathbf{x}\| = 1/\max S_{\mathbf{x}}$ is well-defined and is positive. Moreover, by (ii) and the fact that $\mathbf{0} \in K$, the line segment between $\mathbf{0}$ and any point of K is contained in K. Therefore $\max S_{\mathbf{x}} \geq 1$ if and only if $\mathbf{x} \in K$, which says that (2.4) holds.

It remains to verify Axioms (2) and (3) for a norm. By Property (iii), $\|-\mathbf{x}\| = \|\mathbf{x}\|$. It is left to the reader to check that if $c > 0$, then

$$\max S_{c\mathbf{x}} = \frac{1}{c} \max S_{\mathbf{x}},$$

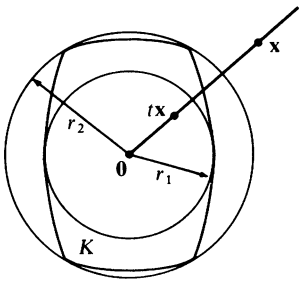


Figure 2.9

and consequently $\|c\mathbf{x}\| = c\|\mathbf{x}\|$. Then Axiom (2) holds. For (3) we may assume that $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$. Let

$$s = \max S_{\mathbf{x}}, \qquad t = \max S_{\mathbf{y}}, \qquad u = \frac{t}{s+t}.$$

Observe that $0 < u < 1$ and

$$\frac{1}{su} = \frac{1}{s} + \frac{1}{t} = \|\mathbf{x}\| + \|\mathbf{y}\|.$$

A little manipulation shows that $su = (1-u)t$. Consequently,

$$su(\mathbf{x} + \mathbf{y}) = u(s\mathbf{x}) + (1-u)t\mathbf{y}.$$

Since $s\mathbf{x}, t\mathbf{y} \in K$ and K is a convex set by Property (ii), $su(\mathbf{x} + \mathbf{y}) \in K$. Therefore $su \leq \max S_{\mathbf{x}+\mathbf{y}}$, and

$$\|\mathbf{x}\| + \|\mathbf{y}\| = \frac{1}{su} \geq \|\mathbf{x} + \mathbf{y}\|.$$

This verifies Axiom (3). □

Problems

1. In E^2 consider the norm $\|(x, y)\| = \max\{|x|, |y|\}$.
(a) What is $K = \{\mathbf{x} : \|\mathbf{x}\| < 1\}$?
(b) Show that the triangle with vertices $\mathbf{0}$, $\mathbf{e}_1$, $\mathbf{e}_2$ is equilateral with respect to the distance which this norm defines.

2. The ellipse $K = \{(x, y) : x^2 + xy + 4y^2 \leq 1\}$ has Properties (i) through (iv) in Theorem 2.13.
(a) For what (quadratic) norm is it the closed unit 2-ball?
(b) Find $\|\mathbf{e}_1 - \mathbf{e}_2\|$.

3. Let a, b, p be real numbers, with $p \geq 1$. Show that $|ta + (1-t)b|^p \leq t|a|^p + (1-t)|b|^p$ for $0 \leq t \leq 1$. In particular, $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$.

4. Let $p \geq 1$, and $K = \{\mathbf{x} : \sum_{i=1}^{n} |x^i|^p \leq 1\}$.
(a) Show that K is convex. [*Hint*: Problem 3.]
(b) Show that K satisfies Properties (i) through (iv) in Theorem 2.13.
(c) Show that the norm associated with K in Theorem 2.13 is $\|\mathbf{x}\| = [\sum_{i=1}^{n} |x^i|^p]^{1/p}$.

Note. For this norm the inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ is called *Minkowski's inequality*. There is a related inequality for integrals, which we prove in Section 5.13.

5. A *seminorm* on E^n is a real valued function f satisfying: $f(\mathbf{x}) \geq 0$ for every $\mathbf{x}$; $f(c\mathbf{x}) = |c| f(\mathbf{x})$ for every c and $\mathbf{x}$; and $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ for every $\mathbf{x}$ and $\mathbf{y}$.
(a) Let f be a seminorm and $K = \{\mathbf{x} : f(\mathbf{x}) \leq 1\}$. Show that K is closed and satisfies Properties (ii) through (iv). Show that K is compact if and only if f is a norm. [*Hint*: First prove that f is continuous.]
(b) Conversely, let K be any closed set satisfying Properties (ii) through (iv). Let $f(\mathbf{x}) = 0$ if $\mathbf{x} = \mathbf{0}$ or if the line through $\mathbf{0}$ and $\mathbf{x}$ is contained in K. Otherwise, let

$$f(\mathbf{x}) = \frac{1}{\max\{t : t\mathbf{x} \in K\}}$$

as in (2.5). Show that f is a seminorm.
(c) Let $n = 3$ and $f(x, y, z) = |x| + 2|y|$. Sketch K and show that f is a seminorm.

3 Differentiation of real valued functions

We now begin the differential calculus for real valued functions of several variables. The first step is to define the notions of directional derivative and partial derivative. Then the concept of differentiable function is introduced, by linear approximation to the increments of a function. Taylor's formula with remainder is obtained for functions of class $C^{(q)}$; such functions have continuous partial derivatives of orders $1, 2, \ldots, q$. It is then applied to problems of relative extrema and to the characterization of convex functions of class $C^{(2)}$.

The chain rule for partial derivatives is postponed to Chapter 4, since it is a natural corollary of the composite function theorem for vector-valued functions to be proved there.

3.1 Directional and partial derivatives

If f is a function of one variable, then its derivative at a point x_0 is defined by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists. The corresponding expression for functions of several variables does not make sense, since $\mathbf{h}$ is then a vector and division by $\mathbf{h}$ is undefined. Therefore we must find an acceptable substitute for it. Let us first consider the derivative of f in various directions.

Let us call any unit vector $\mathbf{v}$ (that is, vector with $|\mathbf{v}| = 1$) a *direction* in E^n. The directions are just the points of the $(n-1)$-dimensional sphere which bounds the unit n-ball. If $n = 1$, the only directions are $\mathbf{e}_1$ and $-\mathbf{e}_1$, which we have identified with the scalars 1 and -1. If $n = 2$, every direction can be written $(\cos\theta, \sin\theta)$ where $0 \leq \theta < 2\pi$. The angle θ determines the

direction. For any $n \geq 2$ the components of a direction $\mathbf{v}$ satisfy $v^i = \cos \theta_i$, $i = 1, \ldots, n$, where θ_i is the angle between $\mathbf{v}$ and $\mathbf{e}_i$.

Given $\mathbf{x}_0$ and a direction $\mathbf{v}$, the line through $\mathbf{x}_0 + \mathbf{v}$ and $\mathbf{x}_0$ is called *the line through* $\mathbf{x}_0$ *with direction* $\mathbf{v}$. According to the definition in Section 1.3, this line is

$$\{\mathbf{x} : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, \quad t \text{ any scalar}\}. \tag{3.1}$$

Let f be a function with domain $D \subset E^n$, and let $\mathbf{x}_0$ be an interior point of D.

Definition. The *derivative of* f *at* $\mathbf{x}_0$ *in the direction* $\mathbf{v}$ is

$$\lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0)}{t} \tag{3.2}$$

if the limit exists.

Since $\mathbf{x}_0$ is an interior point, the δ-neighborhood of $\mathbf{x}_0$ is contained in D for some $\delta > 0$ (Figure 3.1). Since

$$|(\mathbf{x}_0 + t\mathbf{v}) - \mathbf{x}_0| = |t\mathbf{v}| = |t|,$$

$\mathbf{x}_0 + t\mathbf{v} \in D$ provided $|t| < \delta$. The domain of the function ϕ defined by

$$\phi(t) = f(\mathbf{x}_0 + t\mathbf{v})$$

contains the δ-neighborhood of 0. The derivative of f in the direction $\mathbf{v}$ is $\phi'(0)$, if ϕ has a derivative at 0.

The line through $\mathbf{x}_0$ with direction $-\mathbf{v}$ is the same line as the one through $\mathbf{x}_0$ with direction $\mathbf{v}$. However, the derivative in the direction $-\mathbf{v}$ is the negative of the derivative in direction $\mathbf{v}$ (Problem 6). The direction $\mathbf{v}$ defines an *orientation* of this line, and $-\mathbf{v}$ the opposite orientation. When the orientation changes, the directional derivative changes sign. In effect, by assigning the orientation $\mathbf{v}$ we agree that the point $\mathbf{x}_0 + s\mathbf{v}$ precedes $\mathbf{x}_0 + t\mathbf{v}$ on the line if $s < t$.

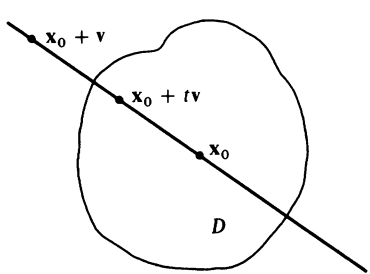


Figure 3.1

EXAMPLE 1. Let $f(x) = |x|$, $n = 1$. The derivative of f at x_0 in the direction $\mathbf{e}_1$ is just $f'(x_0)$. For $x_0 \neq 0$, $f'(x_0) = \operatorname{sgn} x_0$, where $\operatorname{sgn} x_0 = 1$ if $x_0 > 0$ and $\operatorname{sgn} x_0 = -1$ if $x_0 < 0$. The derivative does not exist at $x_0 = 0$.

EXAMPLE 2. Let $f(x, y) = |x^2 - y^2|^{1/2}$, $(x_0, y_0) = (0, 0)$. Consider any direction $\mathbf{v} = (\cos\theta, \sin\theta)$, and let $\phi(t) = f(t\cos\theta, t\sin\theta)$. The directional derivative is $\phi'(0)$, if the derivative $\phi'(0)$ exists. Now

$$\phi(t) = |t^2\cos^2\theta - t^2\sin^2\theta|^{1/2} = |t||\cos^2\theta - \sin^2\theta|^{1/2}.$$

If $\cos^2\theta = \sin^2\theta$, then $\phi(t) = 0$ for all t; and $\phi'(0) = 0$. If $\cos^2\theta \neq \sin^2\theta$, then ϕ has no derivative at $t = 0$ (see Example 1). Thus, the derivative of f at $(0, 0)$ is 0 in the four directions $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)$. The derivative in any other direction $\mathbf{v}$ does not exist.

The *partial derivatives* of f are defined as the derivatives in the directions $\mathbf{e}_1, \ldots, \mathbf{e}_n$, if these directional derivatives exist. There are several equivalent notations in use for partial derivatives. Of these we shall adopt just two. The ith partial derivative of f at $\mathbf{x}$, $i = 1, \ldots, n$, is denoted by

$$f_i(\mathbf{x}) \quad \text{or} \quad \frac{\partial f}{\partial x_i}(\mathbf{x}).$$

Thus

$$f_i(\mathbf{x}) = \lim_{t\to 0} \frac{f(\mathbf{x} + t\mathbf{e}_i) - f(\mathbf{x})}{t} \tag{3.3}$$

provided the limit exists. Stated in less precise terms, $f_i(\mathbf{x})$ is the derivative taken with respect to the ith variable while holding all other variables fixed.

EXAMPLE 3. Let $f(x, y, z) = x^2 + y + \cos(y^2 z)$. Then

$$\begin{aligned} f_1(x, y, z) &= 2x, \\ f_2(x, y, z) &= 1 - 2yz\sin(y^2 z), \\ f_3(x, y, z) &= -y^2\sin(y^2 z). \end{aligned}$$

The symbol f_i denotes the real valued function whose value at $\mathbf{x}$ is $f_i(\mathbf{x})$. Its domain is the set of points where f has an ith partial derivative.

For purposes of brevity, we occasionally abuse the notation by writing f_i for the value $f_i(\mathbf{x})$ at some particular $\mathbf{x}$. In each such instance this abuse is indicated either explicitly or by the context.

EXAMPLE 4. Let $f(\mathbf{x}) = \psi[g(\mathbf{x})]$ for every $\mathbf{x} \in D$. Suppose that the ith partial derivative of g at $\mathbf{x}_0$ and the derivative of ψ at $g(\mathbf{x}_0)$ exist. By the composite function theorem for functions of one variable

$$f_i(\mathbf{x}_0) = \psi'[g(\mathbf{x}_0)]g_i(\mathbf{x}_0). \tag{3.4}$$

This theorem is proved in Section 4.4 as a special case of the composite function theorem for transformations.

In Section 3.3 we define the concept of differentiable function. Since this idea involves linear approximations, we begin in Section 3.2 with linear functions. We see in Section 3.3 that the directional derivatives of a differentiable function are easily computed from the partial derivatives. Disagreeable phenomena of the sort illustrated in Example 2 cannot occur if f is differentiable at $\mathbf{x}_0$.

PROBLEMS

Unless otherwise stated, the domain D of f is E^n for the particular n indicated in the problem.

1. In each case find the partial derivatives of f.
   (a) $f(x, y) = x \log(xy)$, $D = \{(x, y) : xy > 0\}$.
   (b) $f(x, y, z) = (x^2 + 2y^2 + z)^3$.
   (c) $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$.
2. Let $f(x, y) = (x - 1)^2 - y^2$. Find the derivative of f at $\mathbf{e}_2$ in any direction $\mathbf{v}$, using the definition of directional derivative.
3. Let $f(x, y) = 2xy(x^2 + y^2)^{-1/2}$, if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Find the derivative of f at $(0, 0)$ in any direction $\mathbf{v}$.
4. Let $f(x, y) = (xy)^{1/3}$. (a) Using the definition of directional derivative, show that $f_1(0, 0) = f_2(0, 0) = 0$, and that $\pm\mathbf{e}_1$, $\pm\mathbf{e}_2$ are the only directions in which the derivative at $(0, 0)$ exists. (b) Show that f is continuous at $(0, 0)$.
5. Let $f(x, y, z) = |x + y + z|$. If $\mathbf{x}_0 = (x_0, y_0, z_0)$ is such that $x_0 + y_0 + z_0 = 0$, find those directions $\mathbf{v}$ in which the derivative at $\mathbf{x}_0$ exists.
6. Show that the derivative of f at $\mathbf{x}_0$ in the direction $-\mathbf{v}$ is the negative of the derivative at $\mathbf{x}_0$ in the direction $\mathbf{v}$.

3.2 Linear functions

Let L be a real-valued function whose domain is E^n.

Definition. The function L is *linear* if:

(a) $L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in E^n$.
(b) $L(c\mathbf{x}) = cL(\mathbf{x})$ for every $\mathbf{x} \in E^n$ and scalar c.

These two conditions are equivalent to the single condition $L(c\mathbf{x} + d\mathbf{y}) = cL(\mathbf{x}) + dL(\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in E^n$ and scalars c, d. By induction, if L is linear,

$$L\left(\sum_{j=1}^{m} c^j \mathbf{x}_j\right) = \sum_{j=1}^{m} c^j L(\mathbf{x}_j) \tag{3.5}$$

for every m, $\mathbf{x}_1, \ldots, \mathbf{x}_m \in E^n$, and scalars $c^1, \ldots, c^m$. In words, this states that "L of a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_m$ is the corresponding linear combination of $L(\mathbf{x}_1), \ldots, L(\mathbf{x}_m)$."

If $a_1, \ldots, a_n$ are real numbers, then the function L defined by

$$L(\mathbf{x}) = a_1 x^1 + \cdots + a_n x^n, \tag{3.6}$$

for every $\mathbf{x} \in E^n$, is linear. Conversely, if L is a linear function, let

$$a_i = L(\mathbf{e}_i), \qquad i = 1, \ldots, n.$$

For each $\mathbf{x}$ we have

$$\mathbf{x} = x^1 \mathbf{e}_1 + \cdots + x^n \mathbf{e}_n.$$

Applying (3.5) with $c^j = x^j$, $\mathbf{x}_j = \mathbf{e}_j$, and $m = n$, we get

$$L(\mathbf{x}) = x^1 L(\mathbf{e}_1) + \cdots + x^n L(\mathbf{e}_n).$$

This has the form (3.6).

Let us denote the right-hand side of (3.6) by $\mathbf{a} \cdot \mathbf{x}$. We have proved the following.

Proposition 3.1. *A real valued function L is linear if and only if there exist real numbers $a_1, \ldots, a_n$ such that $L(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ for every $\mathbf{x} \in E^n$.* □

The object $\mathbf{a}$ is called a *covector* and $a_1, \ldots, a_n$ are its components. Covectors are not elements of E^n, but belong to the n-dimensional vector space $(E^n)^*$ dual to E^n. This is explained in more detail later in the present section.

Note that we have written a_i with subscripts, while the components x^i of a vector $\mathbf{x}$ are written with superscripts. We call the n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ a *covector*, and $a_1, \ldots, a_n$ are its components. The number $\mathbf{a} \cdot \mathbf{x}$ is called the *scalar product* of the covector $\mathbf{a}$ and vector $\mathbf{x}$. If $\mathbf{a}$ and $\mathbf{b}$ are covectors, then the sum $\mathbf{a} + \mathbf{b}$ and product of $\mathbf{a}$ by a scalar c are defined by

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, \ldots, a_n + b_n), \qquad c\mathbf{a} = (ca_1, \ldots, ca_n).$$

The covectors form a vector space of dimension n, denoted by $(E^n)^*$.

With any vector space $\mathscr{V}$ is associated a dual vector space $\mathscr{V}^*$ (see Appendix A.1). The elements of $\mathscr{V}^*$ are the real-valued linear functions on $\mathscr{V}$. In particular, let $\mathscr{V} = E^n$. Proposition 3.1 enables us to identify the space of covectors with the dual space $(E^n)^*$.

The usefulness of the distinction between E^n and its dual $(E^n)^*$ should gradually become apparent to the reader, especially in Chapters 4, 7, and 8. One important fact about vectors and covectors is that their components x^i and a_i change oppositely with respect to linear transformations (Section 4.1). The reader who is accustomed to the distinction between column and row vectors in matrix algebra may think of a vector $\mathbf{x}$ as a column vector, and a covector $\mathbf{a}$ as a row vector.

The standard basis for $(E^n)^*$

For $i = 1, \ldots, n$ consider the linear function X^i such that $X^i(\mathbf{x}) = x^i$ for each $\mathbf{x} \in E^n$. In particular, we can take for $\mathbf{x}$ any standard basis vector $\mathbf{e}_1, \ldots, \mathbf{e}_n$ for E^n, obtaining

$$X^i(\mathbf{e}_j) = \delta^i_j,$$

where $\delta^i_j = \delta_{ij}$ is Kronecker's delta. Let us denote by $\mathbf{e}^i$ the covector corresponding to the linear function X^i. The ith component of $\mathbf{e}^i$ is 1, and the other components are 0. We call $\{\mathbf{e}^1, \ldots, \mathbf{e}^n\}$ the *standard* basis for $(E^n)^*$. It is dual to the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for E^n, in the sense explained in Appendix A.1.

The notation is chosen so that for every formula about vectors there is a corresponding formula about covectors obtained by interchanging subscripts and superscripts. For instance, the components of a vector $\mathbf{x}$ satisfy $x^i = X^i(\mathbf{x}) = \mathbf{e}^i \cdot \mathbf{x}$. The corresponding formula for the components of a covector $\mathbf{a}$ is $a_i = \mathbf{a} \cdot \mathbf{e}_i$. In $(E^n)^*$ a euclidean inner product and norm are defined in the same way as in E^n. These facts are summarized in the table that follows.

	Vectors	Covectors
Standard bases	$\mathbf{e}_1, \ldots, \mathbf{e}_n$	$\mathbf{e}^1, \ldots, \mathbf{e}^n$
	$\mathbf{x} = \sum_{i=1}^n x^i \mathbf{e}_i$	$\mathbf{a} = \sum_{i=1}^n a_i \mathbf{e}^i$
Euclidean inner product	$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x^i y^i$	$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i$
Euclidean norm	$\lvert\mathbf{x}\rvert^2 = \mathbf{x} \cdot \mathbf{x}$	$\lvert\mathbf{a}\rvert^2 = \mathbf{a} \cdot \mathbf{a}$
Scalar product	$\mathbf{a} \cdot \mathbf{x} = \sum_{i=1}^n a_i x^i$	
	$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j$	
	$\mathbf{e}^i \cdot \mathbf{x} = x^i$	$\mathbf{a} \cdot \mathbf{e}_i = a_i$

For some purposes, one can avoid the use of covectors by the simple device of *raising indices*. If $\mathbf{a} = (a_1, \ldots, a_n)$ is a covector, consider the vector $\mathbf{y} = (y^1, \ldots, y^n)$ such that $y^i = a_i$ for $i = 1, \ldots, n$. Then $\mathbf{a} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{x}$ for any vector $\mathbf{x}$, where $\cdot$ on the left side means the scalar product and on the right side, the euclidean inner product. However, this device is often unsatisfactory. The difference in behavior of vectors and covectors under linear transformations was already mentioned. Moreover, it is sometimes convenient to

give E^n a noneuclidean inner product. This affects the formula for changing covectors into vectors. (See the distinction between the differential and gradient of a function, Section 3.3.)

PROBLEMS

1. Let $n = 3$ and $L(x, y, z) = x + y + 2z$.
   (a) What is the covector $\mathbf{a} = (a_1, a_2, a_3)$ corresponding to L?
   (b) Describe the set $\{(x, y, z) : L(x, y, z) = c\}$ and the intersection of this set with the plane $\{(x, y, z) : y = x\}$.

2. Prove that any linear function L is continuous using Proposition 3.1 and the inequality $|\mathbf{a} \cdot \mathbf{z}| \leq |\mathbf{a}|\,|\mathbf{z}|$

3. Let L be linear on E^n.
   (a) Show that $\{\mathbf{x} : L(\mathbf{x}) = c\}$ is a hyperplane, unless $L(\mathbf{x}) = 0$ for all $\mathbf{x} \in E^n$.
   (b) Give another proof (not the one suggested in Problem 5, Section 1.4) that hyperplanes and closed half-spaces are closed sets, using Problem 2.

4. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for E^n. Define L by the formula

$$L(c^1\mathbf{x}_1 + \cdots + c^n\mathbf{x}_n) = c^n$$

   for every $c^1, \ldots, c^n$. Show that:
   (a) L is a linear function.
   (b) The set $P = \{\mathbf{x} : L(\mathbf{x}) = 0\}$ is the $(n - 1)$-dimensional vector subspace spanned by $\{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$.

5. Let $\| \ \|$ be any norm on E^n (Section 2.11). Define on $(E^n)^*$ the *dual norm* as follows. For every covector $\mathbf{a}$,

$$\|\mathbf{a}\| = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{x}\| = 1\}.$$

   (a) Verify that the dual norm satisfies Properties (1), (2), and (3) (Section 2.9).
   (b) Show that $\|\mathbf{x}\| = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{a}\| = 1\}$. [*Hint*: Problem 7, Section 2.4.]

3.3 Differentiable functions

The existence of a derivative for a function of one variable is a fact of considerable interest. Geometrically, it says that a tangent line exists. However, the fact that a function of several variables has partial derivatives is not in itself of much interest. For one thing, the existence of derivatives in the directions of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ does not imply that derivatives exist in other directions. Moreover, the function need not have a tangent hyperplane even if there is a derivative in every direction (see Example 2 below).

We shall now define a more natural notion, that of differentiability. Geometrically, differentiability means the existence of a tangent hyperplane. It will be shown that most of the basic properties of differentiable functions of one variable remain true for differentiable functions of several variables.

Let us again consider an interior point $\mathbf{x}_0$ of the domain D of a real valued function f.

Definition. The function f is *differentiable at* $\mathbf{x}_0$ if there is a linear function L (depending on $\mathbf{x}_0$) such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - L(\mathbf{h})}{|\mathbf{h}|} = 0. \tag{3.7}$$

Let us show that if f is differentiable at $\mathbf{x}_0$, then f has a derivative at $\mathbf{x}_0$ in every direction $\mathbf{v}$. Taking $\mathbf{h} = t\mathbf{v}$, (3.7) implies that

$$\lim_{t\to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0) - L(t\mathbf{v})}{t} = 0,$$

by Proposition 2.3, and therefore

$$\lim_{t\to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0)}{t} - L(\mathbf{v}) = 0,$$

$$\lim_{t\to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0)}{t} = L(\mathbf{v}).$$

This shows that $L(\mathbf{v})$ is the derivative at $\mathbf{x}_0$ in the direction $\mathbf{v}$.

In Section 3.2 we identified each linear function with a covector. Let $df(\mathbf{x}_0)$ denote the covector corresponding to the linear function L in (3.7).

Definition. $df(\mathbf{x}_0)$ is the *differential* of f at $\mathbf{x}_0$.

By Formula (3.6), $L(\mathbf{h}) = df(\mathbf{x}_0) \cdot \mathbf{h}$ for any vector $\mathbf{h}$. In particular, if f is differentiable at $\mathbf{x}_0$, then the derivative of f in any direction $\mathbf{v}$ equals $df(\mathbf{x}_0) \cdot \mathbf{v}$. In particular, let $\mathbf{v} = \mathbf{e}_i$. The number $a_i = L(\mathbf{e}_i)$ in (3.6) is the ith partial derivative $f_i(\mathbf{x}_0)$. Hence the components of the covector $df(\mathbf{x}_0)$ are the partial derivatives:

$$df(\mathbf{x}_0) = \sum_{i=1}^{n} f_i(\mathbf{x}_0)\mathbf{e}^i, \tag{3.8}$$

$$df(\mathbf{x}_0) \cdot \mathbf{h} = \sum_{i=1}^{n} f_i(\mathbf{x}_0)h^i. \tag{3.9}$$

If $\mathbf{x} \in D$, let us set $\mathbf{x} = \mathbf{x}_0 + \mathbf{h}$. The vector $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$ is often called the *increment between* $\mathbf{x}$ *and* $\mathbf{x}_0$, and in the time-honored notation of calculus one would write $\Delta\mathbf{x}$ for $\mathbf{h}$. The number $f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)$ is the corresponding *increment in* f. However, we use neither the word increment nor the notation $\Delta\mathbf{x}$.

If f is differentiable at $\mathbf{x}_0$, then the differential at $\mathbf{x}_0$ furnishes a linear approximation $df(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$ to $f(\mathbf{x}) - f(\mathbf{x}_0)$ when $\mathbf{x}$ is near $\mathbf{x}_0$. The error in this approximation is $f(\mathbf{x}) - f(\mathbf{x}_0) - df(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$, which is the numerator in (3.7). It is small compared to the distance $|\mathbf{x} - \mathbf{x}_0|$ when $|\mathbf{x} - \mathbf{x}_0|$ is small.

Let us interpret this statement geometrically. Think of f as defining an *n-dimensional surface* in E^{n+1}, namely $M = \{(\mathbf{x}, z) : z = f(\mathbf{x}), \mathbf{x} \in D\}$. A hyperplane $P \subset E^{n+1}$ is called *tangent* to M at $(\mathbf{x}_0, z_0) \in M$ if: for each $(\mathbf{x}, z) \in M$ there exists $(\mathbf{x}, z') \in P$ such that $\lim_{\mathbf{x} \to \mathbf{x}_0} |\mathbf{x} - \mathbf{x}_0|^{-1} |z' - z| = 0$. Let us take

$$P = \{(\mathbf{x}, z') : z' = z_0 + df(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)\}, z_0 = f(\mathbf{x}_0).$$

Then P is the tangent hyperplane to M at $(\mathbf{x}_0, z_0)$ (see Figure 3.2). In Section 4.7 we define the concept of manifold. It turns out that M is a manifold of dimension n; the definition just given of tangent hyperplane is equivalent to the one given there.

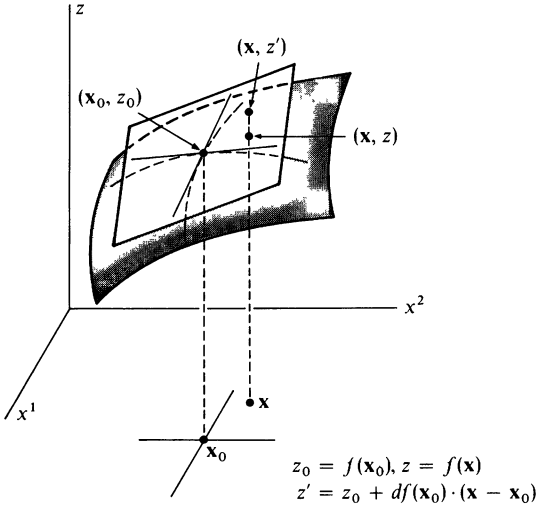


Figure 3.2

EXAMPLE 1. Let $f(x, y) = (xy)^{1/3}$. Find the tangent plane at (1, 1, 1). By elementary calculus

$$f_1(x, y) = \tfrac{1}{3}x^{-2/3}y^{1/3}, \qquad f_2(x, y) = \tfrac{1}{3}x^{1/3}y^{-2/3},$$

except at (0, 0). Moreover, f_1 and f_2 are continuous functions except at (0, 0). By Theorem 3.2 in Section 3.4, f is differentiable at any $(x_0, y_0) \neq (0, 0)$. The components of $df(x_0, y_0)$ are $f_1(x_0, y_0)$ and $f_2(x_0, y_0)$. The equation for the tangent plane at $(x_0, y_0, f(x_0, y_0))$ is

$$z = f(x_0, y_0) + f_1(x_0, y_0)(x - x_0) + f_2(x_0, y_0)(y - y_0).$$

Taking $x_0 = y_0 = 1$, the equation of the tangent plane at (1, 1, 1) is

$$z = 1 + \tfrac{1}{3}(x - 1) + \tfrac{1}{3}(y - 1).$$

The partial derivatives $f_1(0, 0)$ and $f_2(0, 0)$ are both 0, according to Problem 4, Section 3.1. However, there is no tangent plane at (0, 0, 0). If there were a tangent plane at (0, 0, 0), then f would have to be differentiable at (0, 0). Since there is not a derivative in every direction at (0, 0), f is not differentiable there.

Theorem 3.1. *If f is differentiable at $\mathbf{x}_0$ then f is continuous at $\mathbf{x}_0$.*

PROOF. For any $\mathbf{h}$

$$(*) \quad f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) = [f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - df(\mathbf{x}_0) \cdot \mathbf{h}] + df(\mathbf{x}_0) \cdot \mathbf{h}.$$

From the definition of limit (with $\varepsilon = 1$) there is a positive number δ_0 such that if $0 < |\mathbf{h}| < \delta_0$ the quotient in (3.7) has absolute value less than 1,

$$|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - df(\mathbf{x}_0) \cdot \mathbf{h}| < |\mathbf{h}|.$$

By Cauchy's inequality

$$|df(\mathbf{x}_0) \cdot \mathbf{h}| \leq |df(\mathbf{x}_0)|\,|\mathbf{h}|.$$

Applying the triangle inequality to the right side of (*),

$$|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)| \leq |f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - df(\mathbf{x}_0) \cdot \mathbf{h}| + |df(\mathbf{x}_0) \cdot \mathbf{h}|.$$

Consequently, if $0 < |\mathbf{h}| < \delta_0$,

$$|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)| < C|\mathbf{h}|, \tag{3.10}$$

where $C = 1 + |df(\mathbf{x}_0)|$. Given $\varepsilon > 0$, let $\delta = \min\{\delta_0, \varepsilon/C\}$. Then

$$|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)| < \varepsilon$$

for every $\mathbf{h}$ such that $0 < |\mathbf{h}| < \delta$. This shows that

$$\lim_{h \to 0} f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0),$$

in other words, that f is continuous at $\mathbf{x}_0$. □

For $n = 1$, f is differentiable at x_0 if and only if the derivative $f'(x_0)$ exists, since both statements are equivalent to the existence of a tangent line at $(x_0, f(x_0))$. However, for $n \geq 2$ a function f may have a derivative at $\mathbf{x}_0$ in every direction yet not be differentiable or even continuous at $\mathbf{x}_0$. This is shown by the following example.

EXAMPLE 2. Let

$$f(x, y) = \frac{2xy^2}{x^2 + y^4}, \quad \text{if } (x, y) \neq (0, 0), \quad \text{and} \quad f(0, 0) = 0.$$

There are two cases to consider. If $\cos\theta \neq 0$, then the derivative at (0, 0) in the direction $(\cos\theta, \sin\theta)$ is

$$\lim_{t \to 0} \frac{f(t\cos\theta, t\sin\theta) - f(0, 0)}{t} = \lim_{t \to 0} \frac{2\cos\theta \sin^2\theta}{\cos^2\theta + t^2 \sin^4\theta} = \frac{2\sin^2\theta}{\cos\theta}.$$

If $\cos\theta = 0$, then $f(t\cos\theta, t\sin\theta) = 0$ for every t and the directional derivative at $(0, 0)$ is 0. However, $f(y^2, y) = 1$ for every $y \neq 0$. Since $f(0, 0) = 0$, f is not continuous at $(0, 0)$. By Theorem 3.1, f is not differentiable at $(0, 0)$.

Let us next state a proposition which, although of no interest in itself, is useful later.

Proposition 3.2. *Let $\phi(t) = f(\mathbf{x}_0 + t\mathbf{h})$. Then for every t such that f is differentiable at $\mathbf{x}_0 + t\mathbf{h}$,*

$$\phi'(t) = df(\mathbf{x}_0 + t\mathbf{h}) \cdot \mathbf{h}.$$

PROOF. If $\mathbf{h} = \mathbf{0}$ the result is trivial. If $\mathbf{h} \neq \mathbf{0}$, then

$$0 = \lim_{\eta \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{h} + \eta) - f(\mathbf{x}_0 + t\mathbf{h}) - df(\mathbf{x}_0 + t\mathbf{h}) \cdot \eta}{|\eta|}.$$

In particular, let $\eta = \tau\mathbf{h}$. Then

$$0 = \lim_{\tau \to 0} \frac{\phi(t + \tau) - \phi(t) - \tau\, df(\mathbf{x}_0 + t\mathbf{h}) \cdot \mathbf{h}}{\tau},$$

$$0 = \lim_{\tau \to 0} \frac{\phi(t + \tau) - \phi(t)}{\tau} - df(\mathbf{x}_0 + t\mathbf{h}) \cdot \mathbf{h}. \qquad \square$$

Note that if $\mathbf{h}$ is a direction ($|\mathbf{h}| = 1$) and $t = 0$, we again obtain the formula $df(\mathbf{x}_0) \cdot \mathbf{h}$ for the directional derivative.

As a first application of Proposition 3.2 let us extend the mean value theorem to functions of several variables. Consider two points $\mathbf{x}_0$, $\mathbf{x}_0 + \mathbf{h}$. We recall (Section 1.3) that points of the line segment l joining $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{h}$ have the form $\mathbf{x} = \mathbf{x}_0 + t\mathbf{h}$, $t \in [0, 1]$.

Mean value theorem. *Let f be continuous at every point $\mathbf{x}$ of the line segment l joining $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{h}$, with f differentiable at each point of l except perhaps the endpoints. Then there exists a number $s \in (0, 1)$ such that*

$$f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) = df(\mathbf{x}_0 + s\mathbf{h}) \cdot \mathbf{h}.$$

PROOF. Let $\phi(t) = f(\mathbf{x}_0 + t\mathbf{h})$, as in Proposition 3.2. Then ϕ is continuous on $[0, 1]$, and

$$\phi(0) = f(\mathbf{x}_0), \qquad \phi(1) = f(\mathbf{x}_0 + \mathbf{h}),$$
$$\phi'(t) = df(\mathbf{x}_0 + t\mathbf{h}) \cdot \mathbf{h}, \quad \text{if } 0 < t < 1.$$

By the mean value theorem for functions of one variable (Section A.2) there exists $s \in (0, 1)$ such that $\phi(1) - \phi(0) = \phi'(s)$. $\square$

Definition. If f is differentiable at every point of a subset A of its domain D, then we say that f is *differentiable on A*. If D is an open set and f is differentiable at every point of D, then f is called a *differentiable function*.

The mean value theorem has the following corollaries.

Corollary 1. *Let f be differentiable on a convex set K and $C \geq 0$ a number such that $|df(\mathbf{x})| \leq C$ for every $\mathbf{x} \in K$. Then for every $\mathbf{x}, \mathbf{y} \in K$,*

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|.$$

PROOF. By the mean value theorem, with $\mathbf{x}_0 = \mathbf{y}$, $\mathbf{x}_0 + \mathbf{h} = \mathbf{x}$,

$$f(\mathbf{x}) - f(\mathbf{y}) = df(\mathbf{y} + s(\mathbf{x} - \mathbf{y})) \cdot (\mathbf{x} - \mathbf{y}),$$

where $s \in (0, 1)$. By Cauchy's inequality,

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq |df(\mathbf{y} + s(\mathbf{x} - \mathbf{y}))|\,|\mathbf{x} - \mathbf{y}| \leq C|\mathbf{x} - \mathbf{y}|. \qquad \square$$

Corollary 2. *Let f be a differentiable function whose domain D is an open, connected set, such that $df(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x} \in D$. Then f is a constant function.*

PROOF. Let $\mathbf{x}_0$ be some point of D, and let $D_1 = \{\mathbf{x} : f(\mathbf{x}) = f(\mathbf{x}_0)\}$. If $\mathbf{x} \in D_1$ then some neighborhood U of $\mathbf{x}$ is contained in D. Every neighborhood is a convex set. By Corollary 1, with $C = 0$ and $K = U$, $f(\mathbf{y}) = f(\mathbf{x}) = f(\mathbf{x}_0)$ for every $\mathbf{y} \in U$. Hence $U \subset D_1$. This shows that D_1 is an open set.

Since f is differentiable, f is continuous by Theorem 3.1. Therefore $D - D_1 = \{\mathbf{x} : f(\mathbf{x}) \neq f(\mathbf{x}_0)\}$ is also open by Corollary 2 to Theorem 2.6. If $D - D_1$ is not empty, then D is the union of two disjoint, nonempty open sets D_1 and $D - D_1$. Since D is connected, this is impossible. Hence $D - D_1$ is empty, and $D = D_1$. $\square$

Corollary 2 generalizes the result that if $f'(x) = 0$ for every x in an open interval, then f is constant there.

Note. In 1935 H. Whitney (*Duke Math. J.* **1**, 514–517) gave an example of a connected set $A \subset E^2$ and a differentiable function f, such that $df(x,y) = \mathbf{0}$ for every $(x, y) \in A$, but $f(x, y)$ is not constant on A. The set A in Whitney's example has no interior point.

The gradient vector

If $\mathbf{x}$ is any point where f is differentiable, then besides the covector

$$df(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x})\mathbf{e}^i$$

whose components are the partial derivatives $f_i(\mathbf{x})$, it is sometimes more suitable to think instead of the vector with these same components. This is called the *gradient vector* at $\mathbf{x}$ and is denoted by grad $f(\mathbf{x})$. Thus

$$\text{grad } f(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x})\mathbf{e}_i. \tag{3.11}$$

Another common notation for the gradient vector is $\nabla f(\mathbf{x})$.

Note. This definition of the gradient vector is correct only if we use the euclidean inner product in E^n. Suppose that E^n is given some other inner product $B(\mathbf{x}, \mathbf{y}) = \sum_{i,j=1}^{n} c_{ij}x^i y^j$, with (c_{ij}) a symmetric, positive definite matrix (see Section 2.11). Let us denote the gradient vector with respect to this inner product by $\operatorname{grad}_B f(\mathbf{x})$. We require that $\operatorname{grad}_B f(\mathbf{x})$ satisfy

$$B(\operatorname{grad}_B f(\mathbf{x}), \mathbf{h}) = df(\mathbf{x}) \cdot \mathbf{h}, \quad \text{for all } \mathbf{h} \in E^n.$$

To find the components of the vector $\mathbf{z} = \operatorname{grad}_B f(\mathbf{x})$, let $\mathbf{h} = \mathbf{e}_i$ in (3.9). Then

$$\sum_{j=1}^{n} c_{ij}z^j = f_i(\mathbf{x}), \quad i = 1, \ldots, n.$$

This is a system of linear equations for the components $z^1, \ldots, z^n$ of $\operatorname{grad}_B f(\mathbf{x})$. Let (c^{ij}) denote the inverse of the matrix (c_{ij}). Then

$$z^i = \sum_{j=1}^{n} c^{ij}f_j(\mathbf{x}), \quad i = 1, \ldots, n,$$

$$\operatorname{grad}_B f(\mathbf{x}) = \sum_{i,j=1}^{n} c^{ij}f_j(\mathbf{x})\mathbf{e}_i. \tag{3.12}$$

For a noneuclidean inner product, (3.12) replaces (3.11).

Problems

In Problems 1, 2, 3, and 9, assume that f is differentiable. In each case this follows from Theorem 3.2 in Section 3.4.

1. Let $f(x, y) = 3x^2y + 2xy^2$. Find the tangent plane at $(1, -2, 2)$.

2. Using the formula $df(\mathbf{x}_0) \cdot \mathbf{v}$ for directional derivative, find the derivative of f at $\mathbf{x}_0$ in the direction $\mathbf{v}$.
   (a) $f(x, y) = xy$, $\mathbf{x}_0 = (1, 3)$, $\mathbf{v} = (2/\sqrt{5}, -1/\sqrt{5})$.
   (b) $f(x, y) = x \exp(xy)$, $\mathbf{x}_0 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{v} = (1/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_2)$.
   (c) $f(x, y, z) = ax^2 + by^2 + cz^2$, $\mathbf{x}_0 = \mathbf{e}_1$, $\mathbf{v} = \mathbf{e}_3$.

3. Let $f(x, y) = \log(x^2 + 2y + 1) + \int_0^x \cos(t^2)dt$, $y > -\frac{1}{2}$.
   (a) Find $df(x, y)$.
   (b) Find approximately $f(0.03, 0.03)$.

4. Find grad $f(\mathbf{x})$ for each of the following functions:
   (a) $f(\mathbf{x}) = \mathbf{x}_0 \cdot \mathbf{x}$. (b) $f(\mathbf{x}) = |\mathbf{x}|$, $\mathbf{x} \neq \mathbf{0}$. (c) $f(\mathbf{x}) = (\mathbf{x}_0 \cdot \mathbf{x})^2$.

5. In Problem 3, Section 3.1, show that f is continuous at $(0, 0)$, but not differentiable at $(0, 0)$,

6. Let $f(x, y) = 2xy^2/(x^2 + y^4)$, if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$, as in Example 2.
   (a) Show that $-1 \leq f(x, y) \leq 1$ for every (x, y).
   (b) Find $\{(x, y) : f(x, y) = 1\}$ and $\{(x, y) : f(x, y) = -1\}$.
   (c) Find $\{(x, y) : \operatorname{grad} f(x, y) = (0, 0)\}$.
   (d) Find $\{(x, y) : f(x, y) = c\}$ for any c, and illustrate with a sketch.

7. Let f and g be differentiable at $\mathbf{x}_0$. (a) Prove that the sum $f + g$ is differentiable at $\mathbf{x}_0$, and $d(f + g)(\mathbf{x}_0) = df(\mathbf{x}_0) + dg(\mathbf{x}_0)$. (b) Prove that the product fg is differentiable at $\mathbf{x}_0$, and $d(fg)(\mathbf{x}_0) = f(\mathbf{x}_0)dg(\mathbf{x}_0) + g(\mathbf{x}_0)df(\mathbf{x}_0)$. [*Hint*: Recall the proof for $n = 1$.]

8. (*Euler's formula.*) Let p be a real number. A function f is called *homogeneous of degree* p if $f(t\mathbf{x}) = t^p f(\mathbf{x})$ for every $\mathbf{x} \neq \mathbf{0}$ and $t > 0$. Let f be differentiable for all $\mathbf{x} \neq \mathbf{0}$. Show that if f is homogeneous of degree p, then

$$df(\mathbf{x}) \cdot \mathbf{x} = pf(\mathbf{x})$$

for every $\mathbf{x} \neq \mathbf{0}$, and conversely. [*Hint*: Let $\phi(t) = f(t\mathbf{x})$ and use Proposition 3.2 with $\mathbf{x}_0 = \mathbf{0}$. For the converse, show that for fixed $\mathbf{x}$, $\phi(t)t^{-p}$ is a constant.]

9. Let $Q(\mathbf{x}) = \sum_{i,j=1}^{n} C_{ij}x^i x^j$, where $C_{ij} = C_{ji}$ and $Q(\mathbf{x}) > 0$ for every $\mathbf{x} \neq \mathbf{0}$. Let $f(\mathbf{x}) = [Q(\mathbf{x})]^{p/2}$. Calculate $df(\mathbf{x})$ and verify Euler's formula for this function.

10. Let f be continuous at $\mathbf{x}_0$ and g differentiable at $\mathbf{x}_0$ with $g(\mathbf{x}_0) = 0$. Show that the product fg is differentiable at $\mathbf{x}_0$.

3.4 Functions of class $C^{(q)}$

Let f be a function whose domain is an open set $D \subset E^n$.

Definition. If f is continuous on D, then f is said to be a *function of class* $C^{(0)}$. If the partial derivatives $f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})$ exist for every $\mathbf{x} \in D$ and $f_1, \ldots, f_n$ are continuous functions on D, then f is a *function of class* $C^{(1)}$.

The classes $C^{(q)}$ of functions, where $q = 2, 3, \ldots,$ are defined below. We first prove the following sufficient condition for differentiability, which is adequate for most purposes.

Theorem 3.2. *If f is a function of class $C^{(1)}$, then f is a differentiable function.*

PROOF. Let us proceed by induction on the dimension n. If $n = 1$, differentiability means simply that $f'(x)$ exists for every $x \in D$, while if f is of class $C^{(1)}$, then f' is a continuous function. Let us assume that the theorem is true in dimension $n - 1$.

Let $\mathbf{x}_0$ be any point of D and $\delta_0 > 0$ such that the δ_0-neighborhood of $\mathbf{x}_0$ is contained in D. Let us write (Figure 3.3):

$$\hat{\mathbf{x}} = (x^1, \ldots, x^{n-1}), \qquad \hat{\mathbf{x}}_0 = (x_0^1, \ldots, x_0^{n-1}),$$
$$\phi(\hat{\mathbf{x}}) = f(x^1, \ldots, x^{n-1}, x_0^n) = f(\hat{\mathbf{x}}, x_0^n),$$

provided the point $(\hat{\mathbf{x}}, x_0^n)$ is in D. The partial derivatives of ϕ are

$$\phi_i(\hat{\mathbf{x}}) = f_i(\hat{\mathbf{x}}, x_0^n), \qquad i = 1, \ldots, n - 1.$$

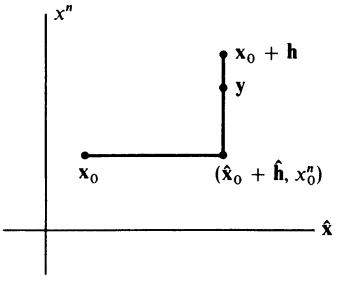


Figure 3.3

Since f is of class $C^{(1)}$, each f_i is continuous. Hence each ϕ_i is continuous and ϕ is of class $C^{(1)}$. By the induction hypothesis ϕ is differentiable at $\hat{\mathbf{x}}_0$. Therefore, given $\varepsilon > 0$ there exists δ_1, $0 < \delta_1 < \delta_0$, such that

$$\left| \phi(\hat{\mathbf{x}}_0 + \hat{\mathbf{h}}) - \phi(\hat{\mathbf{x}}_0) - \sum_{i=1}^{n-1} \phi_i(\hat{\mathbf{x}}_0) h^i \right| < \frac{\varepsilon}{2} |\hat{\mathbf{h}}|$$

whenever $|\hat{\mathbf{h}}| < \delta_1$. Since f_n is continuous, there exists δ_2, $0 < \delta_2 < \delta_0$, such that $|f_n(\mathbf{y}) - f_n(\mathbf{x}_0)| < \varepsilon/2$ whenever $|\mathbf{y} - \mathbf{x}_0| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ and let $|\mathbf{h}| < \delta$. By the mean value theorem,

$$\begin{aligned} f(\mathbf{x}_0 + \mathbf{h}) - \phi(\hat{\mathbf{x}}_0 + \hat{\mathbf{h}}) &= f(\hat{\mathbf{x}}_0 + \hat{\mathbf{h}}, x_0^n + h^n) - f(\hat{\mathbf{x}}_0 + \hat{\mathbf{h}}, x_0^n) \\ &= f_n(\hat{\mathbf{x}}_0 + \hat{\mathbf{h}}, x_0^n + sh^n) h^n \end{aligned}$$

for some $s \in (0, 1)$. Setting $\mathbf{y} = (\hat{\mathbf{x}}_0 + \hat{\mathbf{h}}, x_0^n + sh^n)$,

$$|\mathbf{y} - \mathbf{x}_0| = |(\hat{\mathbf{h}}, sh^n)| \leq |\mathbf{h}| < \delta.$$

Since $f(\mathbf{x}_0) = \phi(\hat{\mathbf{x}}_0)$,

$$\begin{aligned} f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) &= [f(\mathbf{x}_0 + \mathbf{h}) - \phi(\hat{\mathbf{x}}_0 + \hat{\mathbf{h}})] \\ &\quad + [\phi(\hat{\mathbf{x}}_0 + \hat{\mathbf{h}}) - \phi(\hat{\mathbf{x}}_0)], \\ f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - df(\mathbf{x}_0) \cdot \mathbf{h} &= [f_n(\mathbf{y}) h^n - f_n(\mathbf{x}_0) h^n] \\ &\quad + \left[\phi(\hat{\mathbf{x}}_0 + \hat{\mathbf{h}}) - \phi(\hat{\mathbf{x}}_0) - \sum_{i=1}^{n-1} \phi_i(\hat{\mathbf{x}}_0) h^i \right]. \end{aligned}$$

Using the above inequalities, the triangle inequality, and the fact that $|h^n| \leq |\mathbf{h}|$, $|\hat{\mathbf{h}}| \leq |\mathbf{h}|$, we get

$$|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - df(\mathbf{x}_0) \cdot \mathbf{h}| < \frac{\varepsilon}{2} |h^n| + \frac{\varepsilon}{2} |\hat{\mathbf{h}}| \leq \varepsilon |\mathbf{h}|$$

whenever $|\mathbf{h}| < \delta$. This proves that f is differentiable at $\mathbf{x}_0$. □

Corollary. *Every function of class $C^{(1)}$ is of class $C^{(0)}$.*

PROOF. Apply Theorems 3.1 and 3.2. □

EXAMPLE 1. If f and g are functions of class $C^{(1)}$ with the same domain D, then $f + g$ is of class $C^{(1)}$. Using the product rule from elementary calculus, the partial derivatives of the product are

$$(fg)_i = f_i g + f g_i, \qquad i = 1, \ldots, n.$$

Since sums and products of continuous functions are again continuous, $(fg)_i$ is continuous for each $i = 1, \ldots, n$. Hence fg is of class $C^{(1)}$.

EXAMPLE 2. The composite of two functions of class $C^{(1)}$ is also of class $C^{(1)}$. Suppose that $f = \psi \circ g$, where ψ and g are of class $C^{(1)}$. By Formula (3.4), $f_i = (\psi' \circ g) g_i$. Since ψ' and g are continuous, their composite $\psi' \circ g$ is continuous (Theorem 2.7). Since g_i is continuous, the product $(\psi' \circ g) g_i$ is continuous. Thus f is of class $C^{(1)}$.

Higher-order partial derivatives

The partial derivatives $f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})$ are often called the *first-order* partial derivatives of f at $\mathbf{x}$. The functions $f_1, \ldots, f_n$ may themselves possess partial derivatives. If f_i has a jth partial derivative at $\mathbf{x}$, then this partial derivative is called a partial derivative of *order* 2 and is denoted by

$$f_{ij}(\mathbf{x}) \quad \text{or} \quad \frac{\partial^2 f}{\partial x^j \, \partial x^i}(\mathbf{x}).$$

For example, if $f(x, y) = x^2 y^3$ then $f_1(x, y) = 2xy^3$, $f_{11}(x, y) = 2y^3$, $f_{12}(x, y) = 6xy^2$.

If all of the partial derivatives $f_{ij}(\mathbf{x})$, $i, j = 1, \ldots, n$, exist at every $\mathbf{x} \in D$ and each f_{ij} is a continuous function, then f is called a *function of class* $C^{(2)}$. By the corollary to Theorem 3.2, if f is of class $C^{(2)}$ then $f_1, \ldots, f_n$ are continuous. Hence any function of class $C^{(2)}$ is also of class $C^{(1)}$.

The partial derivatives of f of order $q = 3, 4, \ldots$ are defined similarly, wherever they exist. The notation for partial derivative at $\mathbf{x}$, first in the direction $\mathbf{e}_{i_1}$, second in the direction $\mathbf{e}_{i_2}$, and so on, is

$$f_{i_1, \ldots, i_q}(\mathbf{x}), \qquad 1 \le i_l \le n, \qquad l = 1, \ldots, q.$$

Definition. If all of the qth order derivatives of f exist at every $\mathbf{x} \in D$ and each $f_{i_1, \ldots, i_q}$ is a continuous function on D, then f is a *function of class* $C^{(q)}$.

To emphasize that D is the domain of f, we sometimes say *function of class* $C^{(q)}$ *on* D.

By the corollary to Theorem 3.2, any function of class $C^{(q)}$ is also of class $C^{(q-1)}$. As q increases, more and more restrictive conditions are placed on the smoothness of f. In many parts of differential calculus it is sufficient to assume that f is of class $C^{(1)}$ or class $C^{(2)}$. However, for some purposes one needs $C^{(q)}$ for $q > 2$. For instance, in Taylor's formula it is assumed that f is of class $C^{(q)}$.

The sum and product of two functions of class $C^{(q)}$ are of class $C^{(q)}$. If, in Example 2, ψ and g are of class $C^{(q)}$, their composite f is also of class $C^{(q)}$.

EXAMPLE 3. Let $f(x) = x^p$ if $x \geq 0$, and $f(x) = 0$ if $x \leq 0$, where $p > 0$ is given. Then f is of class $C^{(q)}$ if $q < p$ but not if $q > p$. The proof of this is left to the reader (Problem 5). Thus for every q there exist functions of class $C^{(q)}$ that are not of class $C^{(q+1)}$.

EXAMPLE 4. Any polynomial in n variables is a function of class $C^{(q)}$ for every q. If f is a rational function, $f(\mathbf{x}) = P(\mathbf{x})/Q(\mathbf{x})$ where P and Q are polynomials, then f is of class $C^{(q)}$ on any open set where $Q(\mathbf{x}) \neq 0$.

It can happen that a function f has the second partial derivatives f_{ij} and f_{ji}, $i \neq j$, but that $f_{ij} \neq f_{ji}$ (see Problem 7). However, this undesirable phenomenon cannot occur if f is of class $C^{(2)}$. This is even true under the slightly weaker hypotheses of the following theorem in which no assumption is made about the other second-order partial derivatives of f.

Theorem 3.3. *If f is of class $C^{(1)}$ and both f_{ij} and f_{ji} are continuous, $i \neq j$, then $f_{ij} = f_{ji}$.*

PROOF. Suppose first that $n = 2$. We need to show that $f_{12} = f_{21}$. Let (x_0, y_0) be any point of D, and $\delta_0 > 0$ such that the δ_0-neighborhood of (x_0, y_0) is contained in D. For $0 < u < \delta_0/\sqrt{2}$ let

$$A(u) = \frac{1}{u^2}[f(x_0 + u, y_0 + u) - f(x_0, y_0 + u) - f(x_0 + u, y_0) + f(x_0, y_0)].$$

$A(u)$ is sometimes called the *second difference quotient*. Let

$$g(x) = f(x, y_0 + u) - f(x, y_0)$$

for every x such that $(x, y_0 + u), (x, y_0) \in D$. The domain of g is an open subset of E^1 which contains the closed interval $[x_0, x_0 + u]$. Moreover, $g'(x) = f_1(x, y_0 + u) - f_1(x, y_0)$. Thus g is of class $C^{(1)}$ since f_1 is continuous, and

$$A(u) = \frac{1}{u^2}[g(x_0 + u) - g(x_0)].$$

Applying the mean value theorem to g, there exists $\xi \in (x_0, x_0 + u)$ such that

$$A(u) = \frac{1}{u} g'(\xi) = \frac{1}{u}[f_1(\xi, y_0 + u) - f_1(\xi, y_0)]$$

(see Figure 3.4). Of course the number ξ depends on u. Let

$$h(y) = f_1(\xi, y)$$

for every y such that $(\xi, y) \in D$. The domain of h is open and contains $[y_0, y_0 + u]$. Moreover, $h'(y) = f_{12}(\xi, y)$, h is of class $C^{(1)}$ since f_{12} is continuous, and

$$A(u) = \frac{1}{u}[h(y_0 + u) - h(y_0)].$$

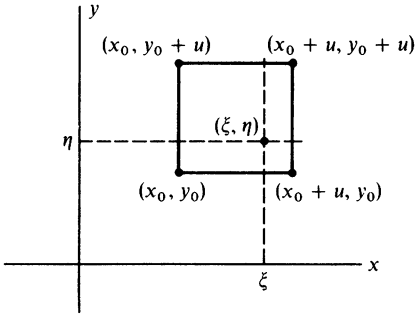


Figure 3.4

Another application of the mean value theorem gives

$$A(u) = f_{12}(\xi, \eta),$$

for some $\eta \in (y_0, y_0 + u)$, depending on u.

By reversing the roles of the first and second variables and repeating the proof, we find that

$$A(u) = f_{21}(\xi^*, \eta^*)$$

for some

$$\xi^* \in (x_0, x_0 + u) \quad \text{and} \quad \eta^* \in (y_0, y_0 + u).$$

Since f_{12} and f_{21} are continuous, given $\varepsilon > 0$ there exists $\delta \in (0, \delta_0)$ such that if $0 < u < \delta/\sqrt{2}$,

$$|f_{12}(\xi, \eta) - f_{12}(x_0, y_0)| < \varepsilon, \qquad |f_{21}(\xi^*, \eta^*) - f_{21}(x_0, y_0)| < \varepsilon.$$

This shows that

$$\lim_{u \to 0^+} A(u) = f_{12}(x_0, y_0) = f_{21}(x_0, y_0),$$

and proves the theorem if $n = 2$.

If $n > 2$ we need consider only the case $i < j$. Given $\mathbf{x}_0 = (x_0^1, \ldots, x_0^n) \in D$, let

$$\phi(x, y) = f(x_0^1, \ldots, x_0^{i-1}, x, x_0^{i+1}, \ldots, x_0^{j-1}, y, x_0^{j+1}, \ldots, x_0^n),$$

for every (x, y) in some open set containing (x_0^i, x_0^j). Applying the theorem to ϕ, we find that

$$f_{ij}(\mathbf{x}_0) = \phi_{12}(x_0^i, x_0^j) = \phi_{21}(x_0^i, x_0^j) = f_{ji}(\mathbf{x}_0). \qquad \square$$

From Theorem 3.3 it follows that in calculating any qth-order partial derivative of a function of class $C^{(q)}$ it is only the number of partial differentiations with respect to each of the variables which matters, and not the order in which they are taken. Thus for functions of class $C^{(3)}$, $f_{123} = f_{132}$, $f_{112} = f_{121}$, and so on.

We are now going to prove a stronger version of the mean-value theorem, which is valid for functions of class $C^{(q)}$. Let f be of class $C^{(q)}$ and $\mathbf{x}, \mathbf{x}_0 \in D$ *such that the line segment joining* $\mathbf{x}$ *and* $\mathbf{x}_0$ *is contained in* D. In particular, if D is convex then $\mathbf{x}$ and $\mathbf{x}_0$ can be any pair of points of D. Let $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$, and define ϕ as in Proposition 3.2 by $\phi(t) = f(\mathbf{x}_0 + t\mathbf{h})$. The domain of ϕ is $\{t : \mathbf{x} + t\mathbf{h} \in D\}$, which is an open subset of E^1 containing the closed interval $[0, 1]$. By repeated application of Proposition 3.2 to f, f_i, $f_{ij}, \ldots$, we find that:

$$\phi'(t) = \sum_{i=1}^{n} f_i(\mathbf{x}_0 + t\mathbf{h})h^i,$$

$$\phi''(t) = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} f_{ij}(\mathbf{x}_0 + t\mathbf{h})h^j \right] h^i,$$

$$\vdots$$

$$\phi^{(q)}(t) = \sum_{i_1, \ldots, i_q = 1}^{n} f_{i_1, \ldots, i_q}(\mathbf{x}_0 + t\mathbf{h})h^{i_1}, \ldots, h^{i_q}.$$

The sum in the last formula has n^q terms. It is taken over all integers i_1, $i_2, \ldots, i_q$, such that $1 \leq i_1, \ldots, i_q \leq n$.

By Taylor's formula for functions of one variable (Section A.2) there exists $s \in (0, 1)$ such that

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \cdots + \frac{1}{(q-1)!}\phi^{(q-1)}(0) + \frac{1}{q!}\phi^{(q)}(s).$$

But $\phi(1) = f(\mathbf{x})$, $\phi(0) = f(\mathbf{x}_0)$, and we have, by substitution, *Taylor's formula with remainder:*

$$(3.13)\quad f(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} f_i(\mathbf{x}_0)(x^i - x_0^i) + \frac{1}{2!}\sum_{i,j=1}^{n} f_{ij}(\mathbf{x}_0)(x^i - x_0^i)(x^j - x_0^i) + \cdots + \frac{1}{(q-1)!}\sum_{i_1, \ldots, i_{q-1}=1}^{n} f_{i_1, \ldots, i_{q-1}}(\mathbf{x}_0)h^{i_1} \cdots h^{i_{q-1}} + R_q(\mathbf{x}),$$

where $h^i = x^i - x_0^i$, $s \in (0, 1)$, and

$$(3.14)\qquad R_q(\mathbf{x}) = \frac{1}{q!}\sum_{i_1, \ldots, i_q = 1}^{n} f_{i_1, \ldots, i_q}(\mathbf{x}_0 + s\mathbf{h})h^{i_1} \cdots h^{i_q}.$$

Notice that the first terms on the right-hand side are just the first degree approximation $f(\mathbf{x}_0) + df(\mathbf{x}_0) \cdot \mathbf{h}$ to $f(\mathbf{x})$ considered in Section 3.3.

If we ignore the remainder $R_q(\mathbf{x})$, the right-hand side is a polynomial in $h^1, \ldots, h^n$ of degree $q - 1$. If the remainder is small this polynomial furnishes an approximation to $f(\mathbf{x})$. One can give an explicit estimate for the error, in terms of bounds for the qth-order partial derivatives of f. Suppose that K is a convex subset of D, such that $\mathbf{x}_0 \in K$ and $\mathbf{x} = \mathbf{x}_0 + \mathbf{h} \in K$. Then $\mathbf{x}_0 + s\mathbf{h} \in K$ for $0 \leq s \leq 1$. Moreover, suppose that all qth-order partial derivatives satisfy

$$|f_{i_1, \ldots, i_q}(\mathbf{x})| \leq C \text{ for all } \mathbf{x} \in K.$$

For any real numbers $y^1, \ldots, y^n$,

$$(y^1 + \cdots + y^n)^q = \sum_{i_1, \ldots, i_q = 1}^{n} y^{i_1} \cdots y^{i_q}.$$

Let $y^i = |h^i|$. From (3.14)

$$|R_q(\mathbf{x})| \leq \frac{C}{q!} \sum_{i_1, \ldots, i_q = 1}^{n} |h^{i_1}| \cdots |h^{i_q}| = \frac{C}{q!} \left(\sum_{i=1}^{n} |h^i| \right)^q.$$

By Problem 4, Section 1.2, $\sum_{i=1}^{n} |h^i| \leq n^{1/2} |\mathbf{h}|$. Therefore, the remainder in Taylor's formula satisfies the estimate

$$|R_q(\mathbf{x})| \leq \frac{C n^{q/2}}{q!} |\mathbf{h}|^q, \qquad \mathbf{h} = \mathbf{x} - \mathbf{x}_0, \tag{3.15}$$

whenever $\mathbf{x}_0, \mathbf{x} \in K$.

EXAMPLE 5. Let $f(x, y) = (1 + y^2)^{1/2} \cos x$, $\mathbf{x}_0 = (0, 1)$, and $q = 2$. Then

$$f_1 = -(1 + y^2)^{1/2} \sin x, f_2 = y(1 + y^2)^{-1/2} \cos x,$$

$$f_{11} = -(1 + y^2)^{1/2} \cos x, f_{12} = -y(1 + y^2)^{-1/2} \sin x,$$

$$f_{22} = (1 + y^2)^{-3/2} \cos x.$$

Here we have written f_1 for short in place of $f_1(x, y)$, and so on. Taylor's formula (3.13) becomes, since $f(0, 1) = 2^{1/2}, f_1(0, 1) = 0$, and $f_2(0, 1) = 2^{-1/2}$,

$$f(x, y) = 2^{1/2} + 2^{-1/2}(y - 1) + R_2(x, y).$$

If the remainder $R_2(x, y)$ is ignored, one gets a first-degree polynomial, which approximates $f(x, y)$ for (x, y) near $(0, 1)$. The second-order partial derivatives obey the estimates

$$|f_{11}| \leq (1 + y^2)^{1/2}, \qquad |f_{12}| \leq |y|, |f_{22}| \leq 1.$$

To apply (3.15), let us choose any $a > 1$ and

$$K = \{(x, y) : |y| \leq a\}, \qquad C = (1 + a^2)^{1/2}.$$

Whenever $|y| \leq a$, we have

$$|R_2(x, y)| \leq (1 + a^2)^{1/2}(x^2 + (y - 1)^2).$$

If f is a polynomial of degree $q - 1$, then the Taylor approximation is exact; in other words, $R_q(\mathbf{x}) = 0$ for every $\mathbf{x}$.

EXAMPLE 6. Let $f(x, y) = x^2 y$ and $(x_0, y_0) = (1, -1)$. Then

$$f_1 = 2xy, \qquad f_2 = x^2, \qquad f_{11} = 2y,$$

$$f_{12} = f_{21} = 2x, \qquad f_{112} = f_{121} = f_{211} = 2,$$

and all other partial derivatives are 0. $R_4(x, y) = 0$, and Taylor's formula becomes

$$f(x, y) = f(1, -1) + f_1(x - 1) + f_2(y + 1) + \frac{1}{2!}[f_{11}(x - 1)^2 + 2f_{12}(x - 1)(y + 1)] + \frac{1}{3!}[3f_{112}(x - 1)^2(y + 1)],$$

where the partial derivatives on the right-hand side are evaluated at $(1, -1)$. Thus

$$x^2y = -1 - 2(x - 1) + (y + 1) - (x - 1)^2 + 2(x - 1)(y + 1) + (x - 1)^2(y + 1).$$

Functions of class $C^{(q)}$ on a set

In many instances either f is not of class $C^{(q)}$ on its entire domain D, or else one is interested only in its values on some subset of D.

Definition. Let A be a nonempty subset of the domain of f. Then f is of *class $C^{(q)}$ on A* if there exists an open set D_1 containing A and a function F of class $C^{(q)}$ with domain D_1 such that $F(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in A$.

If A is open, then we may take $D_1 = A$. In that case $F = f \mid A$, where $f \mid A$ is the restriction of f to A. When A is an open subset of D, f is of class $C^{(q)}$ on A if and only if $f \mid A$ is a function of class $C^{(q)}$.

The function F in the definition is called an *extension of class $C^{(q)}$* of $f \mid A$. It is generally not easy to determine whether there is such an extension F. However, if A has some simple geometrical shape, there is sometimes a method for explicitly constructing extensions.

EXAMPLE 7. In Example 3, let $p > 0$ be an integer. Let $A = [0, \infty)$, $D_1 = E^1$, $F(x) = x^p$ for all x. Then f is of class $C^{(q)}$ on A for any q, and F is an extension of class $C^{(q)}$ of $f \mid A$.

**Whitney's extension theorems*

Some general theorems about extensions of class $C^{(q)}$ were proved by Whitney. Let us cite without proof a result that is a special case of a theorem of Whitney in 1934 (*Ann. Math. 35*:485). Let A be the closure of an open set B, and assume either that A is convex or that its boundary fr A is an $(n - 1)$-manifold of class $C^{(1)}$ (see Section 4.7). Let f be of class $C^{(q)}$ on B, and continuous on A. Moreover, assume that for each $i_1, \ldots, i_q$ there is a function $F_{i_1, \ldots, i_q}$ continuous on A such that $F_{i_1, \ldots, i_q}(\mathbf{x})$ equals the qth-order partial derivative $f_{i_1, \ldots, i_q}(\mathbf{x})$ for every $\mathbf{x} \in B$. Then there exists a function F of

class $C^{(q)}$ on E^n such that $F(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in A$. Hence f is of class $C^{(q)}$ on A.

Actually, what one needs to assume about A to apply Whitney's theorem is the following: Every $\mathbf{x}_0 \in A$ has a neighborhood U such that any pair of points $\mathbf{x}, \mathbf{y} \in U \cap B$ can be joined in B by a polygon of length no more than $c|\mathbf{x} - \mathbf{y}|$, where $c \geq 1$ depends only on U. If A is convex, then the line segment joining $\mathbf{x}$ and $\mathbf{y}$ lies in B, and one may take $c = 1$.

For other extension theorems of Whitney, see *Trans. Amer. Math. Soc. 36* (1934) and *Bull. Amer. Math. Soc. 50* (1944).

**Functions of class $C^{(\infty)}$; real analytic functions*

Let us say that f is of class $C^{(\infty)}$ if f is of class $C^{(q)}$ for every q. If f is of class $C^{(\infty)}$ and $\lim_{q\to\infty} R_q(\mathbf{x}) = 0$, then in place of Taylor's formula with remainder we may put the corresponding infinite series. This infinite series is called the *Taylor series* for $f(\mathbf{x})$ at $\mathbf{x}_0$.

If K is a convex subset of D and $\mathbf{x}_0 \in K$, then the following is a sufficient condition that $f(\mathbf{x})$ be the sum of its Taylor series for every $\mathbf{x} \in K$. Suppose that there is a positive number M whose qth power bounds every qth-order partial derivative of f, namely,

$$|f_{i_1,\ldots,i_q}(\mathbf{x})| \leq M^q \tag{3.16}$$

for every $\mathbf{x} \in K$, $q = 1, 2, \ldots$, and $1 \leq i_1, \ldots, i_q \leq n$. Then (3.15) holds, with $C = M^q$:

$$|R_q(\mathbf{x})| \leq \frac{M^q n^{q/2} |\mathbf{h}|^q}{q!} = \frac{B^q}{q!},$$

where $B = Mn^{1/2}|\mathbf{h}|$. Since $B^q/q! \to 0$ as $q \to \infty$,

$$\lim_{q\to\infty} R_q(\mathbf{x}) = 0$$

for every $\mathbf{x} \in K$, provided inequalities (3.16) hold.

A function f is called *analytic* if every $\mathbf{x}_0 \in D$ has a neighborhood $U_{\mathbf{x}_0}$ such that the Taylor series at $\mathbf{x}_0$ converges to $f(\mathbf{x})$ for every $\mathbf{x} \in U_{\mathbf{x}_0}$. We have just proved the following:

Let f be of class $C^{(\infty)}$, and suppose that every $\mathbf{x}_0 \in D$ has a neighborhood $U_{\mathbf{x}_0}$ in which an estimate (3.16) *holds. Then f is analytic.*

The positive number M in (3.16) may depend on $\mathbf{x}_0$ and on the radius of $U_{\mathbf{x}_0}$. It would lead us away from our main objectives to discuss analytic functions in any detail. Therefore, let us issue just one word of caution, namely, *not every function of class $C^{(\infty)}$ is analytic.* As an example, let $D = E^1$ and let

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x^2}\right) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let us show that this function is of class $C^{(\infty)}$ and that $f^{(q)}(0) = 0$ for every $q = 1, 2, \ldots$. For $x \neq 0$ the derivatives $f^{(q)}(x)$ can be computed by elementary calculus, and each $f^{(q)}$ is continuous on $E^1 - \{0\}$. It is at the point 0 where f must be examined. Now

$$\lim_{u \to +\infty} u^k \exp(-u) = 0 \quad \text{for each } k = 0, 1, 2, \ldots, \tag{3.17}$$

a fact that we prove immediately below. If $x < 0$, then $f(x) = f'(x) = f''(x) = \cdots = 0$. Using (3.17) with $k = 0$, $\exp(-1/x^2) \to 0$ as $x \to 0^+$. Since $f(0) = 0$, f is continuous. If $x > 0$

$$f'(x) = \frac{2}{x^3} \exp\left(-\frac{1}{x^2}\right) = 2x \cdot \frac{1}{x^4} \exp\left(-\frac{1}{x^2}\right).$$

Using (3.17) with $k = 2$, $f'(x) \to 0$ as $x \to 0^+$. Therefore $\lim_{x \to 0} f'(x) = 0$. By Problem 4, $f'(0) = 0$ and f is of class $C^{(1)}$. For each $q = 2, 3, \ldots, f^{(q)}(x)$ is a polynomial in $1/x$ times $\exp(-1/x^2)$ for $x > 0$. Hence $\lim_{x \to 0} f^{(q)}(x) = 0$. By Problem 4 and induction on q, $f^{(q)}(0) = 0$ and $f \in C^{(q)}$ for every q. Thus $f \in C^{(\infty)}$. If we expand f by Taylor's formula about 0, then $f(x) = R_q(x)$ for every x. If $x > 0$ the remainder $R_q(x)$ does not tend to 0 as $q \to \infty$. Hence f is not an analytic function.

PROOF OF (3.17). For each $u < 0$ let $\psi(u) = u^{-k} \exp u$. Then

$$\psi'(u) = (u - k)u^{-k-1} \exp u,$$
$$\psi''(u) = [u^2 - 2ku + k(k+1)]u^{-k-2} \exp u.$$

The expression in brackets has a minimum when $u = k$ and is positive there. Hence $\psi''(u) > 0$ for all $u > 0$. Let us apply Taylor's formula to ψ, with $q = 2$:

$$\psi(u) = \psi(u_0) + \psi'(u_0)(u - u_0) + \tfrac{1}{2}\psi''(v)(u - u_0)^2,$$

with v between u and u_0. Since $\psi''(v) > 0$,

$$\psi(u) \geq \psi(u_0) + \psi'(u_0)(u - u_0).$$

If $u_0 > k$, then $\psi'(u_0) > 0$ and the right-hand side tends to $+\infty$ as $u \to +\infty$. Hence $\psi(u) \to +\infty$ and $1/\psi(u) \to 0$ as $u \to +\infty$. □

PROBLEMS

1. Expand $f(x, y, z) = xyz$ by Taylor's formula about $\mathbf{x}_0 = (1, -1, 0)$, with $q = 4$.

2. Let $f(x, y) = x^{-1} \cos y$, $x > 0$, and $\mathbf{x}_0 = (1, 0)$.
(a) Expand $f(x, y)$ by Taylor's formula about $\mathbf{x}_0$, with $q = 2$, and find an estimate for $|R_2(x, y)|$.
(b) Show that $R_q(x, y) \to 0$ as $q \to \infty$ for (x, y) in some open set containing $\mathbf{x}_0$.

3. Let $f(x, y) = \psi(ax + by)$, where a and b are scalars and ψ is of class $C^{(q)}$ in some open set containing 0. Show that Taylor's formula about (0, 0) becomes

$$f(x, y) = \sum_{m=0}^{q-1} \frac{\psi^{(m)}(0)}{m!} \sum_{j=0}^{m} \binom{m}{j} (ax)^j (by)^{m-j} + R_q(x, y),$$

where $\binom{m}{j}$ is the binomial coefficient (which equals the number of j-element subsets of a set with m elements).

4. Let f be continuous on an open set D and of class $C^{(1)}$ on $D - \{\mathbf{x}_0\}$. Suppose, moreover, that $l_i = \lim_{\mathbf{x} \to \mathbf{x}_0} f_i(\mathbf{x})$ exists for each $i = 1, \ldots, n$. Prove that $l_i = f_i(\mathbf{x}_0)$, and consequently that f is of class $C^{(1)}$ on D. State and prove a corresponding result in case $q > 1$. [*Hint*: Apply the mean value theorem to $f(\mathbf{x}_0 + t\mathbf{e}_i) - f(\mathbf{x}_0)$.

5. Prove the statement made in Example 3. [*Hint*: Problem 4 with $x_0 = 0$.]

6. Let $f(x) = x^k \sin(1/x)$ if $x \neq 0$, and $f(0) = 0$. Show that:
   (a) If $k = 0$, then f is discontinuous at 0.
   (b) If $k = 1$, then f is of class $C^{(0)}$ but not differentiable at 0.
   (c) If $k = 2$, then f is differentiable but not of class $C^{(1)}$.
   (d) What can you say for $k \geq 3$?

7. Let $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$, if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$.
   (a) If $(x, y) \neq (0, 0)$, find $f_{12}(x, y)$ and $f_{21}(x, y)$ by elementary calculus, and verify that they are equal.
   (b) Using Problem 4 show that $f_1(0, 0) = f_2(0, 0) = 0$ and f is of class $C^{(1)}$.
   (c) Using the definition of partial derivative, show that $f_{12}(0, 0)$ and $f_{21}(0, 0)$ exist but are not equal. Why does this not contradict Theorem 3.3?

8. Given n and q, how many solutions of the equation $i_1 + \cdots + i_n = q$ are there with $i_1, \ldots, i_n$ nonnegative integers? With $i_1, \ldots, i_n$ positive integers? What does this say about the number of different qth-order partial derivatives of a function of class $C^{(q)}$?

3.5 Relative extrema

Let A be some subset of E^n and f a function whose domain contains A. Let us consider the problem of minimizing or maximizing $f(\mathbf{x})$ on A.

Definitions. If $\mathbf{x}_0$ is a point of A such that $f(\mathbf{x}_0) \leq f(\mathbf{x})$ for every $\mathbf{x} \in A$, then f has an *absolute minimum* at $\mathbf{x}_0$. The number

$$f(\mathbf{x}_0) = \min\{f(\mathbf{x}) : \mathbf{x} \in A\}$$

is the *minimum value* of f on A. (Of course, there need not be any such point $\mathbf{x}_0$. However, if A is a compact set, then by Theorem 2.5, any continuous function has an absolute minimum at some point of A.) If $f(\mathbf{x}_0) < f(\mathbf{x})$ for every $\mathbf{x} \in A$ except $\mathbf{x}_0$, then f has a *strict absolute minimum at* $\mathbf{x}_0$.

We say that f has a *relative minimum* at $\mathbf{x}_0$ if there is a neighborhood U of $\mathbf{x}_0$ such that $f(\mathbf{x}_0) \leq f(\mathbf{x})$ for every $\mathbf{x} \in A \cap U$. If U can be so chosen that $f(\mathbf{x}_0) < f(\mathbf{x})$ for every $\mathbf{x} \in A \cap U$ except $\mathbf{x}_0$, then f has a *strict relative minimum* at $\mathbf{x}_0$.

The notions of absolute maximum and relative maximum are defined similarly by reversing the inequality signs. We say *extremum* for either maximum or minimum.

In some cases the extrema can be found by inspection. For example, if $A = E^n$ and $f(\mathbf{x}) = |\mathbf{x}|$, then $f(\mathbf{0}) = 0$ and $f(\mathbf{x}) > 0$ for every $\mathbf{x} \neq \mathbf{0}$. Hence f has a strict absolute minimum at $\mathbf{0}$. Since this function is not differentiable at $\mathbf{0}$, the minimum could not have been found through the use of calculus.

If f and A are smooth enough, the relative extrema can be found by using calculus. *In the present section we assume that A is an open set.* In Section 4.8 we learn a technique for finding the extrema when A is a smooth submanifold of E^n.

Definition. A point $\mathbf{x}_0$ is a *critical point* of f if $df(\mathbf{x}_0) = \mathbf{0}$.

If f is a differentiable function, one need look only among the critical points for relative extrema.

Proposition 3.3. *If f has a relative extremum at $\mathbf{x}_0$ and f is differentiable at $\mathbf{x}_0$, then $\mathbf{x}_0$ is a critical point of* $\mathbf{f}$.

PROOF. Given a direction $\mathbf{v}$, let $\phi(t) = f(\mathbf{x}_0 + t\mathbf{v})$ for every t in some open subset of E^1 containing 0. Then ϕ has a relative extremum at 0, and consequently by elementary calculus $\phi'(0) = 0$. But $\phi'(0) = df(\mathbf{x}_0) \cdot \mathbf{v}$ is the derivative at $\mathbf{x}_0$ in the direction $\mathbf{v}$. Hence $df(\mathbf{x}_0) \cdot \mathbf{v} = 0$ for every $\mathbf{v}$, which implies that $df(\mathbf{x}_0) = \mathbf{0}$. □

It is illuminating to look at this result in a slightly different way. In place of the covector $df(\mathbf{x})$ let us consider the vector grad $f(\mathbf{x})$ with the same components (Section 3.3). Suppose that $\mathbf{x}$ is not a critical point. Then grad $f(\mathbf{x}) \neq \mathbf{0}$. Let us find the direction $\mathbf{v}$ for which the directional derivative at $\mathbf{x}$ is maximum.

By Cauchy's inequality,

$$\text{grad } f(\mathbf{x}) \cdot \mathbf{v} \leq |\text{grad } f(\mathbf{x})||\mathbf{v}|;$$

and equality holds if and only if $\mathbf{v} = \mathbf{v}(\mathbf{x})$, where

$$\mathbf{v}(\mathbf{x}) = \frac{1}{|\text{grad } f(\mathbf{x})|} \text{grad } f(\mathbf{x}).$$

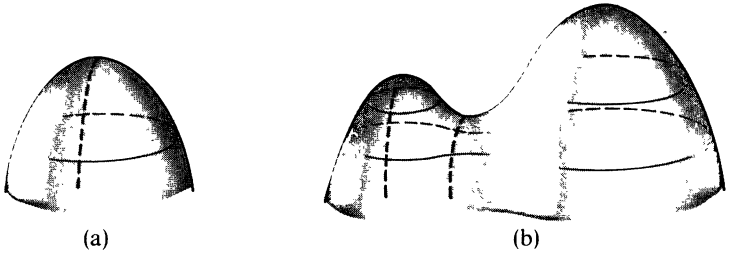
This direction is called the *direction of the gradient* at $\mathbf{x}$, and is the one which maximizes the directional derivative. The maximum value of the directional

derivative is

$$\operatorname{grad} f(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = \frac{1}{|\operatorname{grad} f(\mathbf{x})|} \operatorname{grad} f(\mathbf{x}) \cdot \operatorname{grad} f(\mathbf{x}) = |\operatorname{grad} f(\mathbf{x})|.$$

By going a short distance from $\mathbf{x}$ in the direction $\mathbf{v}(\mathbf{x})$, $f(\mathbf{x})$ is increased. Hence f cannot have a relative maximum at $\mathbf{x}$. The direction $-\mathbf{v}(\mathbf{x})$ minimizes the directional derivative at $\mathbf{x}$. In the same way, f cannot have a relative minimum at $\mathbf{x}$. This confirms the conclusion of Proposition 3.3.

This discussion is the basis for the gradient method (or method of steepest ascent) for finding maxima. A good intuitive picture of the gradient method may be obtained by thinking of an ambitious mountain climber who always takes the steepest direction. Let us suppose that the surface of the mountain can be represented in the form $\{(x, y, f(x, y)) : (x, y) \in A\}$, where f is a smooth function. In particular, no vertical cliffs, overhangs, or sharp ridges are allowed. If the mountain has the shape indicated in Figure 3.5(a), then it appears that the summit will be reached by this technique. However, if the mountain has a more complicated shape, the climber may reach a false summit or a saddle as in Figure 3.5(b). Once he reaches any critical point, the gradient method tells him to stay there. The gradient method will be defined more precisely later (Section 6.5).



(a) (b)

Figure 3.5

There is a second derivative test for relative extrema of functions of one variable. If $f'(x_0) = 0$ and $f''(x_0) \neq 0$, then the sign of $f''(x_0)$ determines whether the critical point x_0 gives a relative minimum or relative maximum. If $f''(x_0) = 0$, then x_0 may give a relative maximum, a relative minimum, or neither.

Let us now state and prove corresponding results for functions of several variables. Let f be a function of class $C^{(2)}$ on A, and let Q be the function with domain $A \times E^n$ defined by the formula

$$Q(\mathbf{x}, \mathbf{h}) = \sum_{i,j=1}^{n} f_{ij}(\mathbf{x}) h^i h^j. \tag{3.18}$$

For fixed $\mathbf{x}$, (3.18) defines a function on E^n, which we denote by $Q(\mathbf{x}, \ )$. Similarly, for fixed $\mathbf{h}$, $Q(\ , \mathbf{h})$ denotes the function on A whose value at $\mathbf{x}$ is $Q(\mathbf{x}, \mathbf{h})$. The function $Q(\mathbf{x}, \ )$ is a quadratic polynomial which in linear

algebra is called the quadratic form corresponding to the $n \times n$ symmetric matrix $(f_{ij}(\mathbf{x}))$ of second partial derivatives. Theorem 3.3 guarantees that this matrix is symmetric.

Let us write $Q(\mathbf{x}, \ ) \geq 0$ if $Q(\mathbf{x}, \mathbf{h}) \geq 0$ for every $\mathbf{h}$, and $Q(\mathbf{x}, \ ) > 0$ if $Q(\mathbf{x}, \mathbf{h}) > 0$ for every $\mathbf{h} \neq \mathbf{0}$. Note that $Q(\mathbf{x}, \mathbf{0}) = 0$. In the theory of quadratic forms $Q(\mathbf{x}, \ )$ is called *positive semidefinite* if $Q(\mathbf{x}, \ ) \geq 0$, and *positive definite* if $Q(\mathbf{x}, \ ) > 0$.

Similarly, we write $Q(\mathbf{x}, \ ) \leq 0$ if $Q(\mathbf{x}, \mathbf{h}) \leq 0$ for every $\mathbf{h}$, and $Q(\mathbf{x}, \ ) < 0$ if $Q(\mathbf{x}, \mathbf{h}) < 0$ for every $\mathbf{h} \neq \mathbf{0}$. The corresponding terms are *negative semidefinite* and *negative definite*.

If $Q(\mathbf{x}, \ )$ has values of both signs, then it is *indefinite*.

In the discussion to follow and in the proof of Theorem 3.4 below we use the following notation. U denotes a neighborhood of a critical point $\mathbf{x}_0$, of radius sufficiently small that $U \subset A$. Points of U are denoted by $\mathbf{x}$. Let $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$. Points on the line segment joining $\mathbf{x}_0$ and $\mathbf{x}$ have the form $\mathbf{x}_0 + s\mathbf{h}, 0 \leq s \leq 1$, with $s = 0$ corresponding to $\mathbf{x}_0$ and $s = 1$ corresponding to $\mathbf{x}$. Since U is a convex set, we can apply Taylor's formula (3.13) with $q = 2$.

By (3.14), the remainder in Taylor's formula when $q = 2$ is $R_2(\mathbf{x}) = \frac{1}{2}Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h})$. By (3.13),

$$f(\mathbf{x}) = f(\mathbf{x}_0) + df(\mathbf{x}_0) \cdot \mathbf{h} + \tfrac{1}{2}Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}).$$

Since $\mathbf{x}_0$ is a critical point, $df(\mathbf{x}_0) = \mathbf{0}$. Taylor's formula becomes

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \tfrac{1}{2}Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}). \tag{3.19}$$

To say that f has a relative minimum at $\mathbf{x}_0$ is equivalent to the statement $Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}) \geq 0$ for all $\mathbf{x} \in U$, where U is some neighborhood of $\mathbf{x}_0$ contained in A. This condition on Q is difficult to apply directly. It is more convenient to use the following theorem, which depends on the sign of Q at the critical point $\mathbf{x}_0$.

Theorem 3.4. *Let f be of class $C^{(2)}$ on an open set A, and $\mathbf{x}_0 \in A$ a critical point. Then:*

(a) $Q(\mathbf{x}_0, \ ) \geq 0$ *is necessary for a relative minimum at* $\mathbf{x}_0$.
(a′) $Q(\mathbf{x}_0, \ ) > 0$ *is sufficient for a strict relative minimum at* $\mathbf{x}_0$.
(b) $Q(\mathbf{x}_0, \ ) \leq 0$ *is necessary for a relative maximum at* $\mathbf{x}_0$.
(b′) $Q(\mathbf{x}_0, \ ) < 0$ *is sufficient for a strict relative maximum at* $\mathbf{x}_0$.

PROOF. Let f have a relative minimum at $\mathbf{x}_0$. Then there exists a neighborhood U of $\mathbf{x}_0$ such that

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) \quad \text{for every } \mathbf{x} \in U \cap A.$$

Since A is open, we may assume that $U \subset A$. Since $df(\mathbf{x}_0) = \mathbf{0}$, Formula (3.19) implies that $Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}) \geq 0$ for all $\mathbf{x} \in U$, where $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$. Since f is of class $C^{(2)}$, the second-order partial derivatives f_{ij} are continuous. Hence $Q(\ , \mathbf{h}_0)$ is continuous on A, for fixed $\mathbf{h}_0$. Suppose that $Q(\mathbf{x}_0, \mathbf{h}_0) < 0$

for some $\mathbf{h}_0 \in E^n$. Then there exists a neighborhood U_1 of $\mathbf{x}_0$, $U_1 \subset U$, such that $Q(\mathbf{y}, \mathbf{h}_0) < 0$ for all $\mathbf{y} \in U_1$. Take $\mathbf{x} = \mathbf{x}_0 + c\mathbf{h}_0$ where $|c|$ is small enough that $\mathbf{x} \in U_1$, and $\mathbf{y} = \mathbf{x}_0 + s\mathbf{h}$. Then $\mathbf{h} = c\mathbf{h}_0$; and since $Q(\mathbf{x}_0 + s\mathbf{h}, \ )$ is quadratic,

$$Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}) = c^2 Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}_0) < 0.$$

But $Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}) \geq 0$, a contradiction. This proves (a).

To prove (a'), suppose that $Q(\mathbf{x}_0, \ ) > 0$. Using Problem 8 and the fact that the functions f_{ij} are continuous at $\mathbf{x}_0$, there exists a neighborhood U of $\mathbf{x}_0$ such that $U \subset A$ and $Q(\mathbf{y}, \ ) > 0$ for every $\mathbf{y} \in U$. Taking $\mathbf{y} = \mathbf{x}_0 + s\mathbf{h}$, we find from (3.19) that $f(\mathbf{x}) > f(\mathbf{x}_0)$ for every $\mathbf{x} \in U$, $\mathbf{x} \neq \mathbf{x}_0$. This proves that f has a strict relative minimum at $\mathbf{x}_0$. Statements (b), (b') follow respectively from (a), (a') by considering $-f$. □

Let $d_n(\mathbf{x}) = \det(f_{ij}(\mathbf{x}))$ denote the determinant of the matrix of second-order partial derivatives. This determinant is called the *Hessian.*

Definition. A critical point $\mathbf{x}$ is *nondegenerate* if the Hessian determinant $d_n(\mathbf{x})$ is not 0.

The behavior of f near a nondegenerate critical point $\mathbf{x}$ is determined by that of the quadratic function $Q(\mathbf{x}, \ )$. Let us state three criteria according to which one can test whether a nondegenerate critical point gives a relative extremum. We begin with a criterion that is special to two dimensions.

Criterion I

In this case, with $n = 2$,

$$Q(x, y, h, k) = f_{11}h^2 + 2f_{12}hk + f_{22}k^2$$

where we have written (h, k) for (h^1, h^2) and f_{ij} for $f_{ij}(x, y)$.

Since (x, y) is a nondegenerate critical point, $d_2(x, y) = f_{11}f_{22} - f_{12}^2 \neq 0$. Consider first the case when $f_{11}f_{22} - f_{12}^2 > 0$. Then the quadratic equation $Q(x, y, h, k) = 0$ has no roots (h, k) except $(0, 0)$. Therefore, $Q(x, y, h, k)$ has the same sign for all $(h, k) \neq (0, 0)$. We have $Q(x, y, h, 0) = f_{11}h^2$, $Q(x, y, 0, k) = f_{22}k^2$. The sign of f_{11} and f_{22} determines whether $Q(x, y, \ , \ ) > 0$ or $Q(x, y, \ , \ ) < 0$.

By Theorem 3.4 (a'), (b') we find that a critical point (x, y) is a point of relative minimum if $f_{11} > 0$, $f_{22} > 0$, $f_{11}f_{22} - f_{12}^2 > 0$, relative maximum if $f_{11} < 0$, $f_{22} < 0$, $f_{11}f_{22} - f_{12}^2 > 0$.

If $f_{11}f_{22} - f_{12}^2 < 0$, then $\{(h, k) : Q(x, y, h, k) = 0\}$ consists of two lines intersecting at $(0, 0)$. They divide the (h, k)-plane into four parts, on two of which $Q(x, y, h, k) > 0$ and on the other two of which $Q(x, y, h, k) < 0$. In this case, $Q(x, y, \ , \ )$ is indefinite. A critical point where $f_{11}f_{22} - f_{12}^2 < 0$ is called a *saddle point.* The function illustrated by Figure 3.5(b) has one point of absolute maximum, one of relative maximum, and one saddle point.

EXAMPLE. Let $f(x, y) = 2y^2 - x(x-1)^2$ for every $(x, y) \in E^2$ and $A = E^2$. This function has two critical points, $(\frac{1}{3}, 0)$ and $(1, 0)$. We find that

$$f_{11} = 4 - 6x, \qquad f_{22} = 4, \qquad f_{11}f_{22} - f_{12}^2 = 16 - 24x.$$

The point $(\frac{1}{3}, 0)$ gives a relative minimum and $(1, 0)$ is a saddle point. In this example it is instructive to find the *level sets* $\{(x, y) : f(x, y) = c\}$. They are indicated in Figure 3.6 for the critical values $-\frac{4}{27} = f(\frac{1}{3}, 0), 0 = f(1, 0)$ and for nearby values of c.

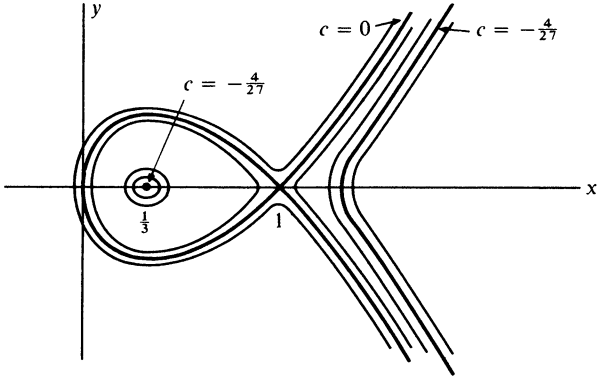


Figure 3.6

The point $(\frac{1}{3}, 0)$ of relative minimum is an isolated point of the level set containing it. For $-\frac{4}{27} < c < 0$ the level set has two parts. The one that encloses $(\frac{1}{3}, 0)$ resembles a small ellipse if c is near $-\frac{4}{27}$. This can be attributed to the fact that near $(\frac{1}{3}, 0)$, $f(x, y)$ is approximated by the first two nonzero terms in its Taylor expansion about $(\frac{1}{3}, 0)$, namely,

$$f(\tfrac{1}{3}, 0) + \tfrac{1}{2}Q(\tfrac{1}{3}, 0, x - \tfrac{1}{3}, y) = -\tfrac{4}{27} + (x - \tfrac{1}{3})^2 + 2y^2.$$

The level sets of this quadratic function are ellipses with center $(\frac{1}{3}, 0)$ if $c > -\frac{4}{27}$. Similarly, $f(x, y)$ is approximated by $-(x-1)^2 + 2y^2$ near the saddle point $(1, 0)$. The level sets $-(x-1)^2 + 2y^2 = c$ are hyperbolas if $c \neq 0$. Near $(1, 0)$ the level sets of f resemble these hyperbolas. For $c = 0$ we get the lines $\sqrt{2}y = \pm(x - 1)$ tangent to the level set of f at $(1, 0)$.

Let us now state two criteria for $Q(\mathbf{x}, \ ) > 0$ (or $Q(\mathbf{x}, \ ) < 0$) that hold for any dimension n. By Theorem 3.4 these imply that a critical point $\mathbf{x}$ gives a relative minimum (or relative maximum).

Criterion II

For any n let

$$d_1(\mathbf{x}) = f_{11}(\mathbf{x}), \qquad d_2(\mathbf{x}) = \det\begin{pmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) \end{pmatrix}, \ldots, d_n(\mathbf{x}) = \det(f_{ij}(\mathbf{x})).$$

These are called the *principal minor determinants* of the matrix $(f_{ij}(\mathbf{x}))$. The mth principal minor $d_m(\mathbf{x})$ is the determinant of the matrix obtained by deleting the last $n-m$ rows and columns. The determinant $d_n(\mathbf{x})$ is the Hessian of f at $\mathbf{x}$.

Let us state without proof the following criterion:

$$Q(\mathbf{x}, \ ) > 0 \quad \textit{iff} \quad d_m(\mathbf{x}) > 0 \qquad \textit{for } m = 1, \ldots, n.$$
$$Q(\mathbf{x}, \ ) < 0 \quad \textit{iff} \quad (-1)^m d_m(\mathbf{x}) > 0 \quad \textit{for } m = 1, \ldots, n.$$

For a proof of the first of these two statements, see the work by Bocher [3], especially pp. 140 and 147. The second follows from the first by considering $-Q$. Here *iff* is an abbreviation for "if and only if."

Criterion II is fairly convenient for small values of n, but becomes unwieldy for larger ones. This is because of the very large number of operations required to calculate the determinant of an $m \times m$ matrix even for moderately small m.

Criterion III

In linear algebra it is shown that any quadratic form can be written as a linear combination of squares by suitably choosing a new orthonormal basis for E^n. This fact is also proved in Section 4.8 below. Therefore

$$Q(\mathbf{x}, \mathbf{h}) = \sum_{i=1}^{n} \lambda_i(\mathbf{x})[\eta^i(\mathbf{x})]^2, \tag{3.20}$$

where for each $\mathbf{h} \in E^n$, $\eta^1(\mathbf{x}), \ldots, \eta^n(\mathbf{x})$ are the components of $\mathbf{h}$ with respect to some orthonormal basis $\{\mathbf{v}_1(\mathbf{x}), \ldots, \mathbf{v}_n(\mathbf{x})\}$ for E^n,

$$\mathbf{h} = \sum_{i=1}^{n} \eta^i(\mathbf{x})\mathbf{v}_i(\mathbf{x}).$$

The numbers $\lambda_1(\mathbf{x}), \ldots, \lambda_n(\mathbf{x})$ are just the characteristic values of the matrix $(f_{ij}(\mathbf{x}))$.

If $\lambda_i(\mathbf{x}) > 0$ for each $i = 1, \ldots, n$, then from (3.20) $Q(\mathbf{x}, \mathbf{h}) > 0$ unless $\eta^i(\mathbf{x}) = 0$ for each i (that is, unless $\mathbf{h} = \mathbf{0}$). In this case $Q(\mathbf{x}, \ )$ is positive definite. Conversely, if $\mathbf{h} = \mathbf{v}_i(\mathbf{x})$, then $Q(\mathbf{x}, \mathbf{h}) = \lambda_i(\mathbf{x})$. Therefore, if $Q(\mathbf{x}, \ ) > 0$, then in particular $Q(\mathbf{x}, \mathbf{v}_i(\mathbf{x})) > 0$, and $\lambda_i(\mathbf{x}) > 0$. This proves the first of the following statements:

$$Q(\mathbf{x}, \ ) > 0 \quad \textit{iff} \quad \lambda_i(\mathbf{x}) > 0 \quad \textit{for } i = 1, \ldots, n.$$
$$Q(\mathbf{x}, \ ) < 0 \quad \textit{iff} \quad \lambda_i(\mathbf{x}) < 0 \quad \textit{for } i = 1, \ldots, n.$$

The second is proved in the same way. Replacing on both sides ">0" by "≥ 0" we get a criterion for nonnegative semidefiniteness, and replacing "<0" by "≤ 0," one for nonpositive semidefiniteness.

If n is fairly large it is better, instead of Criterion II, to try some numerical method for putting $Q(\mathbf{x}, \ )$ in the form (3.20).

Boundary extrema

If f is continuous on a compact set A, then f has absolute extrema on A. They may occur either at interior or at boundary points of A. If an absolute maximum occurs at an interior point $\mathbf{x}_0$ of A, then $\mathbf{x}_0$ is among the relative maxima in int A. We can try to find it by Theorem 3.4. However, Theorem 3.4 does not apply at boundary points of A.

If $\mathbf{x}_0 \in \text{fr } A$ and $\mathbf{x}_0$ gives an absolute maximum, then $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for every $\mathbf{x} \in A$, and in particular for every $\mathbf{x} \in \text{fr } A$. Therefore $\mathbf{x}_0$ also gives an absolute maximum among points of fr A. If fr A is sufficiently smooth, the Lagrange multiplier rule (Section 4.8) can be applied.

In Section 3.6 we discuss extrema of linear functions, for which calculus is of no use.

Problems

In Problems 1 through 6 let $A = E^n$ for the indicated n.

1. Find the critical points, relative extrema, and saddle points. Make a sketch indicating the level sets.

(a) $f(x, y) = x - x^2 - y^2$.
(b) $f(x, y) = (x + 1)(y - 2)$.
(c) $f(x, y) = \sin(xy)$.
(d) $f(x, y) = xy(x - 1)$.

2. Find the critical points, relative extrema, and saddle points.

(a) $f(x, y) = x^3 + x - 4xy - 2y^2$.
(b) $f(x, y) = x(y + 1) - x^2y$.
(c) $f(x, y) = \cos x \cosh y$.

[*Note*: The hyperbolic functions sinh and cosh are defined by

$$\sinh x = \tfrac{1}{2}[\exp x - \exp(-x)],$$
$$\cosh x = \tfrac{1}{2}[\exp x + \exp(-x)].$$

Their derivatives are given by the formulas $\sinh' = \cosh$, $\cosh' = \sinh$.]

3. Let $f(x, y, z) = x^2 + y^2 - z^2$. Show that f has one critical point, which does not give a relative extremum. Describe the level sets.

4. Let $f(x, y, z) = x^2 + 3y^2 + 2z^2 - 2xy + 2xz$. Show that 0 is the minimum value of f.

5. Given $\mathbf{x}_1, \ldots, \mathbf{x}_m$, find the point $\mathbf{x}$ where $\sum_{j=1}^m |\mathbf{x} - \mathbf{x}_j|^2$ has an absolute minimum, and find the minimum value.

6. (a) In Problem 1(a) find the (absolute) maximum and minimum values of f on the circular disk $x^2 + y^2 \leq 1$.
(b) Do the same for 1(c).

7. (a) Let $f(\mathbf{x}) = \psi(\mathbf{a} \cdot \mathbf{x})$, where ψ is of class $C^{(2)}$ and $\mathbf{a} \neq \mathbf{0}$. Find all critical points, and show that every critical point is degenerate.
(b) Illustrate this result in case $f(x, y) = (x - y)^2$.

8. Let $g(\mathbf{h}) = \sum_{i,j=1}^{n} c_{ij}h^ih^j$. Assume that $g > 0$, that is, that $g(\mathbf{h}) > 0$ for every $\mathbf{h} \neq \mathbf{0}$.
   (a) Show that there exists a number $m > 0$ such that $g(\mathbf{h}) \geq m|\mathbf{h}|^2$ for every $\mathbf{h}$. [*Hint*: The polynomial g is continuous and has a positive minimum value m on the unit $(n-1)$-sphere $\{\mathbf{h} : |\mathbf{h}| = 1\}$.]
   (b) Suppose that $|C_{ij} - c_{ij}| < \varepsilon n^{-2}$ for each $i, j = 1, \ldots, n$. Let $G(\mathbf{h}) = \sum_{i,j=1}^{n} C_{ij}h^ih^j$. Show that $G(\mathbf{h}) \geq (m - \varepsilon)|\mathbf{h}|^2$ for every $\mathbf{h}$. Hence $G > 0$ if $\varepsilon < m$.

9. Let $\mathbf{x}_0$ be a nondegenerate critical point of a function f of class $C^{(2)}$. Show that $\mathbf{x}_0$ is isolated, that is, that $\mathbf{x}_0$ has a neighborhood U containing no other critical points of f. [*Hint*: Let $\mathbf{x}$ be another critical point in U. Apply the mean value theorem to each of the functions $f_1, \ldots, f_n$ to find that

$$(*) \qquad 0 = \sum_{j=1}^{n} f_{ij}(\mathbf{y}_i)(x^j - x_0^j), \qquad i = 1, \ldots, n,$$

   where each $\mathbf{y}_i \in U$. Show that if U is small enough, $\det(f_{ij}(\mathbf{y}_i)) \neq 0$ and consequently the system of equations (*) has only the solution $\mathbf{x} - \mathbf{x}_0 = \mathbf{0}$, a contradiction.]

*3.6 Convex and concave functions

Functions that are either concave or convex arise naturally in connection with the study of convex sets. They also occur in a wide variety of applications of calculus. We will see that the theory of maxima and minima is much simpler for them than for functions that are neither concave nor convex.

Let f be a real valued function and K a convex subset of the domain of f.

Definition. The function f is *convex on* K if, for every $\mathbf{x}_1, \mathbf{x}_2 \in K$ and $t \in [0, 1]$,

$$(3.21a) \qquad f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) \leq tf(\mathbf{x}_1) + (1-t)f(\mathbf{x}_2).$$

If strict inequality holds in (3.21a) whenever $\mathbf{x}_1 \neq \mathbf{x}_2$ and $0 < t < 1$, then f is *strictly convex on* K.

The assumption that K is a convex set is needed to ensure that the point $t\mathbf{x}_1 + (1-t)\mathbf{x}_2$ belongs to the domain of f. In order to see the geometric meaning of convexity, let us denote points of E^{n+1} by $(x^1, \ldots, x^n, z)$ or, for short, by $(\mathbf{x}, z)$. Let

$$K^+ = \{(\mathbf{x}, z) : \mathbf{x} \in K, z \geq f(\mathbf{x})\}.$$

If $\mathbf{x}_1 = \mathbf{x}_2$, then (3.21a) holds trivially. Therefore, suppose that $\mathbf{x}_1 \neq \mathbf{x}_2$. Let l denote the line segment in E^{n+1} joining $(\mathbf{x}_1, f(\mathbf{x}_1))$ and $(\mathbf{x}_2, f(\mathbf{x}_2))$. Points of l are of the form

$$(t\mathbf{x}_1 + (1-t)\mathbf{x}_2, tf(\mathbf{x}_1) + (1-t)f(\mathbf{x}_2)),$$

where $t \in [0, 1]$. Inequality (3.21a) says that such points belong to K^+. Therefore, the definition says geometrically that the line segment l is contained in K^+ for every pair of points $\mathbf{x}_1, \mathbf{x}_2 \in K$.

Proposition 3.4. *The function f is convex on K if and only if K^+ is a convex subset of E^{n+1}.*

PROOF. Let f be convex on K. Let $(\mathbf{x}_1, z_1), (\mathbf{x}_2, z_2) \in K^+$, $(\mathbf{x}_1, z_1) \neq (\mathbf{x}_2, z_2)$, and l' be the line segment joining them. Let $(\mathbf{x}, z)$ be any point of l'. Then

$$\mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2,$$
$$z = tz_1 + (1 - t)z_2,$$

where $0 \leq t \leq 1$. Since $z_1 \geq f(\mathbf{x}_1)$ and $z_2 \geq f(\mathbf{x}_2)$, we have (Figure 3.7)

$$z \geq tf(\mathbf{x}_1) + (1 - t)f(\mathbf{x}_2) \geq f(\mathbf{x}).$$

Hence $(\mathbf{x}, z) \in K^+$. This proves that K^+ is a convex set.

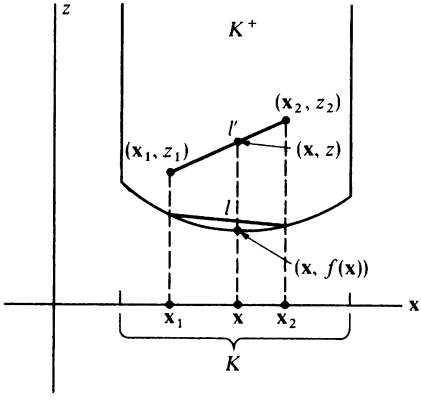


Figure 3.7

Suppose, conversely, that f is not convex on K. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in K$ and $t \in [0, 1]$ such that (3.21a) does not hold. The point $(t\mathbf{x}_1 + (1 - t)\mathbf{x}_2, tf(\mathbf{x}_1) + (1 - t)f(\mathbf{x}_2))$ belongs to the line segment l joining the points $(\mathbf{x}_1, f(\mathbf{x}_1))$ and $(\mathbf{x}_2, f(\mathbf{x}_2))$, but not to K^+. Since $(\mathbf{x}_1, f(\mathbf{x}_1)), (\mathbf{x}_2, f(\mathbf{x}_2)) \in K^+$, the set K^+ is not convex. □

For any real number c, let

$$K_c = \{\mathbf{x} \in K : f(\mathbf{x}) \leq c\}.$$

Proposition 3.5. *If f is convex on K, then K_c is a convex set for every c.*

PROOF. For every $\mathbf{x}_1, \mathbf{x}_2 \in K_c$ and $t \in [0, 1]$,

$$f(t\mathbf{x}_1 + (1 - t)\mathbf{x}_2) \leq tf(\mathbf{x}_1) + (1 - t)f(\mathbf{x}_2) \leq tc + (1 - t)c = c.$$

Hence $t\mathbf{x}_1 + (1 - t)\mathbf{x}_2 \in K_c$. □

The same proof shows that $\{\mathbf{x} : f(\mathbf{x}) < c\}$ is also convex. The converse to Proposition 3.5 is false; for example, let f be any increasing function with domain E^1. Then K_c is either all of E^1, a semiinfinite interval, or the empty set. In each case K_c is convex. However, f need not be a convex function; for instance, if $f(x) = x^3$, then f is not convex on E^1.

EXAMPLE 1. Let A be any nonempty closed subset of E^n. For every $\mathbf{x}$, let $f(\mathbf{x})$ be the distance from $\mathbf{x}$ to A, namely,

$$f(\mathbf{x}) = \min\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in A\}.$$

Let us show that the function f so defined is convex on E^n if and only if A is a convex set. If f is convex on E^n, then $K_0 = A$ and A is convex by Proposition 3.5 with $c = 0$. Conversely, assume that A is convex. Given $\mathbf{x}_1, \mathbf{x}_2 \in E^n$ and $t \in [0, 1]$, let $\mathbf{y}_1$ be a point of A nearest $\mathbf{x}_1$, $\mathbf{y}_2$ a point of A nearest $\mathbf{x}_2$, and

$$\mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2, \qquad \mathbf{y} = t\mathbf{y}_1 + (1 - t)\mathbf{y}_2.$$

(Actually, one can say "the nearest point" rather than "a nearest point" since the set A is convex. This fact is not needed here.) Since A is convex, $\mathbf{y} \in A$. By definition of f, $f(\mathbf{x}) \leq |\mathbf{x} - \mathbf{y}|$. Then

$$f(\mathbf{x}) \leq |t(\mathbf{x}_1 - \mathbf{y}_1) + (1 - t)(\mathbf{x}_2 - \mathbf{y}_2)|,$$
$$f(\mathbf{x}) \leq t|\mathbf{x}_1 - \mathbf{y}_1| + (1 - t)|\mathbf{x}_2 - \mathbf{y}_2| = tf(\mathbf{x}_1) + (1 - t)f(\mathbf{x}_2).$$

Hence f is a convex function on E^n.

In particular, let A consist of a single point $\mathbf{x}_0$. Then $f(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_0|$, and this function is convex on E^n

Concave functions

The definition of concave function is obtained by reversing the inequality sign in (3.21a): f is *concave on* K if, for every $\mathbf{x}_1, \mathbf{x}_2 \in K$ and $t \in [0, 1]$,

$$f(t\mathbf{x}_1 + (1 - t)\mathbf{x}_2) \geq tf(\mathbf{x}_1) + (1 - t)f(\mathbf{x}_2). \tag{3.21b}$$

If strict inequality holds whenever $\mathbf{x}_1 \neq \mathbf{x}_2$ and $0 < t < 1$, then f is *strictly concave on* K.

There are propositions about concave functions corresponding to Propositions 3.4 and 3.5 for convex functions. In them K^+ must be replaced by

$$K^- = \{(\mathbf{x}, z) : \mathbf{x} \in K, z \leq f(\mathbf{x})\},$$

and K_c by

$$K^c = \{\mathbf{x} \in K : f(\mathbf{x}) \geq c\}.$$

A function f is concave on K if and only if $-f$ is convex on K. By using this fact, or by repeating the proofs, it is easy to prove these propositions about concave functions.

Many useful inequalities can be obtained from (3.21a) or (3.21b) by judiciously choosing the function f and the number t.

Continuity of convex functions

Theorem 3.5. *Let K be an open convex set and f convex on K. Then f is continuous on K.*

PROOF. Let $\mathbf{x}_0$ be any point of K, and d the distance from $\mathbf{x}_0$ to the boundary of K ($d = +\infty$ if $K = E^n$). Let C be an n-cube with center $\mathbf{x}_0$ and side length 2δ, where $n^{1/2}\delta < d$. Let V denote the set of vertices of C (see Problem 10, Section 1.5). V is a finite set. Let

$$M = \max\{f(\mathbf{x}) : \mathbf{x} \in V\}. \tag{3.22}$$

By Proposition 3.5, K_M is a convex set. Since C is the convex hull of V and $V \subset K_M$, $C \subset K_M$.

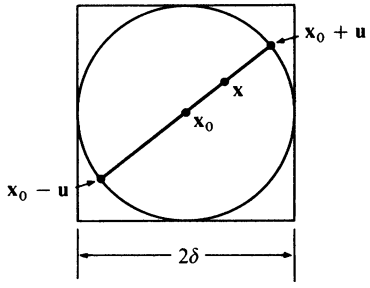


Figure 3.8

Let $\mathbf{x}$ be any point such that $0 < |\mathbf{x} - \mathbf{x}_0| < \delta$, and define $\mathbf{x}_0 + \mathbf{u}$, $\mathbf{x}_0 - \mathbf{u}$ on the line through $\mathbf{x}_0$ and $\mathbf{x}$ as in Figure 3.8. Let us write $\mathbf{x}$ as a convex combination of $\mathbf{x}_0 + \mathbf{u}$ and $\mathbf{x}_0$, and $\mathbf{x}_0$ as a convex combination of $\mathbf{x}$ and $\mathbf{x}_0 - \mathbf{u}$. If $t = \delta^{-1}|\mathbf{x} - \mathbf{x}_0|$, then

$$\mathbf{x} = t(\mathbf{x}_0 + \mathbf{u}) + (1 - t)\mathbf{x}_0,$$

$$\mathbf{x}_0 = \frac{1}{1+t}\mathbf{x} + \frac{t}{1+t}(\mathbf{x}_0 - \mathbf{u}).$$

Since f is convex,

$$f(\mathbf{x}) \leq tf(\mathbf{x}_0 + \mathbf{u}) + (1 - t)f(\mathbf{x}_0) \leq tM + (1 - t)f(\mathbf{x}_0),$$

$$f(\mathbf{x}_0) \leq \frac{1}{1+t}f(\mathbf{x}) + \frac{t}{1+t}f(\mathbf{x}_0 - \mathbf{u}) \leq \frac{f(\mathbf{x}) + tM}{1+t}.$$

The inequalities give

$$-t[M - f(\mathbf{x}_0)] \leq f(\mathbf{x}) - f(\mathbf{x}_0) \leq t[M - f(\mathbf{x}_0)],$$

or

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| \leq \frac{M - f(\mathbf{x}_0)}{\delta} |\mathbf{x} - \mathbf{x}_0|. \tag{3.23}$$

The estimate (3.23) shows that f is continuous at $\mathbf{x}_0$. [*Note*: This proof was suggested by F. J. Almgren.] □

If K is not open, then a convex function f may be discontinuous at boundary points of K. See the example below. The interior of K is an open convex set, and by the theorem f is continuous at every interior point.

EXAMPLE 2. Let $K = [0, 1]$ and $f(x) = x$ if $0 < x \leq 1$, $f(0) = 1$. Then f is convex on K but is discontinuous at the left endpoint 0.

Calculus tests for convexity or concavity of a function

If a function f is sufficiently smooth, then f can be tested for convexity or concavity by using calculus. First let us assume that f is differentiable. Later in the present section we make the stronger assumption that f is of class $C^{(2)}$ and obtain a test, in terms of the second-order partial derivatives (Theorem 3.6).

Figure 3.7 suggests that convexity of a differentiable function f is equivalent to the fact that f lies above its tangent hyperplane at each point $(\mathbf{x}_0, f(\mathbf{x}_0))$. The following proposition shows that this is indeed so.

Proposition 3.6a. *Let f be differentiable on a convex set K. Then f is convex on K if and only if*

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) + df(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \tag{3.24a}$$

for every $\mathbf{x}_0, \mathbf{x} \in K$.

PROOF. Let f be convex on K, and let $\mathbf{x}_0, \mathbf{x}$ be any two points of K. Let $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$ and $t \in (0, 1)$. By definition of convex function,

$$f(\mathbf{x}_0 + t\mathbf{h}) \leq tf(\mathbf{x}_0 + \mathbf{h}) + (1 - t)f(\mathbf{x}_0).$$

This inequality may be rewritten as

$$f(\mathbf{x}_0 + t\mathbf{h}) - f(\mathbf{x}_0) \leq t[f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)]. \tag{3.25}$$

Subtracting $tdf(\mathbf{x}_0) \cdot \mathbf{h}$ from both sides and dividing by t,

$$\frac{f(\mathbf{x}_0 + t\mathbf{h}) - f(\mathbf{x}_0) - tdf(\mathbf{x}_0) \cdot \mathbf{h}}{t} \leq f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - df(\mathbf{x}_0) \cdot \mathbf{h}.$$

The left-hand side tends to 0 as $t \to 0^+$. Hence the right-hand side is nonnegative, which says that (3.24a) holds.

Conversely, assume that (3.24a) holds for every $\mathbf{x}_0, \mathbf{x} \in K$. Let $\mathbf{x}_1, \mathbf{x}_2 \in K$, $\mathbf{x}_1 \neq \mathbf{x}_2$, and let $t \in (0, 1)$. Let

$$\mathbf{x}_0 = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2, \qquad \mathbf{h} = \mathbf{x}_1 - \mathbf{x}_0.$$

A little manipulation shows that

$$\mathbf{x}_2 = \mathbf{x}_0 - \frac{t}{1-t}\mathbf{h}.$$

By (3.24a) we have

$$f(\mathbf{x}_1) \geq f(\mathbf{x}_0) + df(\mathbf{x}_0) \cdot \mathbf{h},$$

$$f(\mathbf{x}_2) \geq f(\mathbf{x}_0) + df(\mathbf{x}_0) \cdot \left(-\frac{t}{1-t}\mathbf{h}\right).$$

Multiplying by $t/(1-t)$ in the first inequality and adding, we get

$$\frac{t}{1-t} f(\mathbf{x}_1) + f(\mathbf{x}_2) \geq \left(\frac{t}{1-t} + 1\right) f(\mathbf{x}_0),$$

or

$$tf(\mathbf{x}_1) + (1-t)f(\mathbf{x}_2) \geq f(\mathbf{x}_0).$$

But this is just the inequality (3.21a) in the definition of convex function. We assumed that $t \in (0, 1)$, but if $t = 0$ or 1, (3.21a) trivially holds. Therefore f is convex on K. □

By sharpening the inequality in (3.24a) we get a necessary and sufficient condition for strict convexity.

Proposition 3.6b. *Let f be differentiable on a convex set K. Then f is strictly convex on K if and only if*

(3.24b) $$f(\mathbf{x}) > f(\mathbf{x}_0) + df(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

for every $\mathbf{x}, \mathbf{x}_0 \in K$ with $\mathbf{x} \neq \mathbf{x}_0$.

PROOF. Let f be strictly convex on K. In particular, f is convex on K and (3.24a) holds for every $\mathbf{x}, \mathbf{x}_0 \in K$. Suppose that $\mathbf{x} \neq \mathbf{x}_0$, and let $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$. For every $t \in (0, 1)$

$$df(\mathbf{x}_0) \cdot (t\mathbf{h}) \leq f(\mathbf{x}_0 + t\mathbf{h}) - f(\mathbf{x}_0),$$

by (3.24a) applied with $\mathbf{x}$ replaced by $\mathbf{x}_0 + t\mathbf{h}$. But according to (3.25), which holds strictly since f is strictly convex,

$$f(\mathbf{x}_0 + t\mathbf{h}) - f(\mathbf{x}_0) < t[f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)].$$

Therefore

$$tdf(\mathbf{x}_0) \cdot \mathbf{h} < t[f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0)].$$

Upon dividing both sides by t we get (3.24b).

The proof of the converse is the same as for Proposition 3.6a, all inequalities now being strict. □

For concave functions the inequality signs must be reversed in (3.24a) and (3.24b). The first of these inequalities then says geometrically that f lies below its tangent hyperplane at $(\mathbf{x}_0, f(\mathbf{x}_0))$, and the second says that this is strictly true except at the point $(\mathbf{x}_0, f(\mathbf{x}_0))$ itself.

In the one-dimensional case there is the following simpler test for convexity, or strict convexity, of f. See Appendix A.4 for a definition of nondecreasing and increasing functions.

Proposition 3.7. ($n = 1$) *Let $K \subset E^1$ be an interval, and f a function that has a derivative $f'(x)$ for every $x \in K$. Then:*

(a) *f is convex on K if and only if f' is nondecreasing on K.*
(b) *f is strictly convex on K if and only if f' is increasing on K.*

PROOF. Let f be convex on K, where $K \subset E^1$ is an interval. Let $x, y \in K$, $y < x$. By (3.24a) applied with $\mathbf{x}_0 = y$,

$$f(x) - f(y) \geq f'(y)(x - y).$$

By (3.24a) applied with $\mathbf{x}_0 = x$,

$$f(y) - f(x) \geq f'(x)(y - x), \qquad f(x) - f(y) \leq f'(x)(x - y).$$

Therefore $f'(y)(x - y) \leq f'(x)(x - y)$, from which $f'(y) \leq f'(x)$. This proves that f' is nondecreasing on K. If f is strictly convex, then each of these inequalities is strict. In particular, $f'(y) < f'(x)$, which shows that f' is increasing on K.

Conversely, assume that f' is nondecreasing on K. Let $x_0, x \in K$, and suppose first that $x_0 < x$. By the mean value theorem there exists $y \in (x_0, x)$ such that

$$f(x) - f(x_0) = f'(y)(x - x_0).$$

Since f' is nondecreasing, $f'(x_0) \leq f'(y)$. Therefore

$$f(x) - f(x_0) \geq f'(x_0)(x - x_0),$$

which is equivalent to inequality (3.24a). Similarly, (3.24a) holds if $x < x_0$. By Proposition 3.6a, f is convex on K. If f' is increasing, then $f'(x_0) < f'(y)$, and the proof shows that (3.24b) holds. By Proposition 3.6b, f is strictly convex on K. □

EXAMPLE 3. Let $p > 1$, p not necessarily an integer. Let $f(x) = |x|^p$ for every $x \in E^1$. Then

$$f'(x) = \begin{cases} p|x|^{p-1} & \text{if } x > 0, \\ -p|x|^{p-1} & \text{if } x < 0, \end{cases}$$

and $f'(0) = 0$. The function f' is increasing. Hence f is strictly convex on E^1. Taking $t = \frac{1}{2}$, we have

$$f[\tfrac{1}{2}(x_1 + x_2)] \leq \tfrac{1}{2}f(x_1) + \tfrac{1}{2}f(x_2).$$

Multiplying both sides of the inequality by 2^p, we get

$$|x_1 + x_2|^p \leq 2^{p-1}(|x_1|^p + |x_2|^p).$$

The inequality is strict unless $x_1 = x_2$.

Let us next prove a theorem that provides a convenient test for concavity or convexity of a function of class $C^{(2)}$. We recall the function Q defined by (3.18).

Theorem 3.6. *Let f be of class $C^{(2)}$ on an open, convex set K. Then:*

(a) *f is convex on K if and only if $Q(\mathbf{x}, \ ) \geq 0$ for every $\mathbf{x} \in K$.*
(a′) *If $Q(\mathbf{x}, \ ) > 0$ for every $\mathbf{x} \in K$, then f is strictly convex on K.*
(b) *f is concave on K if and only if $Q(\mathbf{x}, \ ) \leq 0$ for every $\mathbf{x} \in K$.*
(b′) *If $Q(\mathbf{x}, \ ) < 0$ for every $\mathbf{x} \in K$, then f is strictly concave on K.*

PROOF. Since K is convex we may use Taylor's formula with $q = 2$ and any pair of points $\mathbf{x}_0, \mathbf{x} \in K$:

$$f(\mathbf{x}) = f(\mathbf{x}_0) + df(\mathbf{x}_0) \cdot \mathbf{h} + \tfrac{1}{2}Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}), \tag{3.26}$$

where $s \in (0, 1)$ and $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$. Let us first prove (a′). By hypothesis, $Q(\mathbf{y}, \ ) > 0$ for every $\mathbf{y} \in K$, and in particular for $\mathbf{y} = \mathbf{x}_0 + s\mathbf{h}$. Therefore $Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}) > 0$ if $\mathbf{h} \neq \mathbf{0}$, from which

$$f(\mathbf{x}) > f(\mathbf{x}_0) + df(\mathbf{x}_0) \cdot \mathbf{h}.$$

By Proposition 3.6b, f is strictly convex on K.

Let us next prove (a). If $Q(\mathbf{x}, \ ) \geq 0$ for every $\mathbf{x} \in K$, then the same reasoning shows that

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) + df(\mathbf{x}_0) \cdot \mathbf{h}.$$

By Proposition 3.6a, f is convex on K. On the other hand, if it is not true that $Q(\mathbf{x},) \geq 0$ for every $\mathbf{x} \in K$, then $Q(\mathbf{x}_0, \mathbf{h}_0) < 0$ for some $\mathbf{x}_0 \in K$ and $\mathbf{h}_0 \neq \mathbf{0}$. Since f is of class $C^{(2)}$, $Q(\ , \mathbf{h}_0)$ is continuous on K. Hence there exists $\delta > 0$ such that $Q(\mathbf{y}, \mathbf{h}_0) < 0$ for every $\mathbf{y}$ in the δ-neighborhood of $\mathbf{x}_0$. Let $\mathbf{h} = c\mathbf{h}_0$, where $c > 0$ is small enough that $|\mathbf{h}| < \delta$, and let $\mathbf{x} = \mathbf{x}_0 + \mathbf{h}$. Since $Q(\mathbf{x}_0 + s\mathbf{h}, \ )$ is quadratic,

$$Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}) = c^2 Q(\mathbf{x}_0 + s\mathbf{h}, \mathbf{h}_0) < 0.$$

From (3.26)

$$f(\mathbf{x}) < f(\mathbf{x}_0) + df(\mathbf{x}_0) \cdot \mathbf{h}.$$

By Proposition 3.6a, f is not convex on K.

This proves (a) and (a′). Parts (b) and (b′) follow respectively from (a) and (a′) by considering $-f$. □

If $n = 1$, then $Q(x, h) = f''(x)h^2$. The sign of $Q(x, h)$ is determined by that of $f''(x)$. In that case condition (a) becomes $f''(x) \geq 0$, and (a′) becomes $f''(x) > 0$. The signs are reversed for (b) and (b′).

EXAMPLE 4. Let f be a homogeneous quadratic polynomial,

$$f(\mathbf{x}) = \sum_{i,j=1}^{n} c_{ij}x^i x^j,$$

for each $\mathbf{x} \in E^n$, where the $n \times n$ matrix (c_{ij}) is symmetric. Then $f_{ij}(\mathbf{x}) = 2c_{ij}$ and $Q(\mathbf{x}, \mathbf{h}) = 2f(\mathbf{h})$. Hence f is convex on E^n in case $f(\mathbf{x}) \geq 0$ for every $\mathbf{x}$, and concave in case $f(\mathbf{x}) \leq 0$ for every $\mathbf{x}$. If f has values of both signs, then f is neither convex nor concave.

EXAMPLE 5. Let $f(\mathbf{x}) = \phi[g(\mathbf{x})]$, where: (a) g is convex and of class $C^{(2)}$ on K; (b) ϕ is of class $C^{(2)}$ on an interval I such that $g(K) \subset I$, with $\phi'(u) \geq 0$, $\phi''(u) \geq 0$ for all $u \in I$. By the formula in one-variable calculus for derivatives of composites,

$$f_i(\mathbf{x}) = \phi'[g(\mathbf{x})]g_i(\mathbf{x}).$$

By using this theorem again and the product rule,

$$f_{ij}(\mathbf{x}) = \phi'[g(\mathbf{x})]g_{ij}(\mathbf{x}) + \phi''[g(\mathbf{x})]g_i(\mathbf{x})g_j(\mathbf{x}).$$

Writing for short g_i for $g_i(\mathbf{x})$, and so on,

$$Q(\mathbf{x}, \mathbf{h}) = \phi'[g(\mathbf{x})] \sum_{i,j=1}^{n} g_{ij}h^i h^j + \phi''[g(\mathbf{x})] \sum_{i,j=1}^{n} g_i g_j h^i h^j.$$

The first term on the right-hand side is nonnegative since $\phi'[g(\mathbf{x})] \geq 0$ and g is convex on K. The second term is also nonnegative since $\phi''[g(\mathbf{x})] \geq 0$ and

$$\sum_{i,j=1}^{n} g_i g_j h^i h^j = \left[\sum_{i=1}^{n} g_i h^i\right]^2 \geq 0.$$

Thus $Q(\mathbf{x}, \mathbf{h}) \geq 0$ for all $\mathbf{x} \in K$, $\mathbf{h} \in E^n$. By Theorem 3.6a, f is convex on K.

In Examples 4 and 5, the sign of $Q(\mathbf{x}, \ )$ could be determined by direct calculations. When this is not feasible, one of the tests I, II, or III in Section 3.5 may be applied.

EXAMPLE 6. Let $f(x, y) = \frac{1}{6}(x^3 + y^3) + xy$. Then $f_{11} = x$, $f_{22} = y$, $f_{11}f_{22} - f_{12}^2 = xy - 1$. Hence f is strictly convex on the part of the first quadrant above the hyperbola $xy = 1$, and strictly concave on the part of the third quadrant below this hyperbola.

For convex functions the necessary condition $df(\mathbf{x}_0) = \mathbf{0}$ for a minimum is also sufficient.

Theorem 3.7. *Let f be differentiable and convex on an open convex set K and $\mathbf{x}_0 \in K$ a critical point. Then f has an absolute minimum at $\mathbf{x}_0$.*

PROOF. Since $df(\mathbf{x}_0) = \mathbf{0}$, $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for every $\mathbf{x} \in K$ by Proposition 3.6a. □

Similarly, any differentiable concave function has an absolute maximum at any critical point.

Corollary 1. *A differentiable function which is strictly convex (or strictly concave) has at most one critical point.*

PROOF. Let f be strictly convex on K, and suppose that f has an absolute minimum at distinct points $\mathbf{x}_0, \mathbf{x}_1 \in K$,

$$f(\mathbf{x}_0) = f(\mathbf{x}_1) \leq f(\mathbf{x})$$

for every $\mathbf{x} \in K$. Since $df(\mathbf{x}_0) = 0$, by Proposition 3.6b (with $\mathbf{x} = \mathbf{x}_1$), $f(\mathbf{x}_1) > f(\mathbf{x}_0)$. This is a contradiction. □

Extrema of linear functions

Let f be a linear function. Then calculus is of no help in finding the extrema of f. Since $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} = a_1 x^1 + \cdots + a_n x^n$, the partial derivatives are $f_i(\mathbf{x}) = a_i$. If f has a critical point, then $\mathbf{a} = \mathbf{0}$ and $f(\mathbf{x}) = 0$ for every $\mathbf{x}$.

Let us assume that $\mathbf{a} \neq \mathbf{0}$ and consider the problem of extremum on a convex polytope K (Section 1.5). If K is contained in $\{\mathbf{x} : x^i \geq 0, i = 1, \ldots, n\}$, this is a problem in linear programming and has various interesting applications [9, 13].

For simplicity let us assume that K is compact. The extrema of f must occur on the boundary fr K. Let us show that they can be found by considering only certain points of fr K, called extreme points.

Definition. Let K be a convex set. A point $\mathbf{x} \in K$ is an *extreme point* of K if there do not exist distinct points $\mathbf{x}_1, \mathbf{x}_2 \in K$ and $t \in (0, 1)$ such that $\mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2$.

Stated geometrically, $\mathbf{x}$ is extreme if it is interior to no line segment in K. For example, the extreme points of a simplex are the vertices. If K is a closed n-ball, then every point of fr K is extreme. A half-space has no extreme points.

Proposition 3.8. *Let K be compact and convex. Then every point of K is a convex combination of extreme points of K.*

PROOF. Let us proceed by induction on the dimension n. If $n = 1$, then K is an interval or a single point. Suppose that the proposition is true in dimension $n - 1$. Let $\mathbf{x}_0 \in K$. If $\mathbf{x}_0$ is a boundary point, then by Problem 7, Section 2.4, K has a supporting hyperplane P containing $\mathbf{x}_0$. By an isometry of E^n (see Section 4.2), we may arrange that $\mathbf{x}_0 = \mathbf{0}$ and the equation of P is

$x^n = 0$. The set $K \cap P$ is compact and convex. By the induction hypothesis $\mathbf{x}_0$ is a convex combination of extreme points of $K \cap P$ and hence (Problem 14) of extreme points of K.

If $\mathbf{x}_0 \in \text{int } K$, then any line through $\mathbf{x}_0$ intersects K in a segment with endpoints $\mathbf{x}_1, \mathbf{x}_2 \in \text{fr } K$. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are convex combinations of the set of extreme points, so is $\mathbf{x}_0$ (Problem 8, Section 1.5). □

By Proposition 1.7, taking as S the set of extreme points of K, each point of K is a convex combination of $n + 1$ or fewer extreme points. If S is connected, then $n + 1$ may be replaced by n.

Let C be the maximum value on K of the linear function f and $K_1 = \{\mathbf{x} \in K : f(\mathbf{x}) = C\}$. If K_1 is found, the problem of maximum is solved.

Corollary 2. *K_1 is the convex set spanned by those extreme points of K at which f has an absolute maximum.*

PROOF. Let $\mathbf{x} \in K_1$. By the proposition $\mathbf{x} = \sum t^j \mathbf{x}_j$, where $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are extreme points, each $t^j > 0$, and $\sum t^j = 1$. All sums are from 1 to m. Since C is the maximum value, $f(\mathbf{x}_j) \leq C$. Since f is linear,

$$f(\mathbf{x}) = \sum t^j f(\mathbf{x}_j) \leq \sum t^j C = C.$$

But $f(\mathbf{x}) = C$, and since each $t^j > 0$ we must have $f(\mathbf{x}_j) = C$ for $j = 1, \ldots, m$. Thus $\mathbf{x}_1, \ldots, \mathbf{x}_m \in K_1$. Conversely, if $f(\mathbf{x}_j) = C$ for each j and $\mathbf{x}$ is a convex combination of $\mathbf{x}_1, \ldots, \mathbf{x}_m$, then $f(\mathbf{x}) = C$. □

If K is a convex polytope, then by induction on n the set of extreme points is finite. The problem is no longer one of calculus, but that of maximizing f on this finite set. Except in the simplest situations, the method of unsystematic search among the extreme points is of little value. The best-known systematic method is called the simplex method of linear programming. In a sense it is an adaptation of the gradient method.

PROBLEMS

1. Which fourth-degree polynomials are convex functions on E^1?

2. Use Theorem 3.6 to determine whether f is convex on K, concave on K, or neither. Unless otherwise indicated, $K = E^2$ or E^3.
   (a) $f(x, y, z) = x^2 + y^2 - 4z^2$.
   (b) $f(x, y, z) = x - y^2 - z^2$.
   (c) $f(x, y) = (x + y + 1)^p$, $K = \{(x, y) : x + y + 1 > 0\}$.
   (d) $f(x, y, z) = \exp(x^2 + xy + y^2 + z^2)$.
   (e) $f(x, y) = \exp(xy)$.
   In which cases is the convexity or concavity strict?

3. Let $f(x, y) = \phi(x^2 + y^2)$, where ϕ is of class $C^{(2)}$, increasing and concave. Show that f is convex on the circular disk $x^2 + y^2 \leq a^2$ if and only if $\phi'(u) + 2u\phi''(u) \geq 0$ whenever $0 \leq u \leq a^2$.

4. Using Problem 3, find the largest a such that f is convex on $x^2 + y^2 \leq a^2$.
(a) $f(x, y) = \log(1 + x^2 + y^2)$. (b) $f(x, y) = \sin(x^2 + y^2)$.

5. Using Example 5, show that each of the following functions is convex on E^n:
(a) $f(\mathbf{x}) = |\mathbf{x}|^p$, $p \geq 1$. (b) $f(\mathbf{x}) = (1 + \mathbf{x} \cdot \mathbf{x})^{\mathbf{x} \cdot \mathbf{x}}$.
(c) $f(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{p/2}$, $p \geq 1$. [*Hint*: First consider $p = 1$.]

6. Let $f(x) = x\phi(x)$ and $g(x) = \phi(1/x)$, where ϕ has a second derivative $\phi''(x)$ for every $x > 0$. Show that f is convex on $(0, \infty)$ if and only if g is convex on $(0, \infty)$.

7. Let $x > 0$, $y > 0$, $0 \leq t \leq 1$. Show that $tx + (1 - t)y \geq x^t y^{1-t}$. [*Hint*: Log is an increasing, concave function.]

8. Prove by induction on m that if f is concave on K, then

$$f\left(\sum_{j=1}^{m} t^j \mathbf{x}_j\right) \geq \sum_{j=1}^{m} t^j f(\mathbf{x}_j)$$

for every $\mathbf{x}_1, \ldots, \mathbf{x}_m \in K$ and scalars $t^1, \ldots, t^m$ such that each $t^j \geq 0$ and $t^1 + \cdots + t^m = 1$. [*Note*: For convex functions the sense of the inequality is reversed.]

9. (a) Generalizing Problem 7, show that if $x_1, \ldots, x_m$ are positive numbers, $0 \leq t^j$ for $j = 1, \ldots, m$, and $t^1 + \cdots + t^m = 1$, then

$$t^1 x_1 + \cdots + t^m x_m \geq x_1^{t^1} \cdots x_m^{t^m}.$$

(b) Prove that the geometric mean is no more than the arithmetic mean, namely.

$$\frac{x_1 + \cdots + x_m}{m} \geq (x_1 \cdots x_m)^{1/m}.$$

10. Show that if f and g are convex on K, then $f + g$ is convex on K.

11. (a) Let f and g be convex on K, and let $h(\mathbf{x}) = \max[f(\mathbf{x}), g(\mathbf{x})]$ for every $x \in K$. Show that h is convex on K. [*Hint*: Use Proposition 3.4.]
(b) Illustrate for the case $f(x) = |x - 1|$, $g(x) = x/2$.

12. Let f be both convex and concave on E^n. Show that there exist $\mathbf{a}$ and b such that $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b$ for every $\mathbf{x} \in E^n$.

13. Let f be continuous on K, and assume that $f(\frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)) \leq \frac{1}{2}f(\mathbf{x}_1) + \frac{1}{2}f(\mathbf{x}_2)$ for every $\mathbf{x}_1, \mathbf{x}_2 \in K$. Show that f is convex on K. [*Hint*: First show (3.21a) when $t = j/2^k$ where $j = 0, 1, \ldots, 2^k$ and k is a positive integer.]

14. Let K be closed and convex, and P a supporting hyperplane for K. Show that any extreme point of $K \cap P$ is an extreme point of K.

15. Let K be a closed convex polytope (not necessarily compact) and f be a linear function such that $f(\mathbf{x})$ is bounded above on K. Show that f has an absolute maximum on K.

Vector-valued functions of several variables

4

In this chapter we study the differential calculus of functions of several variables with values in E^n. Among the main results are the theorems about composition and inverses and the implicit function theorem. Later in the chapter, subsets of E^n that are smooth manifolds are considered, and the spaces of tangent and normal vectors at a point of a smooth manifold are found. These ideas are then applied to obtain the Lagrange multiplier rule for constrained extremum problems.

Functions with values in E^n are referred to as *transformations* rather than vector-valued functions. This term has a useful geometric connotation, and it also agrees with rather common usage. Some authors use instead the term "mapping." The differential calculus of transformations is based on local linear approximations, just as for the special case ($n = 1$) of real valued functions considered in Chapter 3. Consequently, it is first necessary to review some results about linear transformations. This is done in Section 4.1.

4.1 Linear transformations

In this section we collect some facts about linear transformations from one euclidean vector space into another. For results stated without proof, references are given to the work by Hoffman and Kunze [12]. However, the results in question are standard in linear algebra and may be found in practically any good book on the subject.

We use the following notation. A linear transformation is generally denoted by $\mathbf{L}$. Its domain is a euclidean space E^r, of dimension r, whose points are denoted by $\mathbf{s}, \mathbf{t}, \ldots$. The transformation $\mathbf{L}$ has values in another linear space E^n, of dimension n, whose points are denoted by $\mathbf{x}, \mathbf{y}, \ldots$.

Definition. A transformation $\mathbf{L}$ from E^r into E^n is *linear* if:

(1) $\mathbf{L}(\mathbf{s} + \mathbf{t}) = \mathbf{L}(\mathbf{s}) + \mathbf{L}(\mathbf{t})$ for every $\mathbf{s}, \mathbf{t} \in E^r$; and
(2) $\mathbf{L}(c\mathbf{s}) = c\mathbf{L}(\mathbf{s})$ for every $\mathbf{s} \in E^r$ and real c.

This is a special case of the definition in Section A.1 of the Appendix.

We also recall from Section A.1 the concept of vector subspace. To verify that a set $P \subset E^n$ is a vector subspace of E^n, one must show that $\mathbf{x}, \mathbf{y} \in P$ implies $\mathbf{x} + \mathbf{y} \in P$ and that $\mathbf{x} \in P$ implies $c\mathbf{x} \in P$ for any real c. A vector subspace P has a dimension ρ, and $0 \leq \rho \leq n$. If $\rho = 0$ then $P = \{\mathbf{0}\}$. If $\rho = 1$, then P is a line containing $\mathbf{0}$; if $\rho = 2$, P is a plane containing $\mathbf{0}$, and so on.

If $\mathbf{L}$ is a linear transformation, then the set $\mathbf{L}(E^r)$ is a vector subspace of E^n. To prove this, let $\mathbf{x}, \mathbf{y} \in \mathbf{L}(E^r)$. Then $\mathbf{x} = \mathbf{L}(\mathbf{s})$, $\mathbf{y} = \mathbf{L}(\mathbf{t})$ for some $\mathbf{s}, \mathbf{t} \in E^r$. But $\mathbf{x} + \mathbf{y} = \mathbf{L}(\mathbf{s}) + \mathbf{L}(\mathbf{t}) = \mathbf{L}(\mathbf{s} + \mathbf{t})$. Hence $\mathbf{x} + \mathbf{y} \in \mathbf{L}(E^r)$. Similarly, if $\mathbf{x} \in \mathbf{L}(E^r)$ then $c\mathbf{x} \in \mathbf{L}(E^r)$ for every scalar c. The dimension ρ of the vector space $\mathbf{L}(E^r)$ is called the *rank* of $\mathbf{L}$.

The *kernel* of $\mathbf{L}$ is $\{\mathbf{t} : \mathbf{L}(\mathbf{t}) = \mathbf{0}\}$. It is a vector subspace of E^r (Problem 2). The dimension ν of the kernel is called the *nullity* of $\mathbf{L}$. The rank and nullity are related by [12, p. 66]

$$\rho + \nu = r. \tag{4.1}$$

The matrix of L

Let us denote the standard basis vectors for E^r by $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_r$ and those for E^n by $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Thus vectors $\mathbf{t} = (t^1, \ldots, t^r)$ and $\mathbf{x} = (x^1, \ldots, x^n)$ can be written, respectively, as

$$\mathbf{t} = \sum_{j=1}^{r} t^j \boldsymbol{\varepsilon}_j, \qquad \mathbf{x} = \sum_{i=1}^{n} x^i \mathbf{e}_i.$$

With these bases is associated a matrix (c_j^i) for $\mathbf{L}$ with n rows and r columns, in the following way. Let $\mathbf{v}_j = \mathbf{L}(\boldsymbol{\varepsilon}_j)$, $j = 1, \ldots, r$. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are the columns of the matrix. Let c_j^i be the element in the ith row of column j.

Let us show that, if $\mathbf{x} = \mathbf{L}(\mathbf{t})$, then the components satisfy

$$x^i = \sum_{j=1}^{r} c_j^i t^j, \; i = 1, \ldots, n. \tag{4.2}$$

To see this, by definition of c_j^i,

$$\mathbf{v}_j = \sum_{i=1}^{n} c_j^i \mathbf{e}_i, j = 1, \ldots, r. \tag{4.3}$$

Since $\mathbf{L}$ is linear,

$$\mathbf{L}(\mathbf{t}) = \mathbf{L}\left(\sum_{j=1}^{r} t^j \boldsymbol{\varepsilon}_j \right) = \sum_{j=1}^{r} t^j \mathbf{L}(\boldsymbol{\varepsilon}_j),$$

$$\mathbf{L}(\mathbf{t}) = \sum_{j=1}^{r} t^j \mathbf{v}_j. \tag{4.4}$$

By (4.3) and (4.4), the components x^i satisfy (4.2).

Equation (4.2) is just a formula for matrix–vector multiplication. If one regards $\mathbf{t}$, $\mathbf{x}$ as column vectors, then (4.2) states that $\mathbf{x}$ is obtained by multiplying $\mathbf{t}$ on the left by the matrix (c_j^i).

Actually, for any pair of bases for E^r and E^n there is a matrix associated with $\mathbf{L}$. It is shown in linear algebra that by suitable choice of bases the associated matrix can be made to have some special form, for instance, the Jordan canonical form if $r = n$ [12, p. 207]. What we have called "the matrix of $\mathbf{L}$ is the matrix corresponding to the standard bases for E^r and E^n.

The rows of the matrix (c_j^i) also have an interesting interpretation. The components $L^1, \ldots, L^n$ of the linear transformation $\mathbf{L}$ are real valued linear functions. The statement $\mathbf{x} = \mathbf{L}(\mathbf{t})$ is equivalent to $x^i = L^i(\mathbf{t})$, $i = 1, \ldots, n$. In Section 3.2 we identified each real valued linear function with a covector. Let $\mathbf{w}^i$ be the covector with which the linear function L^i is identified. Then

$$x^i = L^i(\mathbf{t}) = \mathbf{w}^i \cdot \mathbf{t}, \qquad i = 1, \ldots, n.$$

By (4.2), the components of $\mathbf{w}^i$ are the entries $c_1^i, \ldots, c_r^i$ of the ith row of the matrix (c_j^i). For that reason $\mathbf{w}^1, \ldots, \mathbf{w}^n$ are called the *row covectors* of the matrix (c_j^i).

$$\begin{array}{c} \\ \mathbf{w}^1 \\ \mathbf{w}^2 \\ \mathbf{w}^3 \\ \vdots \\ \mathbf{w}^n \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \ldots & \mathbf{v}_r \end{array} \\ \begin{pmatrix} c_1^1 & c_2^1 & c_3^1 & \ldots & c_r^1 \\ c_1^2 & c_2^2 & \ldots & \ldots & c_r^2 \\ c_1^3 & \ldots & \ldots & \ldots & c_r^3 \\ \vdots & & & & \vdots \\ c_1^n & \ldots & \ldots & \ldots & c_r^n \end{pmatrix} \end{array}$$

By (4.4) the column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ span $\mathbf{L}(E^r)$. The rank ρ equals the largest number of linearly independent column vectors of the matrix. Since row rank equals column rank [12, p. 105], ρ is also the largest number of linearly independent row covectors of the matrix.

Composition

Let $\mathbf{L}$ be linear from E^r into E^n, and $\mathbf{M}$ linear from E^n into E^p. The composite $\mathbf{M} \circ \mathbf{L}$ is linear. Its matrix is the product of the matrices of $\mathbf{M}$ and $\mathbf{L}$ (Problem 4).

The case $r = n$

Let $\mathbf{I}$ denote the identity linear transformation, $\mathbf{I}(\mathbf{t}) = \mathbf{t}$ for every $\mathbf{t} \in E^r$. Its matrix is (δ_j^i), which has 1 for each element of the principal diagonal and 0 elsewhere. $\mathbf{L}$ is *nonsingular* if it has rank $\rho = n$, and *singular* if $\rho < n$. A nonsingular linear transformation $\mathbf{L}$ has an inverse $\mathbf{L}^{-1}$, which is also a linear transformation with

$$\mathbf{L}^{-1} \circ \mathbf{L} = \mathbf{L} \circ \mathbf{L}^{-1} = \mathbf{I}.$$

If $r = n$, then the $n \times n$ matrix (c_j^i) has a determinant, denoted by $\det(c_j^i)$. This number is also called the determinant of $\mathbf{L}$. Thus by definition

$$\det \mathbf{L} = \det(c_j^i).$$

Among the properties of determinants we recall:

$$\det(\mathbf{M} \circ \mathbf{L}) = \det \mathbf{M} \det \mathbf{L} \text{ [12, p. 143]},$$

$$\det \mathbf{L} = 0 \quad \text{if and only if} \quad \mathbf{L} \text{ is singular [12, p. 150]}.$$

By (4.1), $\mathbf{L}$ is singular if and only if $v > 0$. But $v > 0$ means that $\mathbf{L}(\mathbf{t}) = \mathbf{0}$ for some $\mathbf{t} \neq \mathbf{0}$. Therefore, from (4.2) the system of homogeneous linear equations

$$0 = \sum_{j=1}^{n} c_j^i t^j, \qquad i = 1, \ldots, n \tag{4.5}$$

has a nontrivial solution if and only if $\det \mathbf{L} = 0$.

In later chapters we see that the absolute value of the determinant is the ratio of n-dimensional volumes, and the sign of the determinant determines an orientation.

EXAMPLE 1. Let $n = r = p = 2$. Let $\mathbf{L}(s, t) = (2s + t)\mathbf{e}_1 + (3s - t)\mathbf{e}_2$. The matrix of $\mathbf{L}$ is

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}.$$

The linear transformation $\mathbf{L}$ takes the standard basis vectors $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2$ into $\mathbf{v}_1 = 2\mathbf{e}_1 + 3\mathbf{e}_2$, $\mathbf{v}_2 = \mathbf{e}_1 - \mathbf{e}_2$. Since $\det \mathbf{L} = -5 \neq 0$, $\mathbf{L}$ is nonsingular (see Figure 4.1).

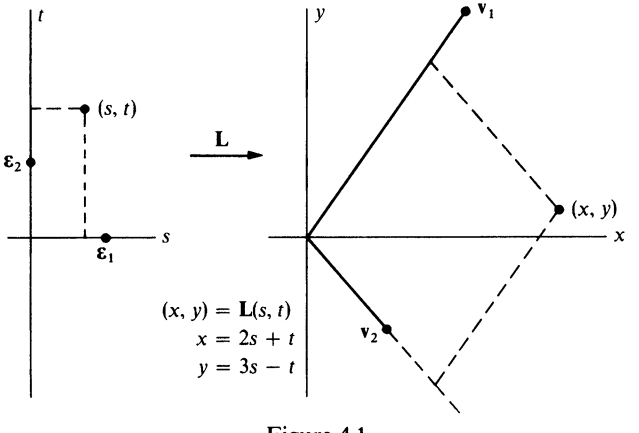


Figure 4.1

Let $\mathbf{M}(x, y) = (2x - 5y)\mathbf{E}_1 - x\mathbf{E}_2$, where $\mathbf{E}_1, \mathbf{E}_2$ denote the standard basis vectors for the plane E^2 in which $\mathbf{M}$ has its values. The matrix of $\mathbf{M}$ is

$$\begin{pmatrix} 2 & -5 \\ -1 & 0 \end{pmatrix}.$$

Since $\det \mathbf{M} = -5 \neq 0$, $\mathbf{M}$ is also nonsingular. The composite is found by

$$(\mathbf{M} \circ \mathbf{L})(s, t) = \mathbf{M}(2s + t, 3s - t) = [2(2s + t) - 5(3s - t)]\mathbf{E}_1 - (2s + t)\mathbf{E}_2,$$
$$(\mathbf{M} \circ \mathbf{L})(s, t) = (-11s + 7t)\mathbf{E}_1 - (2s + t)\mathbf{E}_2.$$

The matrix of $\mathbf{M} \circ \mathbf{L}$ is

$$\begin{pmatrix} -11 & 7 \\ -2 & -1 \end{pmatrix}.$$

As expected, $\det \mathbf{M} \circ \mathbf{L} = 25 = \det \mathbf{M} \det \mathbf{L}$.

EXAMPLE 2. Let $n = r = 3$, and let $\mathbf{L}$ be the linear transformation that takes the standard basis vectors $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3$ respectively into

$$\mathbf{v}_1 = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3, \qquad \mathbf{v}_2 = -\mathbf{e}_1 + \mathbf{e}_2, \qquad \mathbf{v}_3 = -\mathbf{e}_1 + 4\mathbf{e}_2 - \mathbf{e}_3.$$

The matrix is

$$\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 4 \\ -1 & 0 & -1 \end{pmatrix},$$

which has $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as column vectors. The determinant is 0, and therefore $\mathbf{L}$ is singular. In fact, $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, namely, $\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, the rank of $\mathbf{L}$ is 2. $\mathbf{L}(E^3)$ is the plane containing $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2$. By (4.1) the kernel has dimension 1. It is found by solving the system (4.5) of homogeneous linear equations. One solution is $\mathbf{t}_1 = \boldsymbol{\varepsilon}_1 + 2\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_3$. The kernel consists of all scalar multiples of $\mathbf{t}_1$.

The dual L of a linear transformation*

This is the linear transformation from the dual space $(E^n)^*$ into $(E^r)^*$, defined as follows. Let (c_j^i) be the matrix of the linear transformation $\mathbf{L}$, and $\mathbf{w}^1, \ldots, \mathbf{w}^r$ the row covectors of the matrix (c_j^i), as above. Note that $\mathbf{w}^i$ corresponds to the real valued linear function L^i, which has domain E^r. Thus $\mathbf{w}^i \in (E^r)^*$ for each $i = 1, \ldots, n$. The linear transformation $\mathbf{L}^*$ is defined as follows.

Definition. For any covector $\mathbf{a} = (a_1, \ldots, a_n)$ in $(E^n)^*$, let

$$\mathbf{L}^*(\mathbf{a}) = \sum_{i=1}^{n} a_i \mathbf{w}^i. \tag{4.6}$$

The covector $\mathbf{L}^*(\mathbf{a})$ is in $(E^r)^*$, since it is a linear combination of the covectors $\mathbf{w}^1, \ldots, \mathbf{w}^r$ that are in $(E^r)^*$. The components of $\mathbf{L}^*(\mathbf{a})$ can be found

as follows. We have

$$\mathbf{w}^i = \sum_{j=1}^{r} c_j^i \boldsymbol{\varepsilon}^j, \qquad i = 1, \ldots, n,$$

where $\{\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^r\}$ is the standard basis for $(E^r)^*$. If we write $\mathbf{b} = \mathbf{L}^*(\mathbf{a})$, then the components of $\mathbf{b}$ satisfy

$$b_j = \sum_{i=1}^{n} a_i c_j^i, \qquad j = 1, \ldots, r. \tag{4.7}$$

If one regards the covector $\mathbf{a}$ as a row vector, then (4.7) states that $\mathbf{b}$ is obtained by multiplying $\mathbf{a}$ on the right by the matrix (c_j^i).

EXAMPLE 2 (*continued*). In this example the row covectors are $\mathbf{w}^1 = \boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^2 - \boldsymbol{\varepsilon}^3$, $\mathbf{w}^2 = 2\boldsymbol{\varepsilon}^1 + \boldsymbol{\varepsilon}^2 + 4\boldsymbol{\varepsilon}^3$, $\mathbf{w}^3 = -\boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^3$, and $\mathbf{L}^*(\mathbf{a}) = a_1\mathbf{w}^1 + a_2\mathbf{w}^2 + a_3\mathbf{w}^3$.

Note that Formulas (4.6) and (4.7) are statements about covectors corresponding to Formulas (4.4) and (4.2) about vectors. We say that (4.6) is dual to (4.4), and (4.7) dual to (4.2). Further aspects of the duality between $\mathbf{L}$ and $\mathbf{L}^*$ are mentioned in Problem 5.

PROBLEMS

1. Let $r = 3$, $n = 2$, and $\mathbf{L}$ be the linear transformation such that $\mathbf{L}(\boldsymbol{\varepsilon}_1) = \mathbf{e}_1 - 2\mathbf{e}_2$, $\mathbf{L}(\boldsymbol{\varepsilon}_2) = \mathbf{e}_1$, $\mathbf{L}(\boldsymbol{\varepsilon}_3) = 5\mathbf{e}_1 + \mathbf{e}_2$. Find the matrix of $\mathbf{L}$, the rank, and the kernel.

2. Show that the kernel of a linear transformation is a vector subspace of its domain.

3. Let $r = n$, and let $L^i(\mathbf{t}) = c^i t^i$ for every $\mathbf{t} \in E^n$, where $c^1, \ldots, c^n$ are scalars.
   (a) What is the matrix?
   (b) Find $\mathbf{L}^{-1}$ if it exists.
   (c) If $c^1 = \cdots = c^n > 0$, then $\mathbf{L}$ is called *homothetic* about $\mathbf{0}$. Describe $\mathbf{L}$ geometrically. Show that if $\mathbf{L}$ and $\mathbf{M}$ are homothetic about $\mathbf{0}$, then $\mathbf{L}^{-1}$ and $\mathbf{M} \circ \mathbf{L}$ are also homothetic about $\mathbf{0}$.

4. (a) Show directly from the definitions that the composite of two linear transformations is also linear.
   (b) Let (c_j^i), (d_i^l), and (b_j^l) denote respectively the matrices of $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{M} \circ \mathbf{L}$. Show that

$$b_j^l = \sum_{i=1}^{n} d_i^l c_j^i, \quad \text{for } l = 1, \ldots, p, \qquad j = 1, \ldots, r. \tag{4.8}$$

5. Let $\mathbf{L}^*$ be the dual of the linear transformation $\mathbf{L}$. Show that:
   (a) $\mathbf{L}^*(\mathbf{e}^i) = \mathbf{w}^i$, $i = 1, \ldots, n$, where $\{\mathbf{e}^1, \ldots, \mathbf{e}^n\}$ is the standard basis for $(E^n)^*$, Section 3.2.
   (b) $\mathbf{a} \cdot \mathbf{L}(\mathbf{t}) = \mathbf{L}^*(\mathbf{a}) \cdot \mathbf{t}$, for every covector $\mathbf{a} \in (E^n)^*$ and vector $\mathbf{t} \in E^r$.
   (c) $\mathbf{L}(E^r)$, $\mathbf{L}^*[(E^n)^*]$ have the same dimension ρ, and $\mathbf{L}$, $\mathbf{L}^*$ have the same nullity ν. (You may use the fact that row rank of a matrix equals column rank.)

4.2 Affine transformations

If $\mathbf{L}$ is linear, then $\mathbf{L}(\mathbf{0}) = \mathbf{0}$. This fact gives $\mathbf{0}$ a special role that is somewhat unnatural from the geometric viewpoint. Before proceeding to study rather general nonlinear transformations in Section 4.3, let us consider another special class of transformations called *affine*. The main result of the section is a characterization of the isometries of E^n (Theorem 4.1).

The present section can be omitted at first reading. The material in it is mainly used later in the book in examples.

Definition. A transformation $\mathbf{g}$ is *affine* is there exist a linear transformation $\mathbf{L}$ and $\mathbf{x}_0 \in E^n$ such that

$$\mathbf{g}(\mathbf{t}) = \mathbf{L}(\mathbf{t}) + \mathbf{x}_0 \quad \text{for every } \mathbf{t} \in E^r.$$

If $r = n$ and $\mathbf{L} = \mathbf{I}$, then $\mathbf{g}$ is a *translation.*

A translation merely takes each $\mathbf{t}$ into $\mathbf{t} + \mathbf{x}_0$. If $\mathbf{g}$ is affine, then $\mathbf{g}(\mathbf{0}) = \mathbf{x}_0$. Hence an affine transformation $\mathbf{g}$ is linear if and only if $\mathbf{g}(\mathbf{0}) = \mathbf{0}$. Every affine transformation is the composite of a translation and a linear transformation.

Isometries of E^n

Let $\mathbf{g}$ be a transformation from E^n into E^n. If $\mathbf{g}$ preserves the distance between each pair of points, then $\mathbf{g}$ is called an *isometry.*

Definition. If $|\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})| = |\mathbf{s} - \mathbf{t}|$ for every $\mathbf{s}, \mathbf{t} \in E^n$, then $\mathbf{g}$ is an *isometry* of E^n.

Let us first suppose that $\mathbf{g}$ is an isometry of E^n that leaves $\mathbf{0}$ fixed, namely, $\mathbf{g}(\mathbf{0}) = \mathbf{0}$. Then taking $\mathbf{t} = \mathbf{0}$, we have $|\mathbf{g}(\mathbf{s})| = |\mathbf{s}|$ for every $\mathbf{s} \in E^n$. Using the formula $|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2$, we have

$$|\mathbf{g}(\mathbf{s})|^2 - 2\mathbf{g}(\mathbf{s}) \cdot \mathbf{g}(\mathbf{t}) + |\mathbf{g}(\mathbf{t})|^2 = |\mathbf{s}|^2 - 2\mathbf{s} \cdot \mathbf{t} + |\mathbf{t}|^2,$$

for every $\mathbf{s}, \mathbf{t} \in E^n$. Therefore

$$\mathbf{g}(\mathbf{s}) \cdot \mathbf{g}(\mathbf{t}) = \mathbf{s} \cdot \mathbf{t}, \tag{4.9}$$

which says that $\mathbf{g}$ preserves the euclidean inner product. Let $\mathbf{v}_j = \mathbf{g}(\boldsymbol{\varepsilon}_j)$, $j = 1, \ldots, n$. Then $|\mathbf{v}_j| = 1$ and from (4.9)

$$\mathbf{v}_i \cdot \mathbf{v}_j = \boldsymbol{\varepsilon}_i \cdot \boldsymbol{\varepsilon}_j = \delta_{ij}$$

for each $i, j = 1, \ldots, n$ where, as in Section 1.2, δ_{ij} is Kronecker's delta. Hence $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form an orthonormal basis for E^n. Let us show that $\mathbf{g}$ is a linear transformation. For each $\mathbf{s}$, $\mathbf{t}$, we have from (4.9)

$$\mathbf{g}(\mathbf{s}) \cdot \mathbf{v}_j = \mathbf{s} \cdot \boldsymbol{\varepsilon}_j = s^j, \qquad \mathbf{g}(\mathbf{t}) \cdot \mathbf{v}_j = t^j, \qquad \mathbf{g}(\mathbf{s} + \mathbf{t}) \cdot \mathbf{v}_j = s^j + t^j,$$

and hence

$$[\mathbf{g}(\mathbf{s}+\mathbf{t})-\mathbf{g}(\mathbf{s})-\mathbf{g}(\mathbf{t})]\cdot\mathbf{v}_j=0,$$

for each $j=1,\dots n$. The vector $\mathbf{g}(\mathbf{s}+\mathbf{t})-\mathbf{g}(\mathbf{s})-\mathbf{g}(\mathbf{t})$ has component 0 with respect to each basis vector $\mathbf{v}_j$. Hence

$$\mathbf{g}(\mathbf{s}+\mathbf{t})=\mathbf{g}(\mathbf{s})+\mathbf{g}(\mathbf{t}).$$

Similarly, $\mathbf{g}(c\mathbf{s})=c\mathbf{g}(\mathbf{s})$ for every $\mathbf{s}$ and scalar c; thus $\mathbf{g}$ is linear. The column vectors of its matrix are $\mathbf{v}_1,\dots,\mathbf{v}_n$.

Definition. A linear transformation that preserves the euclidean inner product is an *orthogonal transformation.*

Proposition 4.1 ($r=n$). $\mathbf{L}$ *is an orthogonal transformation if and only if the column vectors* $\mathbf{v}_1,\dots,\mathbf{v}_n$ *form an orthonormal basis for* E^n.

PROOF. We have already shown that if $\mathbf{L}$ is orthogonal, then $\mathbf{v}_1,\dots,\mathbf{v}_n$ form an orthonormal basis. To prove the converse, we see from (4.4) that

$$\mathbf{L}(\mathbf{s})\cdot\mathbf{L}(\mathbf{t})=\sum_{i,j=1}^{n}s^it^j\mathbf{v}_i\cdot\mathbf{v}_j.$$

If $\mathbf{v}_1,\dots,\mathbf{v}_n$ is an orthonormal basis, then $\mathbf{v}_i\cdot\mathbf{v}_j=\delta_{ij}$, and

$$\mathbf{L}(\mathbf{s})\cdot\mathbf{L}(\mathbf{t})=\mathbf{s}\cdot\mathbf{t}$$

for every $\mathbf{s},\mathbf{t}\in E^n$. □

Theorem 4.1 ($r=n$). *A transformation* $\mathbf{g}$ *is an isometry of* E^n *if and only if* $\mathbf{g}$ *is an affine transformation of the form* $\mathbf{g}(\mathbf{t})=\mathbf{L}(\mathbf{t})+\mathbf{x}_0$ *for every* $\mathbf{t}\in E^n$, *where* $\mathbf{L}$ *is orthogonal.*

PROOF. Let $\mathbf{g}$ be an isometry of E^n. Let $\mathbf{f}(\mathbf{t})=\mathbf{g}(\mathbf{t})-\mathbf{x}_0$ for every $\mathbf{t}\in E^n$, where $\mathbf{x}_0=\mathbf{g}(\mathbf{0})$. Then

$$|\mathbf{f}(\mathbf{s})-\mathbf{f}(\mathbf{t})|=|\mathbf{g}(\mathbf{s})-\mathbf{g}(\mathbf{t})=|\mathbf{s}-\mathbf{t}|$$

for every $\mathbf{s},\mathbf{t}\in E^n$. Hence $\mathbf{f}$ is an isometry. Moreover, $\mathbf{f}(\mathbf{0})=\mathbf{0}$. We have already shown that $\mathbf{f}$ must be orthogonal.

Conversely, let $\mathbf{L}$ be orthogonal. Then $\mathbf{L}(\mathbf{s})\cdot\mathbf{L}(\mathbf{t})=\mathbf{s}\cdot\mathbf{t}$ for every $\mathbf{s},\mathbf{t}\in E^n$. Taking $\mathbf{s}=\mathbf{t}$, we have

$$|\mathbf{L}(\mathbf{s})|^2=\mathbf{L}(\mathbf{s})\cdot\mathbf{L}(\mathbf{s})=\mathbf{s}\cdot\mathbf{s}=|\mathbf{s}|^2.$$

Hence $|\mathbf{L}(\mathbf{s})|=|\mathbf{s}|$. Replacing $\mathbf{s}$ by $\mathbf{s}-\mathbf{t}$, we have

$$|\mathbf{L}(\mathbf{s})-\mathbf{L}(\mathbf{t})|=|\mathbf{L}(\mathbf{s}-\mathbf{t})|=|\mathbf{s}-\mathbf{t}|$$

for every $\mathbf{s},\mathbf{t}\in E^n$. Hence $\mathbf{L}$ is an isometry of E^n. Since $|\mathbf{g}(\mathbf{s})-\mathbf{g}(\mathbf{t})|=|\mathbf{L}(\mathbf{s})-\mathbf{L}(\mathbf{t})|$, $\mathbf{g}$ is also an isometry of E^n. □

If $\mathbf{L}$ is linear from E^n into E^n, let $\mathbf{L}^t$ denote the linear transformation whose matrix (c_i^j) is obtained by exchanging rows and columns of the matrix (c_j^i) of $\mathbf{L}$. We call $\mathbf{L}^t$ the *adjoint* of $\mathbf{L}$. It is characterized by the formula

$$\mathbf{y} \cdot \mathbf{L}(\mathbf{t}) = \mathbf{L}^t(\mathbf{y}) \cdot \mathbf{t} \tag{4.10}$$

for every $\mathbf{y}, \mathbf{t} \in E^n$ (see Problem 5).

Let us apply (4.10) with $\mathbf{y} = \mathbf{L}(\mathbf{s})$. We get

$$\mathbf{L}(\mathbf{s}) \cdot \mathbf{L}(\mathbf{t}) = (\mathbf{L}^t \circ \mathbf{L})(\mathbf{s}) \cdot \mathbf{t}.$$

From this equation, $\mathbf{L}$ is orthogonal if and only if $\mathbf{s} \cdot \mathbf{t} = (\mathbf{L}^t \circ \mathbf{L})(\mathbf{s}) \cdot \mathbf{t}$ for every $\mathbf{s}, \mathbf{t} \in E^n$. But this is equivalent to the statement that $\mathbf{s} = (\mathbf{L}^t \circ \mathbf{L})(\mathbf{s})$ for every $\mathbf{s}$, in other words, that $\mathbf{I} = \mathbf{L}^t \circ \mathbf{L}$. Hence $\mathbf{L}$ *is orthogonal if and only if*

$$\mathbf{L}^t = \mathbf{L}^{-1}.$$

If $\mathbf{L}$ is orthogonal, then

$$1 = \det \mathbf{I} = \det \mathbf{L}^t \det \mathbf{L}.$$

But $\det \mathbf{L}^t = \det \mathbf{L}$ ([12], p. 146). Hence $1 = (\det \mathbf{L})^2$, and $\det \mathbf{L} = \pm 1$. If $\mathbf{L}$ is orthogonal and $\det \mathbf{L} = 1$, then L is called a *rotation* of E^n about $\mathbf{0}$.

EXAMPLE 1. Any translation is an isometry of E^n, and $\mathbf{L} = \mathbf{I}$.

EXAMPLE 2. Let $\mathbf{S}$ be the orthogonal transformation which takes each $\mathbf{t} = (t^1, \ldots, t^n)$ into $\mathbf{S}(\mathbf{t}) = (t^1, \ldots, t^{n-1}, -t^n)$. $\mathbf{S}$ is a reflection of E^n about the hyperplane $t^n = 0$. Its matrix is

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}.$$

Two such reflections take each $\mathbf{t}$ into itself; that is, $\mathbf{S} \circ \mathbf{S} = \mathbf{I}$. Hence $\mathbf{S} = \mathbf{S}^{-1} = \mathbf{S}^t$. If $\mathbf{M}$ is any orthogonal transformation with $\det \mathbf{M} = -1$, then $\mathbf{L} = \mathbf{S} \circ \mathbf{M}$ is a rotation of E^n about $\mathbf{0}$ and

$$\mathbf{M} = \mathbf{S}^{-1} \circ \mathbf{L} = \mathbf{S} \circ \mathbf{L}.$$

Thus any orthogonal transformation is either a rotation or the composite of $\mathbf{S}$ and a rotation.

EXAMPLE 3. Let $n = 2$, and $\mathbf{L}$ be a rotation of the plane E^2 about $(0, 0)$. Since $|\mathbf{v}_1| = 1$, $\mathbf{v}_1 = (\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2$ for some $\theta \in [0, 2\pi)$. Since $\mathbf{L}$ is a rotation, $\mathbf{v}_2 = (-\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2$. The matrix is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The angle of rotation is θ.

PROBLEMS

1. Let $n = r = 2$.
   (a) Describe geometrically the linear transformation $\mathbf{L}$ with matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
   (b) Find $\mathbf{S} \circ \mathbf{L}$ and $\mathbf{L} \circ \mathbf{S}$, where $\mathbf{S}$ is the same as in Example 2. Show that both are rotations of E^2 about (0, 0).
2. (a) Show that the vectors $\mathbf{v}_1 = (1/\sqrt{5})(\mathbf{e}_1 + 2\mathbf{e}_3), \mathbf{v}_2 = (1/\sqrt{10})(-2\mathbf{e}_1 + \sqrt{5}\mathbf{e}_2 + \mathbf{e}_3)$, $\mathbf{v}_3 = (1/\sqrt{10})(2\mathbf{e}_1 + \sqrt{5}\mathbf{e}_2 - \mathbf{e}_3)$ form an orthonormal basis for E^3.
   (b) Let $\mathbf{L}$ be the orthogonal transformation whose matrix has $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as column vectors. Find $\mathbf{L}'$ and verify that $\mathbf{L}' \circ \mathbf{L} = \mathbf{I}$. Is $\mathbf{L}$ a rotation?
3. Let $\mathbf{L}$ and $\mathbf{M}$ be rotations of E^n about $\mathbf{0}$. Show that $\mathbf{L}^{-1}$ and $\mathbf{M} \circ \mathbf{L}$ are also rotations.
4. (a) Show that the composite of two affine transformations is also affine.
   (b) Which affine transformations are univalent?
5. Prove (4.10).

4.3 Differentiable transformations

We now begin a discussion of differential calculus for transformations from a portion of one euclidean space E^r into another euclidean space E^n. The first step is to define the concept of differentiability.

Let $\mathbf{g}$ be a transformation from $\Delta \subset E^r$ into E^n, and let $\mathbf{t}_0$ be an interior point of Δ. We would like to find a local linear approximation for the difference $\mathbf{g}(\mathbf{t}) - \mathbf{g}(\mathbf{t}_0)$. If there is such an approximation, then $\mathbf{g}$ is said to be differentiable at $\mathbf{t}_0$. More precisely:

Definition. A transformation $\mathbf{g}$ is *differentiable at* $\mathbf{t}_0$ if there exists a linear transformation $\mathbf{L}$ (depending on $\mathbf{t}_0$) such that

$$\lim_{\mathbf{k} \to \mathbf{0}} \frac{1}{|\mathbf{k}|} [\mathbf{g}(\mathbf{t}_0 + \mathbf{k}) - \mathbf{g}(\mathbf{t}_0) - \mathbf{L}(\mathbf{k})] = \mathbf{0}. \tag{4.11}$$

If we set $\mathbf{t} = \mathbf{t}_0 + \mathbf{k}$, then $\mathbf{L}(\mathbf{t} - \mathbf{t}_0)$ is the desired local approximation to $\mathbf{g}(\mathbf{t}) - \mathbf{g}(\mathbf{t}_0)$. If $n = 1$, the definition agrees with the one in Section 3.3 for real valued functions.

Definition. The linear transformation $\mathbf{L}$ in (4.11) is called the *differential* of $\mathbf{g}$ at $\mathbf{t}_0$ and is denoted by $D\mathbf{g}(\mathbf{t}_0)$.

The following proposition is an easy consequence of the definitions, and Proposition 2.1 (1) about limits of sums.

Proposition 4.2. *Let* $\mathbf{g}$ *and* $\mathbf{G}$ *be differentiable at* $\mathbf{t}_0$. *Then the sum* $\mathbf{g} + \mathbf{G}$ *is differentiable at* $\mathbf{t}_0$, *and*

$$D(\mathbf{g} + \mathbf{G})(\mathbf{t}_0) = D\mathbf{g}(\mathbf{t}_0) + D\mathbf{G}(\mathbf{t}_0).$$

For formulas for the differentials of products, see Problem 12, Section 4.4.

In order to find the matrix associated with the differential $D\mathbf{g}(\mathbf{t}_0)$ of a linear transformation $\mathbf{g}$, let us first prove the following proposition. For each $\mathbf{t} \in \Delta$,

$$\mathbf{g}(\mathbf{t}) = g^1(\mathbf{t})\mathbf{e}_1 + \cdots + g^n(\mathbf{t})\mathbf{e}_n.$$

The real valued functions $g^1, \ldots, g^n$ are the components of $\mathbf{g}$. Their partial derivatives are denoted by g^i_j or $\partial g^i/\partial t^j$.

Proposition 4.3. (a) *A transformation* $\mathbf{g}$ *is differentiable at* $\mathbf{t}_0$ *if and only if each each of its components* $g^1, \ldots, g^n$ *is differentiable at* $\mathbf{t}_0$. (b) *If* $\mathbf{g}$ *is differentiable at* $\mathbf{t}_0$, *then the matrix of the linear transformation* $\mathbf{L}$ *is the matrix of partial derivatives* $g^i_j(\mathbf{t}_0)$.

PROOF. Let $L^1, \ldots, L^n$ denote the components of $\mathbf{L}$, and let

$$\boldsymbol{\phi}(\mathbf{k}) = |\mathbf{k}|^{-1}[\mathbf{g}(\mathbf{t}_0 + \mathbf{k}) - \mathbf{g}(\mathbf{t}_0) - \mathbf{L}(\mathbf{k})].$$

The components of $\boldsymbol{\phi}(\mathbf{k})$ are

$$\phi^i(\mathbf{k}) = |\mathbf{k}|^{-1}[g^i(\mathbf{t}_0 + \mathbf{k}) - g^i(\mathbf{t}_0) - L^i(\mathbf{k})], \qquad i = 1, \ldots, n.$$

By Proposition 2.2, $\boldsymbol{\phi}(\mathbf{k}) \to \mathbf{0}$ as $\mathbf{k} \to \mathbf{0}$ if and only if $\phi^i(\mathbf{k}) \to 0$ as $\mathbf{k} \to \mathbf{0}$, $i = 1, \ldots, n$. This proves (a). It also shows that, if $\mathbf{L} = D\mathbf{g}(\mathbf{t}_0)$, then the rows of the matrix of $\mathbf{L}$ are the covectors corresponding to the linear functions $L^1, \ldots, L^n$ (in Section 4.1, these covectors were denoted by $\mathbf{w}^1, \ldots, \mathbf{w}^n$).

From Section 3.3 we recall that $L^i(\mathbf{k}) = dg^i(\mathbf{t}_0) \cdot \mathbf{k}$ for every $\mathbf{k} \in E^r$, where $dg^i(\mathbf{t}_0)$ is the differential of the component g^i at $\mathbf{t}_0$. Therefore, the row covectors of the matrix of $\mathbf{L}$ are $dg^1(\mathbf{t}_0), \ldots, dg^n(\mathbf{t}_0)$. The jth component of $dg^i(\mathbf{t}_0)$ is the partial derivative $g^i_j(\mathbf{t}_0)$; see Formula (3.8). This proves (b). □

The *partial derivatives* of $\mathbf{g}$ are defined just as for real valued functions:

$$\mathbf{g}_j(\mathbf{t}_0) = \lim_{s \to 0} \frac{\mathbf{g}(\mathbf{t}_0 + s\boldsymbol{\varepsilon}_j) - \mathbf{g}(\mathbf{t}_0)}{s}, \qquad j = 1, \ldots, r.$$

By an argument like the proof of Proposition 4.3(a), the components of the vector $\mathbf{g}_j(\mathbf{t}_0)$ are the partial derivatives $g^i_j(\mathbf{t}_0)$, $i = 1, \ldots, n$. Thus $\mathbf{g}_j(\mathbf{t}_0)$ *is the* j*th column vector of the matrix* $(g^i_j(\mathbf{t}_0))$ of partial derivatives. In the proof of Proposition 4.3, we showed that $dg^i(\mathbf{t}_0)$ *is the* i*th row covector of the matrix* $(g^i_j(\mathbf{t}_0))$.

In case $r = n$ the determinant of the linear transformation $D\mathbf{g}(\mathbf{t})$ is called the *Jacobian* of $\mathbf{g}$ at $\mathbf{t}$. It is denoted by $J\mathbf{g}(\mathbf{t})$. Thus

$$J\mathbf{g}(\mathbf{t}) = \det D\mathbf{g}(\mathbf{t}) = \det(g^i_j(\mathbf{t})). \tag{4.12}$$

Another common notation for the Jacobian is

$$\frac{\partial(g^1, \ldots, g^n)}{\partial(t^1, \ldots, t^n)}(\mathbf{t}).$$

We see later that the Jacobian often plays the same role in the calculus of functions of several variables as the derivative does in the case $r = n = 1$. In particular, this is so in the theorems about inverses (Section 4.5) and transforming multiple integrals (Section 5.8). If the Jacobian is 0 at a point $\mathbf{t}_0$, then $D\mathbf{g}(\mathbf{t}_0)$ is singular. This suggests some kind of irregularity in the behavior of $\mathbf{g}$ near $\mathbf{t}_0$. In order to exclude such irregularities we repeatedly make the assumption in later sections that the Jacobian is not 0.

Transformation	$\mathbf{g} = (g^1, \ldots, g^n) = \sum_{i=1}^{n} g^i \mathbf{e}_i$
Differential at $\mathbf{t}$	$D\mathbf{g}(\mathbf{t})$
Matrix of $D\mathbf{g}(\mathbf{t})$	$(g^i_j(\mathbf{t})), \quad i,j = 1, \ldots, n$
Column vectors	$\mathbf{g}_j(\mathbf{t}), \quad j = 1, \ldots, r$
Row covectors	$dg^i(\mathbf{t}), \quad i = 1, \ldots, n$
Jacobian ($r = n$)	$J\mathbf{g}(\mathbf{t}) = \det(g^i_j(\mathbf{t}))$

$$g^i_j(\mathbf{t}) = \frac{\partial g^i}{\partial t^j}, \quad \mathbf{g}_j(\mathbf{t}) = \sum_{i=1}^{n} g^i_j(\mathbf{t})\mathbf{e}_i, \quad dg^i(\mathbf{t}) = \sum_{j=1}^{r} g^i_j(\mathbf{t})\boldsymbol{\varepsilon}^j$$

$\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ standard basis for E^n

$\{\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^r\}$ standard basis for $(E^r)^*$

EXAMPLE 1. If $n = 1$, then $D\mathbf{g}(\mathbf{t})$ has the single row covector $dg(\mathbf{t})$. We may identify $D\mathbf{g}(\mathbf{t})$ with $dg(\mathbf{t})$. If $dg(\mathbf{t}) \neq \mathbf{0}$, the rank of $D\mathbf{g}(\mathbf{t})$ is 1; otherwise it is 0.

EXAMPLE 2. If $r = 1$, then there is a single column vector. It is the derivative, denoted by $\mathbf{g}'(t)$. If $\mathbf{g}'(t) \neq \mathbf{0}$, the rank is 1; otherwise it is 0.

EXAMPLE 3. Let $n = r = 2$. Points of the domain Δ are denoted by (s, t) and those of the image $\mathbf{g}(\Delta)$ by (x, y). It is helpful to think of two copies of the plane E^2. The first contains Δ and is called the st-plane. The second contains $\mathbf{g}(\Delta)$ and is called the xy-plane.

In our example we let Δ be the whole st-plane, and

$$\mathbf{g}(s, t) = (s^2 + t^2)\mathbf{e}_1 + 2st\mathbf{e}_2$$

for every $(s, t) \in E^2$. If $(x, y) \in \mathbf{g}(\Delta)$, then $x = s^2 + t^2$, $y = 2st$ and $x + y \geq 0$, $x - y \geq 0$. Therefore $\mathbf{g}(\Delta)$ is contained in the quadrant Q shown in Figure 4.2. In fact, $\mathbf{g}(\Delta) = Q$. This is seen as follows. Let C be a circle with center $(0, 0)$ and radius $a > 0$. Points of C are given by $s = a \cos \phi$, $t = a \sin \phi$, where $0 \leq \phi \leq 2\pi$. The image $\mathbf{g}(C)$ is the set of points

$$\mathbf{g}(a \cos \phi, a \sin \phi) = a^2\mathbf{e}_1 + (a^2 \sin 2\phi)\mathbf{e}_2, \; 0 \leq \phi \leq 2\pi.$$

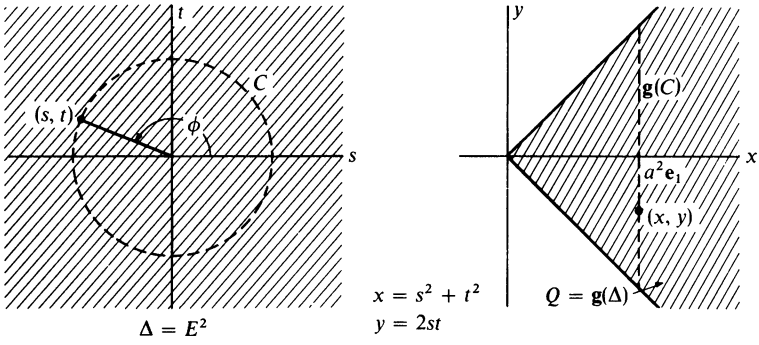


Figure 4.2

This set is the vertical line segment shown in Figure 4.2. By letting a take all possible nonnegative values, one gets a collection of line segments covering Q. If $a = 0$, the line segment degenerates to the point $(0, 0)$. This shows that $Q = \mathbf{g}(\Delta)$.

In Example 3, $g^1(s, t) = s^2 + t^2$, $g^2(s, t) = 2st$. The partial derivatives of g^1 appear in the first row of the matrix of $D\mathbf{g}(s, t)$, and those of g^2 in the second row. This matrix is therefore

$$\begin{pmatrix} 2s & 2t \\ 2t & 2s \end{pmatrix}.$$

The column vectors are the partial derivatives of $\mathbf{g}$, namely,

$$\mathbf{g}_1(s, t) = 2s\mathbf{e}_1 + 2t\mathbf{e}_2,$$
$$\mathbf{g}_2(s, t) = 2t\mathbf{e}_1 + 2s\mathbf{e}_2.$$

The Jacobian is

$$J\mathbf{g}(s, t) = \det\begin{pmatrix} 2s & 2t \\ 2t & 2s \end{pmatrix} = 4(s^2 - t^2).$$

Note that $J\mathbf{g}(s, t) \neq 0$ if $s^2 \neq t^2$. In Section 4.5 we show that $\mathbf{g}$ has a local inverse near any such point (s, t). The points $(s, \pm s)$ where $s^2 = t^2$ map under $\mathbf{g}$ onto the boundary half-lines $y = \pm x, x \geq 0$, on the right-hand side of Figure 4.2. The transformation $\mathbf{g}$ has no local inverse near $(s, \pm s)$.

Let us now suppose that the domain Δ of $\mathbf{g}$ is an open set.

Definition. If the components $g^1, \ldots, g^n$ are of class $C^{(q)}$, $q \geq 0$, then $\mathbf{g}$ is a *transformation of class* $C^{(q)}$. Similarly, if $g^1, \ldots, g^n$ are of class $C^{(q)}$ on $B \subset \Delta$, then $\mathbf{g}$ is *of class* $C^{(q)}$ *on* B.

If $\mathbf{g}$ is differentiable at every $\mathbf{t}_0 \in \Delta$, then $\mathbf{g}$ is a *differentiable* transformation.

Theorem 4.2. *Every differentiable transformation is of class* $C^{(0)}$. *Every transformation of class* $C^{(1)}$ *is differentiable.*

PROOF. Apply Theorems 3.1, 3.2, and Proposition 4.3(a). □

For most theorems in the differential calculus of transformations, one needs to assume that $\mathbf{g}$ is of class $C^{(1)}$ at least. An exception is the composite function theorem (Section 4.4), in which only differentiability need be assumed.

In the remainder of this section we establish several inequalities of a rather technical nature. These inequalities are used in the proofs of theorems to follow.

We first need to introduce a norm that measures the "size" of a linear transformation. Let

$$\|\mathbf{L}\| = \max\{|\mathbf{L}(\mathbf{t})| : |\mathbf{t}| \leq 1\}.$$

The set of all linear transformations with domain E^r and values in E^n forms a vector space of dimension nr. The usual properties of a norm are satisfied (Problem 7).

Let us show that

$$|\mathbf{L}(\mathbf{t})| \leq \|\mathbf{L}\|\,|\mathbf{t}| \tag{4.13}$$

for every $\mathbf{t} \in E^r$. If $\mathbf{t} = \mathbf{0}$, then both sides are $\mathbf{0}$. If $\mathbf{t} \neq \mathbf{0}$, let $c = |\mathbf{t}|^{-1}$. Since $\mathbf{L}$ is linear, $\mathbf{L}(c\mathbf{t}) = c\mathbf{L}(\mathbf{t})$. Since $|c\mathbf{t}| = 1$, $|\mathbf{L}(c\mathbf{t})| \leq \|\mathbf{L}\|$. Thus $|\mathbf{t}|^{-1}|\mathbf{L}(\mathbf{t})| \leq \|\mathbf{L}\|$, which is the same as (4.13).

Since $\mathbf{L}(\mathbf{s}) - \mathbf{L}(\mathbf{t}) = \mathbf{L}(\mathbf{s} - \mathbf{t})$, we have upon replacing $\mathbf{t}$ by $\mathbf{s} - \mathbf{t}$ in (4.13)

$$|\mathbf{L}(\mathbf{s}) - \mathbf{L}(\mathbf{t})| \leq \|\mathbf{L}\|\,|\mathbf{s} - \mathbf{t}|. \tag{4.14}$$

Proposition 4.4. *Let* $\mathbf{g}$ *be differentiable at* $\mathbf{t}_0$. *Then given* $\varepsilon > 0$ *there exists a neighborhood* Ω_0 *of* $\mathbf{t}_0$ *such that* $\Omega_0 \subset \Delta$ *and*

$$|\mathbf{g}(\mathbf{t}) - \mathbf{g}(\mathbf{t}_0)| \leq (\|D\mathbf{g}(\mathbf{t}_0)\| + \varepsilon)|\mathbf{t} - \mathbf{t}_0| \tag{4.15}$$

for every $\mathbf{t} \in \Omega_0$.

PROOF. Let $\mathbf{L} = D\mathbf{g}(\mathbf{t}_0)$ and set $\tilde{\mathbf{g}}(\mathbf{t}) = \mathbf{g}(\mathbf{t}) - \mathbf{L}(\mathbf{t})$. Since $D\mathbf{L}(\mathbf{t}_0) = \mathbf{L}$, $D\tilde{\mathbf{g}}(\mathbf{t}_0) = \mathbf{0}$ (the zero linear transformation). By (4.11), in which $\mathbf{g}$ is replaced by $\tilde{\mathbf{g}}$, there is a neighborhood Ω_0 of $\mathbf{t}_0$ such that

$$|\tilde{\mathbf{g}}(\mathbf{t}) - \tilde{\mathbf{g}}(\mathbf{t}_0)| \leq \varepsilon|\mathbf{t} - \mathbf{t}_0| \tag{*}$$

for every $\mathbf{t} \in \Omega_0$. But

$$\mathbf{g}(\mathbf{t}) - \mathbf{g}(\mathbf{t}_0) = [\mathbf{L}(\mathbf{t}) - \mathbf{L}(\mathbf{t}_0)] + [\tilde{\mathbf{g}}(\mathbf{t}) - \tilde{\mathbf{g}}(\mathbf{t}_0)].$$

From (*), (4.14), and the triangle inequality we get (4.15). □

If $\mathbf{g}$ is of class $C^{(1)}$, there is a stronger version of Proposition 4.4.

Proposition 4.5. *Let* $\mathbf{g}$ *be of class* $C^{(1)}$ *and* $\mathbf{t}_0 \in \Delta$. *Then given* $\varepsilon > 0$ *there exists a neighborhood* Ω *of* $\mathbf{t}_0$ *such that* $\Omega \subset \Delta$ *and*

$$|\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})| \leq (\|D\mathbf{g}(\mathbf{t}_0)\| + \varepsilon)|\mathbf{s} - \mathbf{t}| \tag{4.16}$$

for every $\mathbf{s}, \mathbf{t} \in \Omega$.

PROOF. Let $\tilde{\mathbf{g}}$ be as before. The row covectors $d\tilde{g}^i(\mathbf{t}_0)$ are all $\mathbf{0}$. Since the partial derivatives of $\tilde{g}$ are continuous, given $\varepsilon > 0$ there is a neighborhood Ω of $\mathbf{t}_0$ such that $|d\tilde{g}^i(\mathbf{u})| < \varepsilon/n$ for every $\mathbf{u} \in \Omega$ and $i = 1, \ldots, n$. By Corollary 1, Section 3.3, for every $\mathbf{s}, \mathbf{t} \in \Omega$,

$$|\tilde{g}^i(\mathbf{s}) - \tilde{g}^i(\mathbf{t})| \leq \frac{\varepsilon}{n}|\mathbf{s} - \mathbf{t}|,$$

$$|\tilde{\mathbf{g}}(\mathbf{s}) - \tilde{\mathbf{g}}(\mathbf{t})| \leq \sum_{i=1}^{n} |\tilde{g}^i(\mathbf{s}) - \tilde{g}^i(\mathbf{t})| \leq \varepsilon|\mathbf{s} - \mathbf{t}|. \tag{**}$$

From (**) and (4.14) we obtain (4.16) in the same way as before. □

PROBLEMS

1. In Example 3, find:
   (a) The image of any vertical line $s = c$.
   (b) The inverse image of any line $y = mx$ through the origin.
   (c) The image of the circular disk bounded by C in Figure 4.2.

2. Let $\mathbf{g}(s, t) = |s - t|\mathbf{e}_1 + |s + t|\mathbf{e}_2, \Delta = E^2$. Find $\mathbf{g}(E^2)$ and answer questions (a), (b), and (c) in Problem 1 for this transformation.

3. Let $\mathbf{g}(s, t) = (t \cos 2\pi s)\mathbf{e}_1 + (t \sin 2\pi s)\mathbf{e}_2 + (1 - t)\mathbf{e}_3, \Delta = E^2$.
   (a) Show that $\mathbf{g}(E^2)$ is a cone with vertex $\mathbf{e}_3$.
   (b) What is the image of the square $\{(s, t): 0 \leq s \leq 1, 0 \leq t \leq 1\}$?
   (c) Find $\mathbf{g}^{-1}(\{\mathbf{e}_3\})$ and $\mathbf{g}^{-1}(\{\mathbf{e}_1\})$.

4. Let $\mathbf{g}(s, t) = 1/(s^2 + st + t^2)\mathbf{e}_1 + 1/(s^2 + st + t^2)^2\mathbf{e}_2, \Delta = \{(s, t): 0 < s^2 + t^2 \leq 1\}$.
   (a) Show that $\mathbf{g}(\Delta)$ is part of the parabola $y = x^2$, and find it.
   (b) Find $\mathbf{g}^{-1}(\{(c, c^2)\})$.

5. For each of Problems 2, 3, and 4, find:
   (a) Where $\mathbf{g}$ is differentiable.
   (b) The partial derivatives of $\mathbf{g}$.
   (c) The rank of $D\mathbf{g}(s, t)$.
   (d) The Jacobian $J\mathbf{g}(s, t)$, where applicable.

6. (a) Let $\mathbf{g}$ be affine, $\mathbf{g}(\mathbf{t}) = \mathbf{L}(\mathbf{t}) + \mathbf{x}_0$ for every $\mathbf{t} \in E^r$. Show that $D\mathbf{g}(\mathbf{t}) = \mathbf{L}$ for every $\mathbf{t} \in E^r$.
   (b) Let $\mathbf{g}$ be a differentiable transformation such that $D\mathbf{g}$ is a constant function and Δ is a connected open set. Show that $\mathbf{g}$ is the restriction to Δ of an affine transformation.

7. Show that:
   (a) $\|\mathbf{L}\| > 0$ unless $\mathbf{L}$ has rank 0.
   (b) $\|c\mathbf{L}\| = |c|\,\|\mathbf{L}\|$.
   (c) $\|\mathbf{L}_1 + \mathbf{L}_2\| \leq \|\mathbf{L}_1\| + \|\mathbf{L}_2\|$.
   (d) $\|\mathbf{M} \circ \mathbf{L}\| \leq \|\mathbf{M}\|\,\|\mathbf{L}\|$.
   (e) The vector space $\mathscr{L}(E^r, E^n)$ of all linear $\mathbf{L}$ from E^r into E^n has dimension nr.

8. Another norm for linear transformations, which we denote by $|||\ |||$, is defined as follows:

$$|||\mathbf{L}||| = |\mathbf{w}^1| + \cdots + |\mathbf{w}^n|,$$

   where $\mathbf{w}^1, \ldots, \mathbf{w}^n$ are the row covectors. Show that properties (a)–(d) of Problem 7 hold for this norm. Show that $\|\mathbf{L}\| \leq |||\mathbf{L}|||$.

9. Let $r = n$ and $\mathbf{g}$ be a differentiable transformation. Then $\mathbf{g}$ is called *conformal* if there exists a real-valued function μ such that $\mu(\mathbf{t}) > 0$ and $\mu(\mathbf{t})D\mathbf{g}(\mathbf{t})$ is a rotation of E^n for every $\mathbf{t} \in \Delta$.
   (a) Using Proposition 4.1 show that $\mathbf{g}$ is conformal if and only if, for every $\mathbf{t} \in \Delta$, $J\mathbf{g}(\mathbf{t}) > 0$ and the partial derivatives of $\mathbf{g}$ satisfy:

   (1) $$\mathbf{g}_i(\mathbf{t}) \cdot \mathbf{g}_j(\mathbf{t}) = 0 \quad \text{if } i \neq j,$$

   and

   (2) $$|\mathbf{g}_1(\mathbf{t})| = |\mathbf{g}_2(\mathbf{t})| = \cdots = |\mathbf{g}_n(\mathbf{t})| = 1/\mu(\mathbf{t}).$$

   (b) Show that if $\mathbf{g}$ is conformal, then $\mu(\mathbf{t}) = [J\mathbf{g}(\mathbf{t})]^{-1/n}$.
   (c) Let $n = 2$. Show that $\mathbf{g}$ is conformal if and only if $J\mathbf{g}(\mathbf{t}) > 0$ and $g_1^1(\mathbf{t}) = g_2^2(\mathbf{t})$, $g_2^1(\mathbf{t}) = -g_1^2(\mathbf{t})$ for every $\mathbf{t} \in \Delta$. [*Note*: The partial differential equations $g_1^1 = g_2^2$, $g_2^1 = -g_1^2$ are the Cauchy–Riemann equations in the theory of complex analytic functions; see the work by Nehari [20], for example.]

10. Show that $\mathbf{g}$ is conformal if $g^1(s, t) = \exp(s^2 - t^2)\cos 2st$, $g^2(s, t) = \exp(s^2 - t^2)\sin 2st$, $\Delta = E^2 - \{(0, 0)\}$.

11. Let $\mathbf{g}$ be of class $C^{(1)}$. The maximum rank possible for $D\mathbf{g}(\mathbf{t})$ is $\min(r, n)$, Show that $\{\mathbf{t} : \text{rank } D\mathbf{g}(\mathbf{t}) = \min(r, n)\}$ is open.

4.4 Composition

We now derive a rule for differentiating the composite of two differentiable transformations. As corollaries of the basic formula (4.17) we then obtain a formula for Jacobians and the chain rule for partial derivatives.

Let $\mathbf{g}$ be a transformation from an open set $\Delta \subset E^r$ into an open set $D \subset E^n$, and let $\mathbf{f}$ be a transformation from D into E^p.

Composite function theorem. *Let $\mathbf{g}$ be differentiable at $\mathbf{t}_0$ and $\mathbf{f}$ be differentiable at $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$. Then the composite $\mathbf{F} = \mathbf{f} \circ \mathbf{g}$ is differentiable at $\mathbf{t}_0$ and*

(4.17) $$D\mathbf{F}(\mathbf{t}_0) = D\mathbf{f}(\mathbf{x}_0) \circ D\mathbf{g}(\mathbf{t}_0).$$

PROOF. Let $\mathbf{L} = D\mathbf{g}(\mathbf{t}_0)$, $\mathbf{M} = D\mathbf{f}(\mathbf{x}_0)$. Let us first prove the theorem for two special cases.

Case 1. $\mathbf{M} = \mathbf{0}$. Let us show that

$$(*) \qquad \lim_{\mathbf{k}\to\mathbf{0}} \frac{1}{|\mathbf{k}|}[\mathbf{F}(\mathbf{t}_0 + \mathbf{k}) - \mathbf{F}(\mathbf{t}_0)] = \mathbf{0}.$$

Let $C = \|\mathbf{L}\| + 1$. By Proposition 4.4 with $\varepsilon = 1$, there exists $\delta_0 > 0$ such that

$$|\mathbf{g}(\mathbf{t}_0 + \mathbf{k}) - \mathbf{g}(\mathbf{t}_0)| \leq C|\mathbf{k}|,$$

whenever $|\mathbf{k}| < \delta_0$. Since $\mathbf{M} = \mathbf{0}$, $\|\mathbf{M}\| = \|D\mathbf{f}(\mathbf{x}_0)\| = 0$. Proposition 4.4 (applied this time to $\mathbf{f}$ rather than $\mathbf{g}$ and ε replaced by $C^{-1}\varepsilon$) implies that given $\varepsilon > 0$ there exists $\eta > 0$ such that

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| \leq \frac{\varepsilon}{C}|\mathbf{x} - \mathbf{x}_0|$$

whenever $|\mathbf{x} - \mathbf{x}_0| < \eta$. Let $\delta = \min\{C^{-1}\eta, \delta_0\}$. Then we get by taking $\mathbf{x} = \mathbf{g}(\mathbf{t}_0 + \mathbf{k})$ and using

$$\mathbf{f}[\mathbf{g}(\mathbf{t}_0 + \mathbf{k})] = \mathbf{F}(\mathbf{t}_0 + \mathbf{k}), \mathbf{f}[\mathbf{g}(\mathbf{t}_0)] = \mathbf{F}(\mathbf{t}_0),$$

$$|\mathbf{F}(\mathbf{t}_0 + \mathbf{k}) - \mathbf{F}(\mathbf{t}_0)| \leq \frac{\varepsilon}{C}C|\mathbf{k}| = \varepsilon|\mathbf{k}|,$$

if $|\mathbf{k}| < \delta$. This proves (*). But (*) says that $\mathbf{F}$ is differentiable at $\mathbf{t}_0$ with $D\mathbf{F}(\mathbf{t}_0) = \mathbf{0}$.

Case 2. When $\mathbf{f}$ is a linear transformation, $\mathbf{f}(\mathbf{x}) = \mathbf{M}\mathbf{x}$ for all $\mathbf{x}$. In this case (4.11), together with Propositions 2.1 and 2.2, implies that

$$(**) \qquad \lim_{\mathbf{k}\to\mathbf{0}} \frac{1}{|\mathbf{k}|}\mathbf{M}[\mathbf{g}(\mathbf{t}_0 + \mathbf{k}) - \mathbf{g}(\mathbf{t}_0) - \mathbf{L}(\mathbf{k})] = \mathbf{0}.$$

Since $\mathbf{F}(\mathbf{t}) = \mathbf{M}[\mathbf{g}(\mathbf{t})]$ in the special case when $\mathbf{f}$ is linear, (**) implies in this case that $\mathbf{F}$ is differentiable at $\mathbf{t}_0$, with $D\mathbf{F}(\mathbf{t}_0) = \mathbf{M} \circ \mathbf{L}$.

In the general case, let $\tilde{\mathbf{f}} = \mathbf{f} - \mathbf{M}$. Then $D\tilde{\mathbf{f}}(\mathbf{x}_0) = \mathbf{0}$, and Case 1 applies to the transformation $\tilde{\mathbf{F}} = \tilde{\mathbf{f}} \circ \mathbf{g}$. Moreover, Case 2 applies to $\mathbf{G} = \mathbf{M} \circ \mathbf{g}$. Since $\mathbf{F} = \tilde{\mathbf{F}} + \mathbf{G}$, the theorem now follows by Proposition 4.2. □

By (4.17), the matrix of $D\mathbf{F}(\mathbf{t}_0)$ is the product of the matrices of $\mathbf{M}$ and $\mathbf{L}$. Let us specialize to the case when f and F are real valued ($p = 1$) with $F = f \circ \mathbf{g}$. Moreover, let us abbreviate by writing

$$F_j = F_j(\mathbf{t}_0), f_i = f_i(\mathbf{x}_0), g_j^i = g_j^i(\mathbf{t}_0).$$

The matrix product becomes

$$(F_1, \ldots, F_r) = (f_1, \ldots, f_n)\begin{pmatrix} g_1^1 & \cdots & g_r^1 \\ \vdots & & \\ g_1^n & \cdots & g_r^n \end{pmatrix}.$$

This is called the *chain rule for partial derivatives.* We rewrite it in more compact notation as:

Corollary 1 (*chain rule*)

$$F_j = \sum_{i=1}^{n} f_i g_j^i, \qquad j = 1, \ldots, r. \tag{4.18}$$

Note that $F_1, \ldots, F_r$ are the components of the covector $dF(\mathbf{t}_0)$, and $f_1, \ldots, f_n$ the components of the covector $df(\mathbf{x}_0)$. According to Formula (4.7), the chain rule (4.18) states that $dF(\mathbf{t}_0) = \mathbf{L}^*[df(\mathbf{x}_0)]$ when $\mathbf{L} = D\mathbf{g}(\mathbf{t}_0)$. Thus, the chain rule can be regarded as a particular case of the formula for changing the components of a covector under the dual $\mathbf{L}^*$ of the linear transformation $\mathbf{L}$.

Another suggestive form for the important formula (4.18) is obtained by writing it with the other notation for partial derivatives:

$$\frac{\partial F}{\partial t^j} = \frac{\partial f}{\partial x^1}\frac{\partial g^1}{\partial t^j} + \cdots + \frac{\partial f}{\partial x^n}\frac{\partial g^n}{\partial t^j}, \qquad j = 1, \ldots, r.$$

Let us next consider the case when $\mathbf{F}$ and $\mathbf{g}$ are vector-valued functions of one real variable ($r = 1$). Their derivatives are denoted by $\mathbf{F}'(t_0)$ and $\mathbf{g}'(t_0)$. If we identify the $p \times 1$ matrix of $D\mathbf{F}(t_0)$ with $\mathbf{F}'(t_0)$, regarded as a column vector, and the $n \times 1$ matrix of $D\mathbf{g}(t_0)$ with $\mathbf{g}'(t_0)$, then we get from (4.17):

Corollary 2. *Let* $r = 1$. *Then* $\mathbf{F}'(t_0) = D\mathbf{f}(\mathbf{x}_0)[\mathbf{g}'(t_0)]$.

Corollary 2 is used in the discussion of tangent vectors in Section 4.7.

Again let $p = 1$, and suppose that f and $\mathbf{g}$ are of class $C^{(q)}$ for some $q \geq 1$. In particular, f and $\mathbf{g}$ are differentiable. Formula (4.18) applies at every point $\mathbf{t} \in \Delta$ and the corresponding point $\mathbf{x} = \mathbf{g}(\mathbf{t}) \in D$. Thus

$$F_j(\mathbf{t}) = \sum_{i=1}^{n} (f_i \circ \mathbf{g})(\mathbf{t}) g_j^i(\mathbf{t}),$$

for every $\mathbf{t} \in \Delta$ and $j = 1, \ldots, r$. Since all of the functions f_i, $\mathbf{g}$, g_j^i are continuous, each partial derivative F_j is continuous. Hence F is of class $C^{(1)}$. If $q \geq 2$, then repeated application of the chain rule shows that F is of class $C^{(q)}$ and gives formulas for calculating its partial derivatives of orders $1, 2, \ldots, q$.

In case $p > 1$ and $\mathbf{f}$, $\mathbf{g}$ are of class $C^{(q)}$, the preceding discussion shows that the components $F^1, \ldots, F^p$ of $\mathbf{F}$ are of class $C^{(q)}$, since $F^l = f^l \circ \mathbf{g}$ for each $l = 1, \ldots, p$. Therefore $\mathbf{F}$ is of class $C^{(q)}$. We have proved:

Corollary 3. *If* $\mathbf{f}$ *and* $\mathbf{g}$ *are of class* $C^{(q)}$, *then* $\mathbf{F}$ *is of class* $C^{(q)}$.

EXAMPLE 1. Let $r = p = 1$. The chain rule becomes

$$F'(t) = \sum_{i=1}^{n} f_i[\mathbf{g}(t)]g^{i\prime}(t),$$

which can also be written

$$F'(t) = df[\mathbf{g}(\mathbf{t})] \cdot \mathbf{g}'(t).$$

If in addition $n = 1$, it becomes $F'(t) = f'[g(t)]g'(t)$, which is the composite function rule of elementary calculus.

EXAMPLE 2. Let $F(x) = f[x, g(x)]$, where f and g are of class $C^{(2)}$. In this case $g^1(x) = x, g^2(x) = g(x)$, and the formula in Example 1 becomes

$$F'(x) = f_1[x, g(x)] + f_2[x, g(x)]g'(x).$$

Another application of the chain rule together with the formula for the derivative of a product gives

$$F''(x) = f_{11} + 2f_{12}g'(x) + f_{22}[g'(x)]^2 + f_2g''(x).$$

In this formula the partial derivatives of f are evaluated at $(x, g(x))$.

EXAMPLE 3. Let f be of class $C^{(2)}$ and let

$$F(r, \theta) = f[r\cos\theta, r\sin\theta].$$

Let us show that

$$f_{11} + f_{22} = F_{11} + \frac{1}{r^2}F_{22} + \frac{1}{r}F_1.$$

The expression on the left-hand side is called the *Laplacian* of f. The partial differential equation $f_{11} + f_{22} = 0$ is called *Laplace's equation*. Its solutions are called *harmonic functions*. The formula above expresses the Laplacian in polar coordinates.

In this example

$$g^1(r, \theta) = r\cos\theta, \qquad g^2(r, \theta) = r\sin\theta.$$

Using the chain rule, we get

$$\begin{aligned} F_1 &= f_1g_1^1 + f_2g_1^2 = f_1\cos\theta + f_2\sin\theta, \\ F_2 &= f_1g_2^1 + f_2g_2^2 = f_1(-r\sin\theta) + f_2(r\cos\theta). \end{aligned}$$

Further application of the chain rule gives

$$\begin{aligned} F_{11} &= \cos\theta(f_{11}\cos\theta + f_{12}\sin\theta) + \sin\theta(f_{21}\cos\theta + f_{22}\sin\theta), \\ F_{22} &= -r\sin\theta[f_{11}(-r\sin\theta) + f_{12}(r\cos\theta)] + r\cos\theta[f_{21}(-r\sin\theta) \\ &\quad + f_{22}(r\cos\theta)] - f_1r\cos\theta - f_2r\sin\theta. \end{aligned}$$

Combining terms and using the fact that $f_{21} = f_{12}$, we get

$$F_{11} + \frac{1}{r^2} F_{22} = f_{11} + f_{22} - \frac{1}{r}(f_1 \cos\theta + f_2 \sin\theta),$$

$$F_{11} + \frac{1}{r^2} F_{22} = f_{11} + f_{22} - \frac{1}{r} F_1.$$

This is what we wished to show.

Corollary 4. *Let $n = r = p$, and let* $\mathbf{f}$ *and* $\mathbf{g}$ *be differentiable. Then, for every* $\mathbf{t} \in \Delta$,

$$\frac{\partial(F^1, \ldots, F^n)}{\partial(t^1, \ldots, t^n)} = \frac{\partial(f^1, \ldots, f^n)}{\partial(x^1, \ldots, x^n)} \frac{\partial(g^1, \ldots, g^n)}{\partial(t^1, \ldots, t^n)}, \tag{4.19}$$

the Jacobians being evaluated at $\mathbf{t}$ *and at* $\mathbf{x} = \mathbf{g}(\mathbf{t})$.

PROOF. By (4.17), $D\mathbf{F}(\mathbf{t}) = D\mathbf{f}(\mathbf{x}) \circ D\mathbf{g}(\mathbf{t})$. Hence

$$\det D\mathbf{F}(\mathbf{t}) = \det D\mathbf{f}(\mathbf{x}) \det D\mathbf{g}(\mathbf{t}). \qquad \square$$

EXAMPLE 4. Let $n = r = p = 2$. Let $\mathbf{f}(x, y) = f^1(x, y)\mathbf{E}_1 + f^2(x, y)\mathbf{E}_2$, $\mathbf{g}(r, \theta) = (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2$. As before, $\mathbf{E}_1$, $\mathbf{E}_2$ denote the standard basis vectors for the plane E^2 in which $\mathbf{f}$ takes its values. Then

$$\frac{\partial(g^1, g^2)}{\partial(r, \theta)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r.$$

Hence

$$\frac{\partial(F^1, F^2)}{\partial(r, \theta)} = \frac{\partial(f^1, f^2)}{\partial(x, y)} r.$$

PROBLEMS

Assume that all functions which occur in these problems are of class $C^{(2)}$.

1. Let $F(x, y) = f(x, xy)$. Find the mixed partial derivative F_{12}.

2. Let $F(x, y) = f[x, y, g(x, y)]$. Express the partial derivatives of F of orders 1 and 2 in terms of those of f and g.

3. Let $n = r = p = 2$. Find the Jacobian $\partial(F^1, F^2)/\partial(s, t)$ at the indicated point by means of Corollary 4.
   (a) $\mathbf{f}(x, y) = xy\mathbf{E}_1 + x^2y\mathbf{E}_2$, $\mathbf{g}(s, t) = (s + t)\mathbf{e}_1 + (s^2 - t^2)\mathbf{e}_2$, $(s_0, t_0) = (2, 1)$.
   (b) $\mathbf{f}(x, y) = \phi(x + y)\mathbf{E}_1 + \phi(x - y)\mathbf{E}_2$, $\mathbf{g}(s, t) = (\exp t)\mathbf{e}_1 + \exp(-s)\mathbf{e}_2$, $(s_0, t_0) = (\log 2)\mathbf{e}_1$.

4. (a) Show that the chain rule is still true if $p > 1$, namely,

$$\frac{\partial \mathbf{F}}{\partial t^j} = \sum_{i=1}^{n} \frac{\partial g^i}{\partial t^j} \frac{\partial \mathbf{f}}{\partial x^i}, \qquad j = 1, \ldots, r.$$

   (b) Use it to find $(\partial\mathbf{F}/\partial s)(s_0, t_0)$ and $(\partial\mathbf{F}/\partial t)(s_0, t_0)$ in Problem 3(a).

5. Let $f(x, y) = \phi(x - cy) + \psi(x + cy)$, where c is a scalar. Show that $f_{22} = c^2 f_{11}$.

6. Let

$$n = 4 \qquad \text{and} \qquad \rho = [(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2}.$$

Let

$$f(\mathbf{x}) = [\phi(\rho - cx^4) + \psi(\rho + cx^4)]/\rho.$$

Show that

$$f_{44} = c^2(f_{11} + f_{22} + f_{33}).$$

[*Note*: The partial differential equation $f_{nn} = c^2(f_{11} + \cdots + f_{n-1,n-1})$ is called the *wave equation* in n variables. Problem 5 gives D'Alembert's solution for $n = 2$. Solutions of the type in Problem 6 are called *spherical waves.*]

7. Suppose that f satisfies the partial differential equation $f_2 = f_{11} + bf$, where b is a scalar. Let $F(x, y) = \exp(-by)f(x, y)$. Show that $F_2 = F_{11}$.

8. Let $F = f \circ \mathbf{L}$, where $\mathbf{L}$ is a linear transformation with matrix (c_j^i). Show that the second-order partial derivatives of F satisfy

$$F_{jl} = \sum_{i,k=1}^{n} f_{ik} c_j^i c_l^k, \qquad j, l = 1, \ldots, r.$$

9. Using Problem 8 show that if $r = n$ and $\mathbf{L}$ is orthogonal, then $F_{11} + \cdots + F_{nn} = f_{11} + \cdots + f_{nn}$. In other words, the Laplacian is invariant under orthogonal transformations of E^n. [*Hint*: $\mathbf{L}^t = \mathbf{L}^{-1}$.]

10. Let $n = r$. A linear transformation $\mathbf{L}$ is *a Lorentz* transformation of E^n if $\mathbf{L}^{-1} = \mathbf{S} \circ \mathbf{L}^t \circ \mathbf{S}$, where $\mathbf{S}$ is as in Example 2, Section 4.2.

(a) Show that if $\mathbf{M}$ and $\mathbf{L}$ are Lorentz, then $\mathbf{M} \circ \mathbf{L}$ and $\mathbf{L}^{-1}$ are also Lorentz. [*Hint*: $\mathbf{S}^2 = \mathbf{I}$.]

(b) Show that $\mathbf{L}$ is Lorentz if and only if $\mathbf{S} = \mathbf{L}^t \circ \mathbf{S} \circ \mathbf{L}$.

(c) Show that $\mathbf{L}$ is Lorentz if and only if

$$\sum_{i=1}^{n-1} [L^i(\mathbf{t})]^2 - [L^n(\mathbf{t})]^2 = \sum_{i=1}^{n-1} (t^i)^2 - (t^n)^2$$

for every $\mathbf{t}$. [*Hint*: The right-hand side is $\mathbf{S}(\mathbf{t}) \cdot \mathbf{t}$. Use (b).]

11. Show that if $\mathbf{L}$ is Lorentz, then $F_{11} + \cdots + F_{n-1,n-1} - F_{nn} = f_{11} + \cdots + f_{n-1,n-1} - f_{nn}$. In other words, the wave operator is invariant under Lorentz transformations ($c = 1$).

12. Use the composite function theorem to establish the following rule for differentials of inner products. Let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be differentiable at $\mathbf{t}_0$, both with values in E^n. Show that $\boldsymbol{\phi} \cdot \boldsymbol{\psi}$ is differentiable at $\mathbf{t}_0$ and that $d(\boldsymbol{\phi} \cdot \boldsymbol{\psi})(\mathbf{t}_0) = \boldsymbol{\phi}(\mathbf{t}_0) \cdot D\boldsymbol{\psi}(\mathbf{t}_0) + \boldsymbol{\psi}(\mathbf{t}_0) \cdot D\boldsymbol{\phi}(\mathbf{t}_0)$. [*Note*: If $\mathbf{v} \in E^n$ and $\mathbf{L}$ is a linear transformation from E^r into E^n, then $\mathbf{v} \cdot \mathbf{L}$ denotes the real valued linear function such that $(\mathbf{v} \cdot \mathbf{L})(\mathbf{k}) = \mathbf{v} \cdot \mathbf{L}(\mathbf{k})$ for all $\mathbf{k} \in E^r$.] [*Hint*: Take $\mathbf{g} = (\boldsymbol{\phi}, \boldsymbol{\psi})$ with values in E^{2n} and $f(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.]

4.5 The inverse function theorem

In this section we suppose that $r = n$. Thus $\mathbf{g}$ denotes a transformation from an open set $\Delta \subset E^n$ into E^n. We recall that $\mathbf{g}$ is called *univalent* if $\mathbf{s} \neq \mathbf{t}$ implies $\mathbf{g}(\mathbf{s}) \neq \mathbf{g}(\mathbf{t})$. A univalent transformation $\mathbf{g}$ has an inverse, denoted by $\mathbf{g}^{-1}$. The inverse $\mathbf{g}^{-1}$ has as domain the set $\mathbf{g}(\Delta)$. It is defined by $\mathbf{t} = \mathbf{g}^{-1}(\mathbf{x})$, where $\mathbf{t}$ is the unique point of Δ such that $\mathbf{x} = \mathbf{g}(\mathbf{t})$. It occasionally happens that $\mathbf{g}^{-1}$ can be found explicitly by solving the system of equations $x^i = g^i(\mathbf{t})$, $i = 1, \ldots, n$, for the components $t^1, \ldots, t^n$ in terms of $\mathbf{x}$. However, the more common situation is either that these equations cannot be explicitly solved, or that it is inconvenient to solve them explicitly. One would like a criterion which guarantees that the inverse $\mathbf{g}^{-1}$ exists, and a formula for its differential, without explicitly finding $\mathbf{g}^{-1}$ itself.

Let us begin by reviewing the situation when $n = 1$. Let g be a real valued function of class $C^{(1)}$, with domain an open interval Δ. The inverse g^{-1} exists if and only if g is strictly monotone on Δ (Appendix A.4). It is proved in elementary calculus that the inverse is differentiable provided $g'(t) \neq 0$ for all $t \in \Delta$, and that

$$g^{-1\prime}(x) = \frac{1}{g'(t)}, \quad \text{if } x = g(t). \tag{4.20}$$

In two or more dimensions the Jacobian $J\mathbf{g}(\mathbf{t})$ takes the place of the derivative $g'(t)$. However, the situation is by no means as simple as before. First of all, we have to assume that $\mathbf{g}$ is at least of class $C^{(1)}$, a stronger condition than differentiability. Second, and more important, the fact that $J\mathbf{g}(\mathbf{t}) \neq 0$ does not imply that $\mathbf{g}$ has an inverse. (see Example 2 below). However, the nonvanishing of the Jacobian $J\mathbf{g}(\mathbf{t}_0)$ at a point $\mathbf{t}_0$ implies that the restriction $\mathbf{g}|\Delta_0$ has an inverse for some open set Δ_0 containing $\mathbf{t}_0$. This is part of the statement of the inverse function theorem. We sometimes call $(\mathbf{g}|\Delta_0)^{-1}$ a *local inverse* of $\mathbf{g}$.

Inverse function theorem. *Let $\mathbf{g}$ be a transformation of class $C^{(q)}$, $q \geq 1$, from an open set $\Delta \subset E^n$ into E^n. If $J\mathbf{g}(\mathbf{t}_0) \neq 0$, then there exists an open set Δ_0 containing $\mathbf{t}_0$, such that:*

(1) *The restriction $\mathbf{g}|\Delta_0$ is univalent.*
(2) *The set $\mathbf{g}(\Delta_0)$ is open.*
(3) *The inverse $\mathbf{f}$ of $\mathbf{g}|\Delta_0$ is of class $C^{(q)}$.*
(4) $D\mathbf{f}(\mathbf{x}) = [D\mathbf{g}(\mathbf{t})]^{-1}$, *if* $\mathbf{x} = \mathbf{g}(\mathbf{t})$, $\mathbf{t} \in \Delta_0$.

The following argument should make the existence of a local inverse plausible. For $\mathbf{t}$ near $\mathbf{t}_0$, $\mathbf{g}(\mathbf{t})$ is approximated by $\mathbf{G}(\mathbf{t}) = \mathbf{g}(\mathbf{t}_0) + \mathbf{L}(\mathbf{t} - \mathbf{t}_0)$, where $\mathbf{L} = D\mathbf{g}(\mathbf{t}_0)$. Since the determinant of $\mathbf{L}$, namely $J\mathbf{g}(\mathbf{t}_0)$, is not 0, $\mathbf{L}$ has an inverse. Hence, the affine transformation $\mathbf{G}$ also has an inverse. Since $\mathbf{G}$

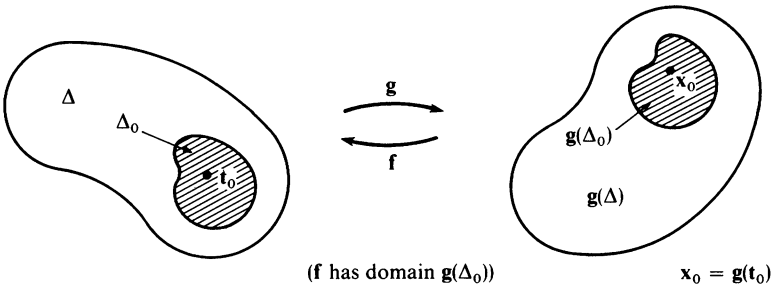


Figure 4.3

is a good approximation to $\mathbf{g}$ near $\mathbf{t}_0$, this suggests that the restriction of $\mathbf{g}$ to a small enough open set Δ_0 containing $\mathbf{t}_0$ should also have an inverse. This plausibility argument does not, of course, constitute a proof. In fact, it is a moderately difficult task to prove the inverse function theorem. We give a proof at the end of the section. In the meantime, we state some consequences of this theorem and examples.

The determinant of the inverse $\mathbf{L}^{-1}$ of a linear transformation $\mathbf{L}$ is $(\det \mathbf{L})^{-1}$. Therefore, part (4) of the inverse function theorem implies that:

$$J\mathbf{f}(\mathbf{x}) = \frac{1}{J\mathbf{g}(\mathbf{t})}, \qquad \mathbf{t} = \mathbf{f}(\mathbf{x}). \tag{4.21}$$

The inverse function theorem has the following corollary.

Corollary. *Let* $\mathbf{g}$ *be of class* $C^{(1)}$, *and suppose that* $J\mathbf{g}(\mathbf{t}) \neq 0$ *for all* $\mathbf{t} \in \Delta$. *Then the image* $\mathbf{g}(B)$ *of any open set* $B \subset \Delta$ *is an open set.*

PROOF. Let $B \subset \Delta$ be open, and consider any $\mathbf{t}_1 \in B$. We apply the inverse function theorem, with Δ replaced by B. There exists an open set Δ_1 containing $\mathbf{t}_1$, with $\Delta_1 \subset B$, such that $\mathbf{g}(\Delta_1)$ is open. Therefore the point $\mathbf{x}_1 = \mathbf{g}(\mathbf{t}_1)$ has a neighborhood U_1 such that $U_1 \subset \mathbf{g}(\Delta_1)$. But $\mathbf{g}(\Delta_1) \subset \mathbf{g}(B)$, and hence $\mathbf{x}_1$ is an interior point of $\mathbf{g}(B)$. Since this is true for each $\mathbf{x}_1 \in \mathbf{g}(B)$, $\mathbf{g}(B)$ is open. □

EXAMPLE 1. Let $\mathbf{g}$ be as in Example 3, Section 4.3. We saw that $J\mathbf{g}(s,t) \neq 0$ provided $s^2 \neq t^2$. The lines $s = \pm t$ divide the plane into four open quadrants. The restriction of $\mathbf{g}$ to any of these quadrants has an inverse, which can be found by explicitly solving the equations $x = s^2 + t^2$, $y = 2st$ for (s,t) in terms of (x,y). For instance, consider the quadrant $\Delta_0 = \{(s,t) : |t| < s, s > 0\}$. Then

$$x + y = (s + t)^2, \qquad x - y = (s - t)^2,$$

$$(*) \qquad s = \frac{\sqrt{x+y} + \sqrt{x-y}}{2}, \qquad t = \frac{\sqrt{x+y} - \sqrt{x-y}}{2}.$$

The restriction $\mathbf{g}|\Delta_0$ has all of the properties (1)–(4) of the inverse function theorem. Its inverse is $\mathbf{f} = (f^1, f^2)$ where $f^1(x, y)$ and $f^2(x, y)$ are on the right side of equations (*). The open set $\mathbf{g}(\Delta_0)$ is the interior of the quadrant Q in Figure 4.2.

EXAMPLE 2. Let $n = 2$, and

$$\mathbf{g}(s, t) = (\cosh s \cos t)\mathbf{e}_1 + (\sinh s \sin t)\mathbf{e}_2,$$

where cosh and sinh are hyperbolic functions. Then

$$g_1^1(s, t) = \sinh s \cos t, \qquad g_2^1(s, t) = -\cosh s \sin t,$$
$$g_1^2(s, t) = \cosh s \sin t, \qquad g_2^2(s, t) = \sinh s \cos t.$$

The Jacobian is $\sinh^2 s \cos^2 t + \cosh^2 s \sin^2 t$, which simplifies because $\cosh^2 s = 1 + \sinh^2 s$ and $\cos^2 t + \sin^2 t = 1$ to

$$J\mathbf{g}(s, t) = \sinh^2 s + \sin^2 t.$$

If we take for Δ the right half-plane $s > 0$, then $\sinh s > 0$ and $J\mathbf{g}(s, t) > 0$. The hypotheses of the inverse function theorem are satisfied; hence local inverses exist. Since cos and sin are periodic, $\mathbf{g}(s, t + 2\pi) = \mathbf{g}(s, t)$. The transformation $\mathbf{g}$ is not univalent, and consequently has no inverse. By the corollary, $\mathbf{g}(\Delta)$ is an open set which, as we shall soon see, is E^2 with a line segment removed.

Let $\tilde{\Delta} = \{(s, t) : s > 0, 0 < t < 2\pi\}$, and let $\tilde{\mathbf{g}}$ be the restriction of $\mathbf{g}$ to $\tilde{\Delta}$. Let us show that $\tilde{\mathbf{g}}$ has an inverse. It is not easy to solve the equations

$$x = g^1(s, t) = \cosh s \cos t, \qquad y = g^2(s, t) = \sinh s \sin t$$

explicitly for s and t. However, let us consider what happens on vertical straight lines $s = c$. For each $c > 0$, $\mathbf{g}(c, t)$ represents on $[0, 2\pi]$ an ellipse with major semiaxis of length $\cosh c > 1$ and minor semiaxis of length $\sinh c$. Each of these ellipses has $\pm\mathbf{e}_1$ as foci, and $\mathbf{g}(c, 0) = \mathbf{g}(c, 2\pi) = (\cosh c)\mathbf{e}_1$. If $s_1 \neq s_2$, then the points $\tilde{\mathbf{g}}(s_1, t_1)$ and $\tilde{\mathbf{g}}(s_2, t_2)$ lie on different ellipses.

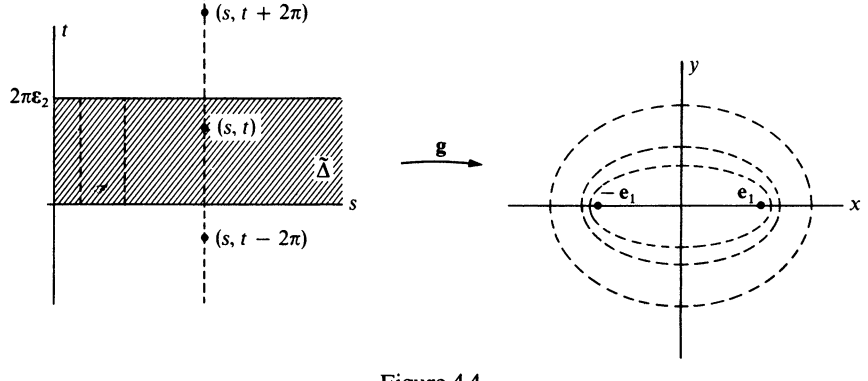


Figure 4.4

Moreover, $\tilde{\mathbf{g}}(s, t_1) = \tilde{\mathbf{g}}(s, t_2)$ implies $t_1 = t_2$. Hence $\tilde{\mathbf{g}}(s_1, t_1) = \tilde{\mathbf{g}}(s_2, t_2)$ implies that $(s_1, t_1) = (s_2, t_2)$, and $\tilde{\mathbf{g}}$ is univalent. The image of $\tilde{\Delta}$ is E^2 with the semi-infinite line on the x-axis from $-\mathbf{e}_1$ to ∞ deleted. The part of the boundary of $\tilde{\Delta}$ on the s-axis is transformed onto the part of the line from $\mathbf{e}_1$ to ∞, and the vertical part of the boundary onto the part from $-\mathbf{e}_1$ to $\mathbf{e}_1$. Hence $\mathbf{g}(\text{cl } \tilde{\Delta}) = E^2$. By periodicity each value which $\mathbf{g}$ takes on Δ is also taken somewhere on $\tilde{\Delta}$ or its lower boundary. Hence $\mathbf{g}(\Delta)$ is E^2 with the line segment joining $-\mathbf{e}_1$ and $\mathbf{e}_1$ removed (see Figure 4.4).

Regular transformations

By adding to the hypotheses of the corollary above the assumption that $\mathbf{g}$ is univalent we obtain a property that we call *regularity*.

Definition ($r = n$). A transformation $\mathbf{g}$ is *regular* if:

(1) $\mathbf{g}$ is of class $C^{(1)}$,
(2) $\mathbf{g}$ is univalent, and
(3) $J\mathbf{g}(\mathbf{t}) \neq 0$ for every $\mathbf{t} \in \Delta$.

A regular transformation $\mathbf{g}$ has an inverse $\mathbf{g}^{-1}$ which is also of class $C^{(1)}$. Regular transformations are called by many authors *diffeomorphisms of class* $C^{(1)}$. A transformation of class $C^{(0)}$ which has an inverse of class $C^{(0)}$ is called a *homeomorphism* (Section 2.6). By Theorem 4.2 every regular transformation is a homeomorphism.

One might expect naively that at worst a transformation distorts shapes, and that the image of a set has basically the same structure as the original. For instance, the image of a line segment should be a smooth curve with no self intersections, the interior of a set should transform onto the interior of the image, and so on. From various examples we know by now that this need not be the case at all. However, it is so for regular transformations. They are the ones that behave properly throughout calculus.

We turn now to a proof of the inverse function theorem. In preparation, we first establish two lemmas. In Lemma 1, $\boldsymbol{\phi}$ denotes a transformation from some neighborhood Ω_1 of $\mathbf{0}$ of radius δ_1 into E^n. We assume that

$$\boldsymbol{\phi}(\mathbf{0}) = \mathbf{0}, \quad |\boldsymbol{\phi}(\mathbf{s}) - \boldsymbol{\phi}(\mathbf{t})| \leq c|\mathbf{s} - \mathbf{t}| \tag{4.22}$$

for all $\mathbf{s}, \mathbf{t} \in \Omega_1$, where $0 < c < 1$. Condition (4.22) implies in particular that $\boldsymbol{\phi}$ is continuous on Ω_1.

Lemma 1. *Let* $\boldsymbol{\phi}$ *satisfy* (4.22); *and let* Ω *be the* δ*-neighborhood of* $\mathbf{0}$, *where* $\delta \leq (1 - c)\delta_1$. *Define* $\boldsymbol{\psi}_m(t)$ *for* $m = 0, 1, 2, \ldots$ *by* $\boldsymbol{\psi}_0(\mathbf{t}) = \mathbf{0}$ *and*

$$\boldsymbol{\psi}_{m+1}(\mathbf{t}) = \boldsymbol{\phi}[\boldsymbol{\psi}_m(\mathbf{t})] + \mathbf{t}, \quad \mathbf{t} \in \Omega. \tag{4.23}$$

Then $\boldsymbol{\psi}_m(t)$ *tends to a limit* $\boldsymbol{\psi}(\mathbf{t})$ *as* $m \to \infty$. *Moreover, for all* $\mathbf{t} \in \Omega$,

$$|\boldsymbol{\psi}(\mathbf{t})| \leq \frac{|\mathbf{t}|}{1 - c}, \qquad \boldsymbol{\psi}(\mathbf{t}) - \boldsymbol{\phi}[\boldsymbol{\psi}(\mathbf{t})] = \mathbf{t}.$$

PROOF. Let us first show by induction on m that, for all $\mathbf{t} \in \Omega$,

$$\text{(a)} \quad |\psi_m(\mathbf{t}) - \psi_{m-1}(\mathbf{t})| \le c^{m-1}|\mathbf{t}|; \qquad \text{(b)} \quad |\psi_m(\mathbf{t})| \le \frac{|\mathbf{t}|}{1-c}.$$

Note that (b) implies that $\psi_m(\mathbf{t}) \in \Omega_1$ since $\delta \le (1-c)\delta_1$. For $m = 1$, (a) and (b) are immediate from $\psi_0(\mathbf{t}) = \mathbf{0}$, $\psi_1(\mathbf{t}) = \mathbf{t}$. Suppose that (a) and (b) hold for $m \le r$. Then

$$\psi_{r+1}(\mathbf{t}) - \psi_r(\mathbf{t}) = \phi[\psi_r(\mathbf{t})] - \phi[\psi_{r-1}(\mathbf{t})],$$

$$|\psi_{r+1}(\mathbf{t}) - \psi_r(\mathbf{t})| \le c|\psi_r(\mathbf{t}) - \psi_{r-1}(\mathbf{t})| \le c \cdot c^{r-1}|\mathbf{t}|,$$

$$\psi_{r+1}(\mathbf{t}) = \sum_{m=0}^{r} [\psi_{m+1}(\mathbf{t}) - \psi_m(\mathbf{t})],$$

$$|\psi_{r+1}(\mathbf{t})| \le \left(\sum_{m=0}^{r} c^m\right)|\mathbf{t}| \le \frac{|\mathbf{t}|}{1-c}.$$

This gives (a) and (b) for $m = r + 1$, as required.

We also have for $m < l$

$$\psi_l(\mathbf{t}) - \psi_m(\mathbf{t}) = \sum_{r=m}^{l-1} [\psi_{r+1}(\mathbf{t}) - \psi_r(\mathbf{t})],$$

$$|\psi_l(\mathbf{t}) - \psi_m(\mathbf{t})| \le \left(\sum_{r=m}^{l-1} c^r\right)|\mathbf{t}| \le \frac{c^m}{1-c}|\mathbf{t}|.$$

Since the right side tends to 0 as $m \to \infty$, the sequence $[\psi_m(\mathbf{t})]$ is Cauchy. By Theorem 2.2, $\psi_m(\mathbf{t})$ tends to a limit $\psi(\mathbf{t})$ as $m \to \infty$, for each $\mathbf{t} \in \Omega$. Moreover, $\psi_{m+1}(\mathbf{t})$ tends to $\psi(\mathbf{t})$, and since ϕ is continuous $\phi[\psi_m(\mathbf{t})] \to \phi[\psi(\mathbf{t})]$ as $m \to \infty$. From (4.23) and (b), ψ has the properties stated in Lemma 1. □

Lemma 2. *Let* $\mathbf{g}$ *be of class* $C^{(1)}$, *from* Δ *into* E^n, *with* $J\mathbf{g}(\mathbf{t}_1) \neq 0$, $\mathbf{t}_1 \in \Delta$. *Then there exist a neighborhood* Ω *of* $\mathbf{t}_1$ *with* $\Omega \subset \Delta$, *a neighborhood* U *of* $\mathbf{x}_1 = \mathbf{g}(\mathbf{t}_1)$, *and a function* $\mathbf{F}$ *defined on* U, *such that:*

(i) *The restriction* $\mathbf{g}|\Omega$ *is univalent.*
(ii) $U \subset \mathbf{g}(\Omega)$ *and* $\mathbf{F}(U) \subset \Omega$.
(iii) $\mathbf{g}[\mathbf{F}(\mathbf{x})] = \mathbf{x}$ *for all* $\mathbf{x} \in U$.
(iv) $\mathbf{F}$ *is differentiable at* $\mathbf{x}_1$, *and*

$$D\mathbf{F}(\mathbf{x}_1) = [D\mathbf{g}(\mathbf{t}_1)]^{-1}.$$

PROOF. We may assume that $\mathbf{t}_1 = \mathbf{0}$ and $\mathbf{x}_1 = \mathbf{0}$. If this is not the case, we first make translations in both $\mathbf{t}$-space and $\mathbf{x}$-space, replacing $\mathbf{g}(\mathbf{t})$ by $\tilde{\mathbf{g}}(\mathbf{t}) = \mathbf{g}(\mathbf{t} + \mathbf{t}_1) - \mathbf{x}_1$. Let $\mathbf{L} = D\mathbf{g}(\mathbf{0})$, and $\phi = \mathbf{I} - \mathbf{L}^{-1} \circ \mathbf{g}$. Note that the inverse $\mathbf{L}^{-1}$ exists since $J\mathbf{g}(\mathbf{0}) \neq \mathbf{0}$. Moreover, $\phi(\mathbf{0}) = \mathbf{0}$ and

$$D\phi(\mathbf{0}) = \mathbf{I} - \mathbf{L}^{-1} \circ D\mathbf{g}(\mathbf{0}) = \mathbf{I} - \mathbf{L}^{-1} \circ \mathbf{L} = \mathbf{0}.$$

We apply Proposition 4.4 to $\boldsymbol{\phi}$, with $\varepsilon = \frac{1}{2}$. Since $D\boldsymbol{\phi}(\mathbf{0}) = \mathbf{0}$ we have, in some neighborhood Ω_0 of $\mathbf{0}$ of radius δ_0, $|\boldsymbol{\phi}(\mathbf{t})| \leq \frac{1}{2}|\mathbf{t}|$. In (4.22) let $c = \frac{1}{2}$. Choose Ω to be the δ-neighborhood of $\mathbf{0}$, with $\delta \leq \frac{1}{2}\delta_0$ sufficiently small that Proposition 4.5 applies to $\boldsymbol{\phi}$ when $\varepsilon = \frac{1}{2}$.

To prove (i), suppose that $\mathbf{g}(\mathbf{s}) = \mathbf{g}(\mathbf{t})$ with $\mathbf{s}, \mathbf{t} \in \Omega$. Then

$$\boldsymbol{\phi}(\mathbf{s}) = \mathbf{s} - \mathbf{L}^{-1}[\mathbf{g}(\mathbf{s})], \boldsymbol{\phi}(\mathbf{t}) = \mathbf{t} - \mathbf{L}^{-1}[\mathbf{g}(\mathbf{t})].$$

By subtracting, we get

$$\boldsymbol{\phi}(\mathbf{s}) - \boldsymbol{\phi}(\mathbf{t}) = \mathbf{s} - \mathbf{t}.$$

But $|\boldsymbol{\phi}(\mathbf{s}) - \boldsymbol{\phi}(\mathbf{t})| \leq \frac{1}{2}|\mathbf{s} - \mathbf{t}|$ by Proposition 4.5. Therefore, $\mathbf{s} = \mathbf{t}$. This proves that $\mathbf{g}|\Omega$ is univalent.

Let $\eta = \delta(2\|L^{-1}\|)^{-1}$, and let U be the η-neighborhood of $\mathbf{0}$. Define $\mathbf{F}$ by

$$\mathbf{F}(\mathbf{x}) = \boldsymbol{\psi}[\mathbf{L}^{-1}(\mathbf{x})] \quad \text{for all } \mathbf{x} \in U,$$

with $\boldsymbol{\psi}$ as in Lemma 1. Since $c = \frac{1}{2}$,

$$|\mathbf{F}(\mathbf{x})| \leq 2|\mathbf{L}^{-1}(\mathbf{x})| \leq 2\|\mathbf{L}^{-1}\|\,|\mathbf{x}| < \delta$$

if $\mathbf{x} \in U$. Thus $\mathbf{F}(U) \subset \Omega$. By definition of $\boldsymbol{\phi}$ and Lemma 1

$$\mathbf{I} - \boldsymbol{\phi} = \mathbf{L}^{-1} \circ \mathbf{g}, \qquad (\mathbf{I} - \boldsymbol{\phi}) \circ \boldsymbol{\psi} = \mathbf{I},$$

the transformations in these equations being restricted to Ω. Then

$$\mathbf{L}^{-1} \circ \mathbf{g} \circ \boldsymbol{\psi} = \mathbf{I};$$

and $\mathbf{g} \circ \boldsymbol{\psi} = \mathbf{L}$. In other words,

$$\mathbf{g}[\boldsymbol{\psi}(\boldsymbol{\tau})] = \mathbf{L}(\boldsymbol{\tau}), \quad \text{for all } \boldsymbol{\tau} \in \Omega.$$

Set $\boldsymbol{\tau} = \mathbf{L}^{-1}(\mathbf{x})$, $\mathbf{x} \in U$. Then $\boldsymbol{\tau} \in \Omega$ and

$$\mathbf{g}[\mathbf{F}(\mathbf{x})] = \mathbf{g}[\boldsymbol{\psi}(\boldsymbol{\tau})] = \mathbf{L}[\mathbf{L}^{-1}(\mathbf{x})] = \mathbf{x}.$$

This shows that $U \subset \mathbf{g}(\Omega)$ and that $\mathbf{g}[\mathbf{F}(\mathbf{x})] = \mathbf{x}$, completing the proof of (ii) and (iii).

To prove (iv), we have by Lemma 1

$$\boldsymbol{\psi}(\boldsymbol{\tau}) - \boldsymbol{\tau} = \boldsymbol{\phi}[\boldsymbol{\psi}(\boldsymbol{\tau})], \quad \text{for all } \boldsymbol{\tau} \in \Omega.$$

Let $0 < c \leq \frac{1}{2}$. Since $\boldsymbol{\phi}(\mathbf{0}) = \mathbf{0}$ and $D\boldsymbol{\phi}(\mathbf{0}) = \mathbf{0}$, inequality (4.22) holds in some neighborhood $\tilde{\Omega}_0$ of $\mathbf{0}$, of radius $\tilde{\delta}_0$. Let $\tilde{\delta} \leq (1 - c)\tilde{\delta}_0$. Then

$$|\boldsymbol{\psi}(\boldsymbol{\tau}) - \boldsymbol{\tau}| \leq c|\boldsymbol{\psi}(\boldsymbol{\tau})| \leq \frac{c}{1 - c}|\boldsymbol{\tau}|, \quad \text{if } |\boldsymbol{\tau}| < \tilde{\delta}.$$

Since $|\boldsymbol{\tau}| \leq \|\mathbf{L}^{-1}\|\,|\mathbf{x}|$ and $\mathbf{F}(\mathbf{x}) = \boldsymbol{\psi}(\boldsymbol{\tau})$, we have

$$|\mathbf{F}(\mathbf{x}) - \mathbf{L}^{-1}(\mathbf{x})| \leq \frac{c\|\mathbf{L}^{-1}\|\,|\mathbf{x}|}{1 - c}, \quad \text{if } |\mathbf{x}| < \tilde{\eta},$$

where $\tilde{\eta} = \tilde{\delta}(\|\mathbf{L}^{-1}\|)^{-1}$. Given any $\varepsilon > 0$ choose c such that $0 < c \leq \frac{1}{2}$ and

$$\frac{c\|\mathbf{L}^{-1}\|}{1-c} \leq \varepsilon.$$

For the corresponding $\tilde{\eta}$ we have

$$\frac{1}{|\mathbf{x}|}|\mathbf{F}(\mathbf{x}) - \mathbf{L}^{-1}(\mathbf{x})| \leq \varepsilon, \quad \text{if } 0 < |\mathbf{x}| < \tilde{\eta}.$$

Since $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, this shows that

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{1}{|\mathbf{x}|}|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{0}) - \mathbf{L}^{-1}(\mathbf{x})| = 0.$$

Thus $\mathbf{F}$ is differentiable at $\mathbf{0}$, and $\mathbf{L}^{-1} = D\mathbf{F}(\mathbf{0})$. □

Proof of inverse function theorem

Given $\mathbf{t}_0$, let us take for Δ_0 a neighborhood of $\mathbf{t}_0$ such that $\mathbf{g}|\Delta_0$ is univalent. Such a neighborhood exists by Lemma 2. Since $\mathbf{g}$ is of class $C^{(q)}$, $q \geq 1$, $J\mathbf{g}$ is continuous. Hence we can choose Δ_0 such that $J\mathbf{g}(\mathbf{t}) \neq 0$ for all $\mathbf{t} \in \Delta_0$. Let $\mathbf{f} = (\mathbf{g}|\Delta_0)^{-1}$; its domain is $\mathbf{g}(\Delta_0)$. Consider any $\mathbf{t}_1 \in \Delta_0$; and let $\mathbf{x}_1 = \mathbf{g}(\mathbf{t}_1)$, Ω, U, $\mathbf{F}$ be as in Lemma 2, with $\Omega \subset \Delta_0$. Since $U \subset \mathbf{g}(\Omega) \subset \mathbf{g}(\Delta_0)$ and $\mathbf{g}|\Delta_0$ is univalent, $\mathbf{F}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in U$. Since each $\mathbf{x}_1 \in \mathbf{g}(\Delta_0)$ has such a neighborhood U, $\mathbf{g}(\Delta_0)$ is an open set. Moreover, by (iv) (Lemma 2) $\mathbf{f}$ is differentiable at $\mathbf{x}_1$ and

$$D\mathbf{f}(\mathbf{x}_1) = [D\mathbf{g}(\mathbf{t}_1)]^{-1}.$$

To complete the proof it remains only to show that $\mathbf{f}$ is of class $C^{(q)}$. By Theorem 4.2, $\mathbf{f}$ is continuous since $\mathbf{f}$ is differentiable at every point of $\mathbf{g}(\Delta_0)$. Since $D\mathbf{f}(\mathbf{x}) = [D\mathbf{g}(\mathbf{t})]^{-1}$, the product of the corresponding matrices of first-order partial derivatives is the identity:

$$\delta_j^i = \sum_{l=1}^{n} f_l^i(\mathbf{x}) g_j^l(\mathbf{t}), \qquad i, j = 1, \ldots, n,$$

where $\mathbf{t} = \mathbf{f}(\mathbf{x})$. Since each g_j^l is continuous and $\mathbf{f}$ is continuous, the composite $g_j^l \circ \mathbf{f}$ is continuous. Let (y_j^l) be a nonsingular matrix. Cramer's rule [12] expresses the solution $(z_1, \ldots, z_n)$ of the system of linear equations

$$b_j = \sum_{l=1}^{n} z_l y_j^l, \qquad j = 1, \ldots, n$$

as a rational function (quotient of two polynomials) in $y_1^1, \ldots, y_n^n$, with denominator not 0. Let us take $b_j = \delta_j^i$, $z_l = f_l^i(\mathbf{x})$, $y_j^l = g_j^l(\mathbf{t})$. Since $J\mathbf{g}(\mathbf{t}) \neq 0$, the matrices $(g_j^l(\mathbf{t}))$ are nonsingular. Thus f_l^i is the composite of a rational function, with nonzero denominator, and the continuous functions $g_j^l \circ \mathbf{f}$. Thus, each partial derivative f_l^i is continuous. This shows that $\mathbf{f}$ is of class

$C^{(1)}$. If $\mathbf{g}$ is of class $C^{(2)}$, then each g_j^i is of class $C^{(1)}$. By Corollary 3, Section 4.4, $g_j^l \circ \mathbf{f}$ is of class $C^{(1)}$. Hence each f_j^i is of class $C^{(1)}$, and $\mathbf{f}$ of class $C^{(2)}$. By repeating this argument, we find that $\mathbf{f}$ is of class $C^{(q)}$ if $\mathbf{g}$ is of class $C^{(q)}$. □

PROBLEMS

1. Determine whether $J\mathbf{g}(\mathbf{t}) \neq 0$ for all $\mathbf{t} \in \Delta$. Find $\mathbf{g}(\Delta)$. If $\mathbf{g}$ is univalent, find $\mathbf{g}^{-1}$ explicitly.
   (a) $\mathbf{g}(\mathbf{t}) = \mathbf{t} + \mathbf{x}_0$ (a translation), $\Delta = E^n$.
   (b) $\mathbf{g}(s, t) = (s + 2t)\mathbf{e}_1 + (s - t)\mathbf{e}_2$, $\Delta = E^2$.
   (c) $\mathbf{g}(s, t) = (s^2 - s - 2)\mathbf{e}_1 + 3t\mathbf{e}_2$, $\Delta = E^2$.
   (d) $\mathbf{g}(s, t) = (s^2 - t^2)\mathbf{e}_1 + st\mathbf{e}_2$, $\Delta = E^2 - \{(0, 0)\}$.
   (e) $\mathbf{g}(s, t) = (\log st)\mathbf{e}_1 + 1/(s^2 + t^2)\mathbf{e}_2$, $\Delta = \{(s, t) : 0 < t < s\}$.

2. Let $g(t) = t^4 + 2t^2$, $\Delta = (0, \infty)$. Find g^{-1}.

3. Let $r = n = 3$, and $\mathbf{g}(s, t, u) = (u \cos st)\mathbf{e}_1 + (u \sin st)\mathbf{e}_2 + (s + u)\mathbf{e}_3$. Then $\mathbf{g}(\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_3) = \mathbf{e}_1 + 2\mathbf{e}_3$. Let $\mathbf{f}$ be a local inverse of $\mathbf{g}$ such that $\mathbf{f}(\mathbf{e}_1 + 2\mathbf{e}_3) = \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_3$. Find $D\mathbf{f}(\mathbf{e}_1 + 2\mathbf{e}_3)$ using (4) of the inverse function theorem.

4. In Example 2, what are the images of horizontal straight lines? Show that $\mathbf{g}$ is a conformal transformation (Problem 9, Section 4.3), and hence that the images of vertical and horizontal straight lines intersect at right angles. Illustrate with a sketch.

5. Let $\mathbf{g}(s, t) = (\exp s \cos t)\mathbf{e}_1 + (\exp s \sin t)\mathbf{e}_2$ and $\Delta = E^2$.
   (a) Show that $J\mathbf{g}(s, t) \neq 0$ for all (s, t), but $\mathbf{g}$ is not univalent.
   (b) Let $\tilde{\Delta} = \{(s, t) : 0 < t < 2\pi\}$. Show that the restriction of $\mathbf{g}$ to $\tilde{\Delta}$ is univalent, and find its inverse.
   (c) Find $\mathbf{g}(E^2)$.
   (d) Show that $\mathbf{g}$ is conformal.

6. Let Δ be an open convex set and $\mathbf{g}$ a differentiable transformation such that $\sum_{i,j=1}^n g_j^i(\mathbf{t})h^i h^j > 0$ for every $\mathbf{t} \in \Delta$ and $\mathbf{h} \neq \mathbf{0}$. Show that $\mathbf{g}$ is univalent. [*Hint*: Suppose that $\mathbf{g}(\mathbf{t}_1) = \mathbf{g}(\mathbf{t}_2)$. Let $\mathbf{h} = \mathbf{t}_2 - \mathbf{t}_1$, $f(\mathbf{t}) = [\mathbf{g}(\mathbf{t}) - \mathbf{g}(\mathbf{t}_1)] \cdot \mathbf{h}$, and apply the mean value theorem to f.] This result is due to H. Nikaidô.

7. Suppose that $\mathbf{g}$ is of class $C^{(1)}$, with $J\mathbf{g}(\mathbf{t}) \neq 0$, for all $\mathbf{t} \in \Delta$. Given $\mathbf{x} \notin \mathbf{g}(\Delta)$, let $\psi(\mathbf{t}) = |\mathbf{x} - \mathbf{g}(\mathbf{t})|^2$. Show that $d\psi(\mathbf{t}) \neq \mathbf{0}$ for all $\mathbf{t} \in \Delta$.

8. Let $\mathbf{g}$ be of class $C^{(1)}$. Suppose that there is a number $c > 0$ such that $|\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})| \geq c|\mathbf{s} - \mathbf{t}|$ for all $\mathbf{s}, \mathbf{t} \in E^n$. Show that:
   (a) $\mathbf{g}$ is univalent.
   (b) $J\mathbf{g}(\mathbf{t}) \neq 0$ for all $\mathbf{t} \in E^n$.
   (c) $\mathbf{g}(E^n) = E^n$. [*Hint*: Use Problem 7, and show that $\psi(\mathbf{t})$ has a minimum.]

4.6 The implicit function theorem

In elementary calculus there is a principle, often carelessly stated, that an equation $\Phi(x, y) = 0$ "implicitly determines one of the variables x or y as a function of the other." Actually, this statement is correct in a neighborhood U of any point (x_0, y_0) such that $\Phi(x_0, y_0) = 0$ and at least one of the partial

derivatives $\Phi_1(x_0, y_0)$, $\Phi_2(x_0, y_0)$ is not 0. This is a special case of the implicit function theorem below.

More generally, suppose that $1 \leq m < n$. Consider a transformation

$$\mathbf{\Phi} = (\Phi^1, \ldots, \Phi^m) \tag{4.24}$$

from an open set $D \subset E^n$ into E^m. The implicit function theorem ensures that the equation $\mathbf{\Phi}(\mathbf{x}) = \mathbf{0}$ determines m components of $\mathbf{x}$ as functions of the remaining $n - m$ components, in a neighborhood of any $\mathbf{x}_0$ such that $\mathbf{\Phi}(\mathbf{x}_0) = \mathbf{0}$ and $D\mathbf{\Phi}(\mathbf{x}_0)$ has maximum rank m.

The matrix of $D\mathbf{\Phi}(\mathbf{x}_0)$ is $m \times n$, with elements the partial derivatives $\Phi^i_j(\mathbf{x}_0)$, $i = 1, \ldots, m, j = 1, \ldots, n$.

If $D\mathbf{\Phi}(\mathbf{x}_0)$ has maximum rank m, then some set of m columns of its matrix is linearly independent. For the present, let us assume that the last m columns are linearly independent. Let $r = n - m$, and let

$$\tilde{J}\mathbf{\Phi}(\mathbf{x}) = \frac{\partial(\Phi^1, \ldots, \Phi^m)}{\partial(x^{r+1}, \ldots, x^n)} \tag{4.25}$$

denote the determinant of the sub matrix with these columns. We also let

$$\hat{\mathbf{x}} = (x^1, \ldots, x^r), \qquad \hat{\mathbf{x}}_0 = (x^1_0, \ldots, x^r_0)$$

denote the vectors obtained by taking only the first r components of $\mathbf{x}$ and $\mathbf{x}_0$. We seek to write the remaining components as $x^{r+l} = \phi^l(\hat{\mathbf{x}})$ for $l = 1, \ldots, m$, and for $\hat{\mathbf{x}}$ in some open set R containing $\hat{\mathbf{x}}_0$. When this is possible we write for brevity $\boldsymbol{\phi} = (\phi^1, \ldots, \phi^m)$ and $\mathbf{x} = (\hat{\mathbf{x}}, \boldsymbol{\phi}(\hat{\mathbf{x}}))$.

Implicit function theorem. *Let $\mathbf{\Phi}$ be of class $C^{(q)}$ from an open set $D \subset E^n$ into E^m, where $q \geq 1$ and $1 \leq m < n$. Let $\mathbf{x}_0 \in D$ be such that $\mathbf{\Phi}(\mathbf{x}_0) = \mathbf{0}$ and $\tilde{J}\mathbf{\Phi}(\mathbf{x}_0) \neq 0$. Then there exist a neighborhood U of $\mathbf{x}_0$, an open set $R \subset E^r$ containing $\hat{\mathbf{x}}_0$, and $\boldsymbol{\phi} = (\phi^1, \ldots, \phi^m)$ of class $C^{(q)}$ on R such that:*

$$\tilde{J}\mathbf{\Phi}(\mathbf{x}) \neq 0 \quad \textit{for all } \mathbf{x} \in U;$$

and

$$\{\mathbf{x} \in U : \mathbf{\Phi}(\mathbf{x}) = \mathbf{0}\} = \{(\hat{\mathbf{x}}, \boldsymbol{\phi}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in R\}.$$

PROOF. Since $\mathbf{\Phi}$ is at least of class $C^{(1)}$, the Jacobian $\tilde{J}\mathbf{\Phi}$ is a continuous function. By assumption it is not zero at $\mathbf{x}_0$, and therefore is not zero for $\mathbf{x}$ in some neighborhood U_0 of $\mathbf{x}_0$.

Let us consider the transformation $\mathbf{f}$, with domain U_0 and values in E^n,

$$\begin{aligned} f^i(\mathbf{x}) &= x^i, & i &= 1, \ldots, r, \\ f^{r+l}(\mathbf{x}) &= \Phi^l(\mathbf{x}), & l &= 1, \ldots, m, \quad m + r = n. \end{aligned}$$

The transformation $\mathbf{f}$ is of class $C^{(q)}$. Its matrix of partial derivatives is

$$\left(\begin{array}{cccc|ccc} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & & \dots & \dots & \vdots \\ \vdots & & & & & & \vdots \\ 0 & & \dots & 1 & 0 & \dots & 0 \\ \hline \Phi_1^1 & & \dots & \Phi_r^1 & \Phi_{r+1}^1 & \dots & \Phi_n^1 \\ \vdots & & & & & & \vdots \\ \Phi_1^m & & \dots & \Phi_r^m & \Phi_{r+1}^m & \dots & \Phi_n^m \end{array}\right)$$

By properties of determinants, the Jacobian $J\mathbf{f}(\mathbf{x})$ equals the determinant $\tilde{J}\boldsymbol{\Phi}(\mathbf{x})$ of the $m \times m$ block in the lower right-hand corner. Therefore $J\mathbf{f}(\mathbf{x}) \neq 0$. By the inverse function theorem, there is a neighborhood U of $\mathbf{x}_0$ such that $\mathbf{f}(U)$ is an open set and the restriction $\mathbf{f}|U$ has an inverse $\mathbf{g}$ of class $C^{(q)}$. Note here that the roles of the symbols $\mathbf{f}$ and $\mathbf{g}$ in Section 4.5 have been reversed.

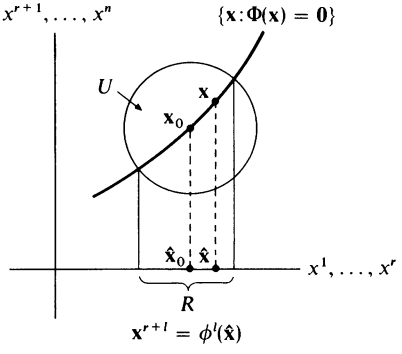


Figure 4.5

Writing $(\hat{\mathbf{x}}, \mathbf{0})$ for $(x^1, \dots, x^r, 0, \dots, 0)$, let (see Figure 4.5)

$$R = \{\hat{\mathbf{x}} : (\hat{\mathbf{x}}, \mathbf{0}) \in \mathbf{f}(U)\}.$$

Since $\mathbf{f}(U)$ is an open set, R is open. For every $\hat{\mathbf{x}} \in R$, let

$$\phi^l(\hat{\mathbf{x}}) = g^{r+l}(\hat{\mathbf{x}}, \mathbf{0}), \qquad l = 1, \dots, m.$$

Then $\mathbf{x} \in U$ and $\boldsymbol{\Phi}(\mathbf{x}) = \mathbf{0}$ if and only if $\hat{\mathbf{x}} \in R$ and $\mathbf{f}(\mathbf{x}) = (\hat{\mathbf{x}}, \mathbf{0})$. Since $\mathbf{f}|U$ and $\mathbf{g}$ are inverses, $\mathbf{f}(\mathbf{x}) = (\hat{\mathbf{x}}, \mathbf{0})$ if and only if $\mathbf{x} = \mathbf{g}(\hat{\mathbf{x}}, \mathbf{0})$. □

EXAMPLE 1. Let $\Phi(x, y) = x^2 - y^2 - 1$. The set $H = \{(x, y) : \Phi(x, y) = 0\}$ is a hyperbola. If $(x_0, y_0) \in H$ and $y_0 \neq 0$, then $\Phi_2(x_0, y_0) = -2y_0 \neq 0$. The implicit function theorem states that (x_0, y_0) has a neighborhood U such that $H \cap U = \{(x, \phi(x)) : x \in R\}$, where R is an open interval containing x_0 and ϕ is a suitable function. In this example, ϕ can be found explicitly by

solving the equation $\Phi(x, y) = 0$ for y. We get $\phi(x) = \pm(x^2 - 1)^{1/2}$, the sign depending on whether $y_0 > 0$ or $y_0 < 0$. The interval R is the projection of $H \cap U$ onto the x-axis.

If $y_0 = 0$, $x_0 = \pm 1$, then $\Phi_2(x_0, y_0) = 0$ and the reasoning above fails. However, at these points, $\Phi_1(x_0, y_0) \neq 0$ and (x_0, y_0) has a neighborhood U such that $H \cap U = \{(\psi(y), y) : y \in R\}$, where $\psi(y) = \pm(y^2 + 1)^{1/2}$ and R the projection of $H \cap U$ onto the y axis.

The partial derivatives of $\phi^1, \ldots, \phi^m$ in the implicit function theorem can be calculated in terms of those of $\Phi^1, \ldots, \Phi^m$ by means of the chain rule and Cramer's rule. We illustrate the technique in two special cases. Let us suppose that $q \geq 2$.

We first consider $n = 3, m = 1$. Suppose as in the implicit function theorem that $\Phi(x_0, y_0, z_0) = 0$ and $\Phi_3(x_0, y_0, z_0) \neq 0$. Then there exist an open set R containing (x_0, y_0) and ϕ such that

$$\Phi[x, y, \phi(x, y)] = 0, \tag{*}$$

and $\Phi_3[x, y, \phi(x, y)] \neq 0$ for every $(x, y) \in R$. Applying the chain rule to (*), we get

$$\begin{aligned} \Phi_1 + \Phi_3\phi_1 = 0, \qquad \phi_1 = -\frac{\Phi_1}{\Phi_3}, \\ \Phi_2 + \Phi_3\phi_2 = 0, \qquad \phi_2 = -\frac{\Phi_2}{\Phi_3}. \end{aligned} \tag{**}$$

In the formulas (**) the partial derivatives of Φ are evaluated at $(x, y, \phi(x, y))$.

To calculate the second-order partial derivatives $\phi_{11}, \phi_{12}, \phi_{22}$, the chain rule is applied again. For instance, taking the partial derivative with respect to the second variable in the first equations (**), we get

$$\Phi_{12} + \Phi_{13}\phi_2 + [\Phi_{32} + \Phi_{33}\phi_2]\phi_1 + \Phi_3\phi_{12} = 0.$$

Substituting the expressions for ϕ_1, ϕ_2 obtained above and solving for ϕ_{12}, we get

$$\phi_{12} = -\frac{(\Phi_3)^2\Phi_{12} - \Phi_2\Phi_3\Phi_{13} - \Phi_1\Phi_3\Phi_{32} + \Phi_1\Phi_2\Phi_{33}}{(\Phi_3)^3}.$$

Let $m = 2$, $n = 3$, $r = n - m = 1$. Writing $\boldsymbol{\Phi} = (\Phi, \Psi)$ rather than (Φ^1, Φ^2), and ϕ, ψ rather than ϕ^1, ϕ^2, we have

$$\begin{aligned} \Phi[x, \phi(x), \psi(x)] = 0, \\ \Psi[x, \phi(x), \psi(x)] = 0, \end{aligned}$$

and $\Phi_2\Psi_3 - \Phi_3\Psi_2 \neq 0$ for every $x \in R$. The partial derivatives in question are evaluated at $(x, \phi(x), \psi(x))$. By the chain rule

$$\begin{aligned} \Phi_1 + \Phi_2\phi' + \Phi_3\psi' = 0, \\ \Psi_1 + \Psi_2\phi' + \Psi_3\psi' = 0, \end{aligned}$$

and by Cramer's rule

$$\phi' = \frac{\Phi_3\Psi_1 - \Phi_1\Psi_3}{\Phi_2\Psi_3 - \Phi_3\Psi_2}, \qquad \psi' = \frac{\Phi_1\Psi_2 - \Phi_2\Psi_1}{\Phi_2\Psi_3 - \Phi_3\Psi_2}.$$

The second derivatives ϕ'', ψ'' can be found by another application of the chain rule.

For convenience we assumed in (4.25) that the last m columns of the $m \times n$ matrix of partial derivatives $(\Phi_i^j(\mathbf{x}_0))$ were linearly independent. More generally, one need merely suppose that some set of m columns is linearly independent, in other words, that the linear transformation $D\mathbf{\Phi}(\mathbf{x}_0)$ has maximum rank m. Let us suppose that columns $j_1, j_2, \ldots, j_m$ form a linearly independent set, where we may suppose that $j_1 < j_2 < \cdots < j_m$. Let $i_1, \ldots, i_r$ be those integers between 1 and n not included among $j_1, \ldots, j_m$, with $i_1 < \cdots < i_r$. For brevity let us write λ for the r-tuple of integers $(i_1, \ldots, i_r)$ and $\mathbf{x}^\lambda$ for the r-tuple $(x^{i_1}, \ldots, x^{i_r})$.

The implicit function theorem now states, roughly speaking, that locally the equation $\mathbf{\Phi}(\mathbf{x}) = \mathbf{0}$ determines $x^{j_1}, \ldots, x^{j_m}$ as functions of $\mathbf{x}^\lambda$. More precisely, U, R, and $\phi^1, \ldots, \phi^m$ exist as before such that

$$\frac{\partial(\Phi^1, \ldots, \Phi^m)}{\partial(x^{j_1}, \ldots, x^{j_m})} \neq 0 \quad \text{at every } \mathbf{x} \in U,$$

and

$$\{\mathbf{x} \in U : \mathbf{\Phi}(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} : \mathbf{x}^\lambda \in R,\ x^{j_l} = \phi^l(\mathbf{x}^\lambda) \text{ for } l = 1, \ldots, m\}.$$

In the case we considered above, $j_1, \ldots, j_m$ are the integers $r+1$, $r+2, \ldots, n$, $\lambda = (1, 2, \ldots, r)$, and then $\mathbf{x}^\lambda = \hat{\mathbf{x}}$.

EXAMPLE 2. Suppose that $m = 2$, $n = 5$, and

$$\frac{\partial(\Phi^1, \Phi^2)}{\partial(x^1, x^4)} \neq 0 \quad \text{at } \mathbf{x}_0.$$

Then we can take $j_1 = 1, j_2 = 4$, $\lambda = (2, 3, 5)$.

Let $m = 1$. Then $D\Phi(\mathbf{x}_0) = d\Phi(\mathbf{x}_0)$, and $D\Phi(\mathbf{x}_0)$ has maximum rank 1 if and only if at least one partial derivative $\Phi_j(\mathbf{x}_0)$ is not zero. If $\Phi_j(\mathbf{x}_0) \neq 0$, then we can take $j_1 = j$ and $\mathbf{x}^\lambda = (x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^n)$.

EXAMPLE 3. Let

$$\Phi(x, y, z) = x^2 + y^2 + z^2 - 2xz - 4.$$

Then

$$\Phi_1(x, y, z) = 2x - 2z,$$
$$\Phi_2(x, y, z) = 2y,$$
$$\Phi_3(x, y, z) = 2z - 2x.$$

If $d\Phi(x, y, z) = \mathbf{0}$, then $y = 0$, $x = z$, and $\Phi(x, y, z) = -4 \neq 0$. The implicit function theorem applies at any (x_0, y_0, z_0) where $\Phi(x_0, y_0, z_0) = 0$. If $x_0 \neq z_0$, then $\Phi_3(x_0, y_0, z_0) \neq 0$. We may take $j_1 = 3$, $\lambda = (1, 2)$, and proceed as above. We may equally well take $j_1 = 1$, $\lambda = (2, 3)$. However, if $x_0 = z_0$, then we must take $j_1 = 2$, $\lambda = (1, 3)$.

Problems

In each problem assume that Φ is of class $C^{(2)}$.

1. Let $\Phi[x, \phi(x)] = 0$ and $\Phi_2[x, \phi(x)] \neq 0$ for every $x \in R$. Find ϕ' and ϕ''.

2. Let $\Phi[\phi(y, z), y, z] = 0$ and $\Phi_1[\phi(y, z), y, z] \neq 0$ for every $(y, z) \in R$. Find ϕ_{11}.

3. Let $m = 2, n = 4, \Phi(\mathbf{x}) = (x^2)^2 + (x^4)^2 - 2x^1x^3, \Psi(\mathbf{x}) = (x^2)^3 + (x^4)^3 + (x^1)^3 - (x^3)^3$, and $\mathbf{\Phi} = (\Phi, \Psi)$. Let $\mathbf{x}_0 = (1, -1, 1, 1), j_1 = 1, j_2 = 3$.

(a) Show that the hypotheses of the implicit function theorem are satisfied.

(b) Write $\phi^1 = \phi$, $\phi^2 = \psi$, where according to the theorem,

$$(x^2)^2 + (x^4)^2 - 2\phi(x^2, x^4)\psi(x^2, x^4) = 0,$$
$$(x^2)^3 + (x^4)^3 + [\phi(x^2, x^4)]^3 - [\psi(x^2, x^4)]^3 = 0$$

for every $(x^2, x^4) \in R$. Find the first-order partial derivatives of ϕ and ψ at $\mathbf{x}_0^\lambda = (-1, 1)$.

4. Let $\Phi(x, y, z) = x^2 + 4y^2 - 2yz - z^2$, $\mathbf{x}_0 = 2\mathbf{e}_1 + \mathbf{e}_2 - 4\mathbf{e}_3$.

(a) Verify the hypotheses of the implicit function theorem.

(b) Find the largest neighborhood U of $\mathbf{x}_0$ such that $\Phi_3(x, y, z) \neq 0$ for every $(x, y, z) \in U$.

(c) Find the largest neighborhood of $\mathbf{x}_0$ containing no critical point of Φ.

5. Let $\Phi(x, y) = x^2 - y^2$, $\mathbf{x}_0 = (0, 0)$.

(a) Let U be any neighborhood of $(0, 0)$, of radius a, and $R = (-a/\sqrt{2}, a/\sqrt{2})$. Find a function ϕ such that $\Phi[x, \phi(x)] = 0$ for every $x \in R$.

(b) Show that no ϕ exists such that $\{(x, y) \in U: \Phi(x, y) = 0\} = \{(x, \phi(x)): x \in R\}$.

6. (a) Let $m = 2$. Let $\mathbf{\Phi} = (\Phi, \Psi)$, where $\Psi(\mathbf{x}) = \theta(\mathbf{x})\Phi(\mathbf{x})$ for every $\mathbf{x} \in D$ and θ is a real valued function. Show that $D\mathbf{\Phi}(\mathbf{x}_0)$ has rank less than 2 at any $\mathbf{x}_0$ such that $\mathbf{\Phi}(\mathbf{x}_0) = \mathbf{0}$.

(b) State and prove a corresponding result for $m > 2$.

7. Give an alternate proof of the implicit function theorem, in case $m = 1$, $n = 2$, by carrying out the following steps. Let Φ be of class $C^{(1)}$ and suppose that $\Phi(x_0, y_0) = 0$, $\Phi_2(x_0, y_0) \neq 0$. For definiteness assume that $\Phi_2(x_0, y_0) > 0$.

(a) Show that there exists $\varepsilon > 0$ such that $\Phi(x_0, y) < 0$ if $y_0 - \varepsilon \leq y < y_0$ and $\Phi(x_0, y) > 0$ if $y_0 < y \leq y_0 + \varepsilon$.

(b) Show that there exists $\delta > 0$ such that $\Phi(x, y_0 - \varepsilon) < 0$ and $\Phi(x, y_0 + \varepsilon) > 0$ if $|x - x_0| < \delta$.

(c) Let $I = \{(x, y): |x - x_0| < \delta, |y - y_0| < \varepsilon\}$. The numbers ε and δ in (a) and (b) may be so chosen that $\Phi_2(x, y) > 0$ for every $(x, y) \in I$. Show that if $|x_1 - x_0| < \delta$ the equation $\Phi(x_1, y) = 0$ has exactly one solution y_1 with $(x_1, y_1) \in I$. Set $y_1 = \phi(x_1)$. This defines ϕ on the open interval $(x_0 - \delta, x_0 + \delta)$.

(d) Show that ϕ is differentiable and that

$$\phi'(x) = -\frac{\Phi_1[x, \phi(x)]}{\Phi_2[x, \phi(x)]}.$$

In this proof of the theorem, the rectangle I replaces the circular neighborhood U, but this is unimportant. Can you extend this proof to the case $m = 1, n > 2$?

4.7 Manifolds

The word *manifold* is used in mathematics to describe a topological space that locally is "like" euclidean E^r, for some r called the dimension of the manifold. For instance, a circle is locally like E^1. Such geometric figures in E^3 as ellipsoids, cylinders, and tori are locally like E^2. A cone is not locally like E^2 near its vertex.

We approach the idea of manifold from a rather concrete viewpoint. For us, a manifold M is a subset of some euclidean E^n that can locally be described by an equation $\mathbf{\Phi}(\mathbf{x}) = \mathbf{0}$, where $D\mathbf{\Phi}(\mathbf{x})$ must have maximum rank. Another definition of manifold can be given abstractly in terms of coordinate systems. It has the advantage that one need not presuppose that M is a subset of some euclidean space. This will be discussed in Chapter 8.

Definition. Let $1 \leq r < n, q \geq 1$. A nonempty set $M \subset E^n$ is a *manifold of dimension r and class* $C^{(q)}$ if M has the property that for every $\mathbf{x}_0 \in M$ there exist a neighborhood U of $\mathbf{x}_0$ and $\mathbf{\Phi} = (\Phi^1, \ldots, \Phi^{n-r})$ of class $C^{(q)}$ on U, such that $D\mathbf{\Phi}(\mathbf{x})$ has rank $n - r$ for every $\mathbf{x} \in U$ and

$$M \cap U = \{\mathbf{x} \in U : \mathbf{\Phi}(\mathbf{x}) = \mathbf{0}\}.$$

Throughout the following discussion we take $q = 1$. For brevity we say *r-manifold* instead of "manifold of dimension r and class $C^{(1)}$." If $r = n$, let us call any open subset of E^n an *n-manifold*.

Let us indicate how an r-manifold M is locally like E^r. First assume that $\tilde{J}\mathbf{\Phi}(\mathbf{x}_0) \neq 0$ with $\tilde{J}\mathbf{\Phi}(\mathbf{x})$ as in (4.25), and define $\mathbf{f}$ as in the proof of the implicit function theorem. The neighborhood of $\mathbf{x}_0$ chosen in Section 4.6 need not coincide with the neighborhood U in the definition of manifold. In this section let us denote the former neighborhood by U_1 rather than U, and let us denote the set R in Section 4.6 by R_1. We may suppose that $U_1 \subset U$. Now $\mathbf{f} \mid U_1$ is a regular transformation (Section 4.5), and

$$\mathbf{f}(M \cap U_1) = \{(\hat{\mathbf{x}}, \mathbf{0}) : \hat{\mathbf{x}} \in R_1\}$$

is a relatively open subset of the r-dimensional subspace of E^n spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_r$. Therefore it is reasonable to say that $M \cap U_1$ is "like" E^r (Figure 4.6). In case $\tilde{J}\mathbf{\Phi}(\mathbf{x}_0) = 0$, one must replace the r-tuple of integers $1, 2, \ldots, r$ by some other r-tuple λ, as indicated in Section 4.6.

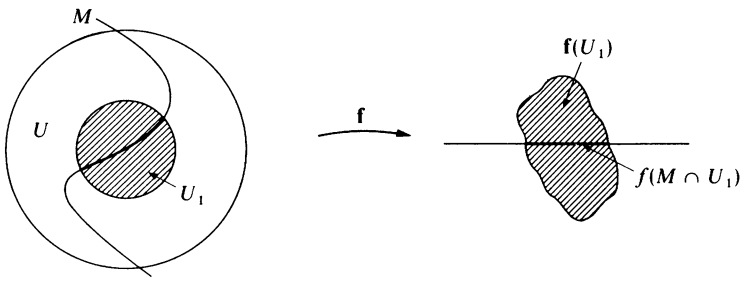


Figure 4.6

EXAMPLE 1. Let $n = 2, r = 1$, and H be the hyperbola $x^2 - y^2 = 1$. To show that H is a 1-manifold, let us take $\Phi(x, y) = x^2 - y^2 - 1$. Then $D\Phi(x, y) = d\Phi(x, y) = 2x\mathbf{e}^1 - 2y\mathbf{e}^2$. If $(x, y) \neq (0, 0)$, then $d\Phi(x, y) \neq \mathbf{0}$ and the rank is 1. Given $(x_0, y_0) \in H$, let U be any neighborhood of (x_0, y_0) that does not contain $(0, 0)$. In this example the choice of Φ does not depend on (x_0, y_0).

EXAMPLE 2. Let M be the union of H and one of its asymptotes,

$$M = \{(x, y) : x^2 - y^2 = 1\} \cup \{(x, y) : y = x\}.$$

To show that M is a 1-manifold we must show that given $(x_0, y_0) \in M$, there exist U and Φ such that $d\Phi(x, y) \neq \mathbf{0}$ in U and

$$M \cap U = \{(x, y) \in U : \Phi(x, y) = 0\}.$$

If $(x_0, y_0) \in H$, then we let $\Phi(x, y) = x^2 - y^2 - 1$ as before, and let U be any neighborhood of (x_0, y_0) that does not meet the asymptote $y = x$. However, if (x_0, y_0) is on the asymptote, we take $\Phi(x, y) = y - x$ and U any neighborhood of (x_0, y_0) that does not meet H. In this example our choice of Φ depends on (x_0, y_0).

EXAMPLE 3. Let $M = \{(x, y) : x^2 = y^2\}$. This set consists of the two lines $y = \pm x$, and is not a 1-manifold. Roughly speaking, M is not like E^1 near the crossing point $(0, 0)$. More precisely, if M were a 1-manifold, then by the implicit function theorem the following would be true: Each $(x_0, y_0) \in M$ has a neighborhood U_1 such that either $M \cap U_1 = \{(x, \phi(x)) : x \in R_1\}$ or $M \cap U_1 = \{(\psi(y), y) : y \in R_2\}$, where R_1, R_2 are open. In the present example, $(0, 0)$ has no such neighborhood.

Most examples of manifolds that we consider are obtained in the following way. Let $\mathbf{\Phi}$ be a transformation of class $C^{(1)}$, from an open set $D \subset E^n$ into E^m. Let

$$M = \{\mathbf{x} : \mathbf{\Phi}(\mathbf{x}) = \mathbf{0} \text{ and } D\mathbf{\Phi}(\mathbf{x}) \text{ has rank } m\}. \tag{4.26}$$

If M is not empty, then it is an r-manifold, where $m = n - r$. To show that M is an r-manifold, in the definition of manifold let us choose this same $\mathbf{\Phi}$ for

every $\mathbf{x}_0 \in M$. By Problem 11, Section 4.3, $\{\mathbf{x} : D\mathbf{\Phi}(\mathbf{x})$ has rank $m\}$ is open. Hence any $\mathbf{x}_0 \in M$ has a neighborhood U such that $D\mathbf{\Phi}(\mathbf{x})$ has rank m for every $\mathbf{x} \in U$, and $M \cap U = \{\mathbf{x} \in U : \mathbf{\Phi}(\mathbf{x}) = \mathbf{0}\}$.

Definition. When (4.26) holds, we say that M is the *r-manifold determined by* $\mathbf{\Phi}$.

EXAMPLE 4. The $(n-1)$-sphere $\{\mathbf{x} : |\mathbf{x}| = 1\}$ is an $(n-1)$-manifold. In fact, it is the $(n-1)$-manifold determined by Φ, where $\Phi(\mathbf{x}) = |\mathbf{x}|^2 - 1$. The only critical point of Φ is $\mathbf{0}$, which is not on the $(n-1)$-sphere.

EXAMPLE 5. Let F be real-valued and of class $C^{(1)}$. Consider a level set

$$B_c = \{\mathbf{x} : F(\mathbf{x}) = c\}.$$

Let $\Phi(\mathbf{x}) = F(\mathbf{x}) - c$. Then $d\Phi(\mathbf{x}) = dF(\mathbf{x})$. If B_c is not empty and contains no critical points of F, then B_c is the $(n-1)$-manifold determined by Φ. If B_c contains critical points, then the $(n-1)$-manifold determined by Φ is

$$M_c = B_c - (\text{set of critical points of } F),$$

unless M_c happens to be empty. We saw in Section 3.5 that B_c need not resemble E^{n-1} near a critical point contained in B_c.

EXAMPLE 6. Let $F(x, y) = \exp(xy)$. The partial derivatives are $F_1(x, y) = y\exp(xy)$ and $F_2(x, y) = x\exp(xy)$. The only critical point is $(0, 0)$, and $F(0, 0) = 1$. If $c \leq 0$, then B_c is empty. If $c > 0, c \neq 1$, then B_c is a 1-manifold. In fact, B_c is the hyperbola $xy = \log c$. The level set B_1 is the union of the x- and y-axes, and is not a 1-manifold. $M_1 = B_1 - \{(0, 0)\}$ is a 1-manifold.

EXAMPLE 7. As in Example 5, if $\mathbf{F}$ is of class $C^{(1)}$ and has values in E^m, $m = n - r$, then

$$M_c = \{\mathbf{x} : \mathbf{F}(\mathbf{x}) = \mathbf{c} \text{ and } D\mathbf{F}(\mathbf{x}) \text{ has rank } m\}$$

is either empty or an r-manifold.

Tangent vectors to a manifold

Let M be a manifold and $\mathbf{x}_0 \in M$.

Definition. A vector $\mathbf{h}$ is a *tangent vector* to M at $\mathbf{x}_0$ if there exists a function $\boldsymbol{\psi}$ from an interval $(-\delta, \delta)$ into M such that $\boldsymbol{\psi}(0) = \mathbf{x}_0$ and $\boldsymbol{\psi}'(0) = \mathbf{h}$.

The definition can be restated in a way that is more appealing geometrically. For brevity let us set $\mathbf{x}_t = \mathbf{x}_0 + t\mathbf{h}$. Then $\mathbf{h}$ is a tangent vector if, for some $\delta > 0$, there exists $\mathbf{y}_t \in M$ whenever $0 < |t| < \delta$, such that

$$(*) \qquad \lim_{t \to 0} \frac{|\mathbf{y}_t - \mathbf{x}_t|}{|t|} = 0.$$

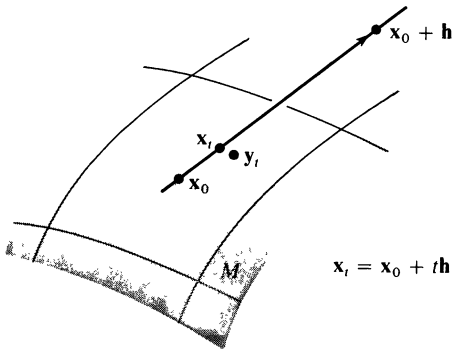


Figure 4.7

If we set $\psi(t) = \mathbf{y}_t$ for $0 < |t| < \delta$, and $\psi(0) = \mathbf{x}_0$, then (*) states that $\psi'(0) = \mathbf{h}$ (see Figure 4.7).

Let $T(\mathbf{x}_0)$ denote the set of all tangent vectors at $\mathbf{x}_0$. It is called the *tangent space* to M at $\mathbf{x}_0$. If r is the dimension of M, then it is plausible that the tangent space is a vector space of dimension r. Let us show that this is true. Let U and Φ be the same as in the definition of manifold.

Theorem 4.3. *The tangent space $T(\mathbf{x}_0)$ is the kernel of the linear transformation $D\Phi(\mathbf{x}_0)$.*

Since $D\Phi(\mathbf{x}_0)$ has rank m, the kernel $T(\mathbf{x}_0)$ is a vector subspace of E^n with dimension $r = n - m$.

PROOF OF THEOREM 4.3. Let $\mathbf{L} = D\Phi(\mathbf{x}_0)$. We must show that $\mathbf{h}$ is a tangent vector if and only if $\mathbf{L}(\mathbf{h}) = \mathbf{0}$.

Let $\mathbf{h} \in T(\mathbf{x}_0)$. Let ψ be as in the definition of tangent vector. Then $\Phi[\psi(t)] = \mathbf{0}$ for every $t \in (-\delta, \delta)$. Calculating the derivative of $\Phi \circ \psi$ by Corollary 2, Section 4.4, we have

$$\mathbf{0} = \mathbf{L}[\psi'(0)] = \mathbf{L}(\mathbf{h}).$$

Conversely, let $\mathbf{L}(\mathbf{h}) = \mathbf{0}$. For simplicity let us assume, as in Section 4.6, that the last m columns of the matrix of $\mathbf{L}$ are linearly independent. Let $\mathbf{f}$ be as in the proof of the implicit function theorem, and let U_1 be a neighborhood of $\mathbf{x}_0$ such that the restriction of $\mathbf{f}$ to U_1 is regular. There exists $\delta > 0$ such that $\hat{\mathbf{x}}_0 + t\hat{\mathbf{h}} \in R_1$ for every $t \in (-\delta, \delta)$. Let $\mathbf{g} = (\mathbf{f} \mid U_1)^{-1}$ and

$$\psi(t) = \mathbf{g}(\hat{\mathbf{x}}_0 + t\hat{\mathbf{h}}, \mathbf{0}).$$

Then $\psi(t) \in M$ and ψ is of class $C^{(1)}$ (Figure 4.8). We must show that $\psi'(0) = \mathbf{h}$.

Let $\Lambda = D\mathbf{f}(\mathbf{x}_0)$. Then Λ^{-1} is the differential of $\mathbf{g}$ at the point $(\hat{\mathbf{x}}_0, \mathbf{0}) = \mathbf{f}(\mathbf{x}_0)$. Therefore

$$\psi'(0) = \Lambda^{-1}(\hat{\mathbf{h}}, \mathbf{0}).$$

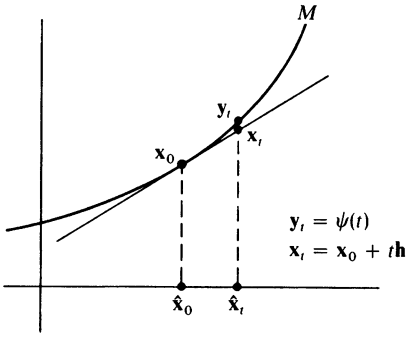


Figure 4.8

By definition of $\mathbf{f}$,

$$\Lambda^i(\mathbf{h}) = h^i, \qquad i = 1, \ldots, r,$$
$$\Lambda^{l+r}(\mathbf{h}) = d\Phi^l(\mathbf{x}_0) \cdot \mathbf{h}, \qquad l = 1, \ldots, m.$$

Since $\mathbf{h}$ is in the kernel of $D\mathbf{\Phi}(\mathbf{x}_0)$, $\Lambda^{l+r}(\mathbf{h}) = 0$. Thus $\mathbf{\Lambda}(\mathbf{h}) = (\hat{\mathbf{h}}, \mathbf{0})$, and

$$\mathbf{h} = \mathbf{\Lambda}^{-1}(\hat{\mathbf{h}}, \mathbf{0}) = \psi'(0). \qquad \square$$

Normal vectors to a manifold

A vector $\mathbf{n}$ is called *normal* to M at $\mathbf{x}_0$ if $\mathbf{n} \cdot \mathbf{h} = 0$ for every $\mathbf{h} \in T(\mathbf{x}_0)$. The normal vectors form a vector space of dimension $m = n - r$, the orthogonal complement of the tangent space $T(\mathbf{x}_0)$. The gradient $\operatorname{grad} \Phi^l(\mathbf{x}_0)$ is the vector with the same components as the covector $d\Phi^l(\mathbf{x}_0)$. By Theorem 4.3

$$\operatorname{grad} \Phi^l(\mathbf{x}_0) \cdot \mathbf{h} = d\Phi^l(\mathbf{x}_0) \cdot \mathbf{h} = 0$$

for every $\mathbf{h} \in T(\mathbf{x}_0)$ and $l = 1, \ldots, m$. Hence $\operatorname{grad} \Phi^1(\mathbf{x}_0), , \ldots, \operatorname{grad} \Phi^m(\mathbf{x}_0)$ are normal vectors to M at $\mathbf{x}_0$. Since $D\mathbf{\Phi}(\mathbf{x}_0)$ has rank m, these vectors are linearly independent.

Thus, we have proved the following.

Corollary 1. *The gradients* $\operatorname{grad} \Phi^1(\mathbf{x}_0), \ldots, \operatorname{grad} \Phi^m(\mathbf{x}_0)$ *form a basis for the space of normal vectors to* M *at* $\mathbf{x}_0$.

In particular, if M is an $(n-1)$-manifold, then $m = 1$; $\operatorname{grad} \Phi(\mathbf{x}_0)$ is a normal vector, and all others are scalar multiples of it.

Tangent r-planes

The *r-plane tangent to* M *at* $\mathbf{x}_0$ is

$$\{\mathbf{x} : \mathbf{x} = \mathbf{x}_0 + \mathbf{h}, \mathbf{h} \in T(\mathbf{x}_0)\}$$

(see Figure 4.9). The terms *tangent line*, *tangent plane*, and *tangent hyperplane* are used when $r = 1, 2$, and $n - 1$, respectively. The tangent r-plane is

$$\{\mathbf{x} : \operatorname{grad} \Phi^l(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0 \quad \text{for } l = 1, \ldots, m\}.$$

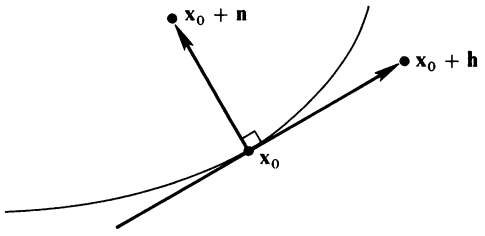


Figure 4.9

EXAMPLE 8. Let

$$M = \{(x, y, z) : x^2 + y^2 + z^2 - 2xz - 4 = 0\}.$$

According to Example 3, Section 4.6, M is the 2-manifold determined by the function Φ of that example. Then

$$\operatorname{grad} \Phi(x, y, z) = 2(x - z)\mathbf{e}_1 + 2y\mathbf{e}_2 + 2(z - x)\mathbf{e}_3.$$

Let us find the spaces of tangent and normal vectors to M at $(2, \sqrt{3}, 1)$.

$$\operatorname{grad} \Phi(2, \sqrt{3}, 1) = 2\mathbf{e}_1 + 2\sqrt{3}\mathbf{e}_2 - 2\mathbf{e}_3.$$

This is a normal vector, and any other is a scalar multiple of it. The tangent vectors $\mathbf{h}$ satisfy

$$0 = \operatorname{grad} \Phi(2, \sqrt{3}, 1) \cdot \mathbf{h} = 2h^1 + 2\sqrt{3}h^2 - 2h^3.$$

Two linearly independent solutions of this equation are $\mathbf{e}_1 + \mathbf{e}_3$ and $\sqrt{3}\mathbf{e}_1 - \mathbf{e}_2$. These vectors form a basis for $T(2, \sqrt{3}, 1)$. The equation of the tangent plane is $2(x - 2) + 2\sqrt{3}(y - \sqrt{3}) - 2(z - 1) = 0$, or $x + \sqrt{3}y - z = 4$.

EXAMPLE 9. Let $M \subset E^3$ be a 1-manifold. Let us find the tangent space at $(x_0, y_0, z_0) \in M$. Let us write $\mathbf{\Phi} = (\Phi, \Psi)$. The tangent vectors satisfy

$$0 = \operatorname{grad} \Phi(x_0, y_0, z_0) \cdot \mathbf{h} = \operatorname{grad} \Psi(x_0, y_0, z_0) \cdot \mathbf{h}.$$

From Cramer's rule one solution of this pair of linear equations is the vector $\mathbf{d}$ with components (Problem 8)

$$d^1 = \Phi_2\Psi_3 - \Phi_3\Psi_2, \qquad d^2 = \Phi_3\Psi_1 - \Phi_1\Psi_3, \qquad d^3 = \Phi_1\Psi_2 - \Phi_2\Psi_1,$$

the partial derivatives being evaluated at (x_0, y_0, z_0). The tangent space $T(x_0, y_0, z_0)$ consists of all scalar multiples of $\mathbf{d}$.

In Example 9 one can think of M as the intersection of two surfaces, obtained by setting $\Phi = 0$ and $\Psi = 0$ separately. Since grad $\Phi(x, y, z)$ and grad $\Psi(x, y, z)$ are normal to these surfaces and are linearly independent for

(x, y, z) near (x_0, y_0, z_0), the tangent planes cannot coincide at points on the intersection M.

This idea can be extended to the intersection of manifolds of other dimensions.

**Intersections of manifolds*

Let M be an r-manifold and N an s-manifold, with $M \cap N$ nonempty. Let us assume that $r + s > n$. Let $T_1(\mathbf{x}_0)$ denote the tangent space to M at a point $\mathbf{x}_0 \in M \cap N$, and $T_2(\mathbf{x}_0)$ the tangent space to N at $\mathbf{x}_0$. There exist a neighborhood U_1 of $\mathbf{x}_0$ and $\mathbf{\Phi} = (\Phi^1, \ldots, \Phi^{n-r})$ such that $D\mathbf{\Phi}(\mathbf{x})$ has rank $n - r$ for every $\mathbf{x} \in U_1$ and

$$M \cap U_1 = \{\mathbf{x} \in U_1 : \mathbf{\Phi}(\mathbf{x}) = \mathbf{0}\}.$$

In the same way, there exists a neighborhood U_2 of $\mathbf{x}_0$ and $\mathbf{\Psi} = (\Psi^1, \ldots, \Psi^{n-s})$ such that $D\mathbf{\Psi}(\mathbf{x})$ has rank $n - s$ for every $\mathbf{x} \in U_2$ and

$$N \cap U_2 = \{\mathbf{x} \in U_2 : \mathbf{\Psi}(\mathbf{x}) = \mathbf{0}\}.$$

Let $U = U_1 \cap U_2$, and $\mathbf{\Theta} = (\Phi^1, \ldots, \Phi^{n-r}, \Psi^1, \ldots, \Psi^{n-s})$. Then

$$(M \cap N) \cap U = \{\mathbf{x} \in U : \mathbf{\Theta}(\mathbf{x}) = \mathbf{0}\}.$$

Let $T_1(x_0) \cap T_2(x_0)$ have dimension $r + s - n$ at each $x_0 \in M \cap N$. The kernel of $D\mathbf{\Theta}(\mathbf{x}_0)$ is $T_1(\mathbf{x}_0) \cap T_2(\mathbf{x}_0)$. Hence $D\mathbf{\Theta}(\mathbf{x}_0)$ has the desired rank $n - (r + s - n) = (n - r) + (n - s)$. From the definition, $M \cap N$ is an $(r + s - n)$-manifold.

EXAMPLE 10. Let $n = 3, r = s = 2$. If the tangent planes of M and N do not coincide at any point of $M \cap N$, then $M \cap N$ is a 1-manifold. The tangent line at $\mathbf{x}_0 \in M \cap N$ is the intersection of the tangent planes to M and N at $\mathbf{x}_0$.

PROBLEMS

In Problems 1–6 and 11, one can show that the set in question is a manifold by verifying (4.26) for suitable $\mathbf{\Phi}$.

1. Let $F(x, y) = \exp(x^2 + 2y^2 + 2)$. Find the level sets and determine which are 1-manifolds.

2. (a) Show that if $c \neq 0$, then the hyperboloid $x^2 + y^2 - 4z^2 = c$ is a 2-manifold. Is the cone $x^2 + y^2 = 4z^2$ a 2-manifold?
   (b) Find the tangent plane at $2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$ to the hyperboloid $x^2 + y^2 - 4z^2 = 1$.

3. Let $M = \{(x, y, z) : xy = 0, x^2 + y^2 + z^2 = 1, z \neq \pm 1\}$. Show that M is a 1-manifold. Sketch M.

4. Let f be of class $C^{(1)}$ on an open set $A \subset E^2$. Let $M = \{(x, y, f(x, y)) : (x, y) \in A\}$.
   (a) Show that M is a 2-manifold.
   (b) Show that $(f_1(x, y), f_2(x, y), -1)$ is a normal vector to M at $(x, y, f(x, y))$.
   (c) Show that the equation for a tangent plane agrees with the one in Section 3.3.
   (d) State the corresponding results when f is a function of n variables.

5. Let $A \subset E^1$ be open, and f, g be real valued functions of class $C^{(1)}$ on A.
(a) Show that $M = \{(x, f(x), g(x)) : x \in A\}$ is a 1-manifold.
(b) Show that $(1, f'(x), g'(x))$ is a tangent vector to M at $(x, f(x), g(x))$.

6. Let $M = \{(x, y) : x^y = y^x, x > 0, y > 0, (x, y) \neq (e, e)\}$, where e is the base for natural logarithms. Show that M is a 1-manifold. Make a sketch.

7. Let $M = \{(x, y, z) : xy = xz = 0\}$. Is M a 1-manifold?

8. In Example 9 show that $\mathbf{d}$ is a tangent vector, using Cramer's rule.

9. Let M and N be r-manifolds such that (cl M) $\cap$ N and $M \cap$ cl N are empty, $M \subset E^n$, $N \subset E^n$. Prove that $M \cup N$ is an r-manifold.

10. Let M be an r-manifold and A an open set such that $M \cap A$ is not empty. Prove that $M \cap A$ is an r-manifold.

11. Let $M = \{\mathbf{x} : \sum_{i,j=1}^{n} c_{ij} x^i x^j = 1\}$, where the matrix (c_{ij}) has rank n and is symmetric.
(a) Show that M is an $(n - 1)$-manifold.
(b) Show that the equation of the tangent hyperplane at $\mathbf{x}_0 \in M$ is

$$\sum_{i,j=1}^{n} c_{ij} x^i x_0^j = 1.$$

12. (*Product manifolds.*) Let $M \subset E^n$ be an r-manifold and $N \subset E^m$ an s-manifold. Regarding E^{m+n} as the cartesian product $E^n \times E^m$, show that $M \times N$ is an $(r + s)$-manifold. Show that the tangent space at a point of $M \times N$ is the cartesian product of the tangent spaces at the corresponding points of M and N.

13. Let points of E^4 be denoted by (x, y, u, v). Let $C = \{\mathbf{x} : x^2 + y^2 = 1, u^2 + v^2 = 1\}$, $K = \{\mathbf{x} : x^2 + y^2 \leq 1, u^2 + v^2 \leq 1\}$, and $B = \text{fr } K$ (boundary of K).
(a) Show that C is a 2-manifold.
(b) Show that $B - C$ is a 3-manifold. [*Hint*: Use Problems 9 and 12.]

14. Let M, $\mathbf{\Phi}$, and U be as in the definition of "manifold." For each $l = 1, \ldots, m$, let $\Psi^l(\mathbf{x}) = g^l(\mathbf{x})\Phi^l(\mathbf{x})$, where g^l is of class $C^{(1)}$ and $g^l(\mathbf{x}_0) \neq 0$. Show that there is a neighborhood U_0 of $\mathbf{x}_0$ such that $D\mathbf{\Psi}(\mathbf{x})$ has rank m for every $\mathbf{x} \in U_0$ and $M \cap U_0 = \{\mathbf{x} \in U_0 : \mathbf{\Psi}(\mathbf{x}) = \mathbf{0}\}$. [*Hint*: Show that $d\Psi^1(\mathbf{x}_0), \ldots, d\Psi^m(\mathbf{x}_0)$ are linearly independent.]

15. Let f be a function of class $C^{(2)}$ on E^2, each of whose critical points is isolated. Suppose that $A = \{\mathbf{x} : f(\mathbf{x}) > 0\}$ is connected and contains two points of relative maximum. Show that A contains a saddle point.

16. Let us identify E^{n^2} with the set of all linear transformations from E^n into itself by associating with each linear transformation $\mathbf{L}$ the vector

$$\sum_{i,j=1}^{n} c_j^i \mathbf{e}_{i+(j-1)n},$$

where (c_j^i) is the matrix of $\mathbf{L}$. Let $O(n)$ be the set of orthogonal transformations of E^n.
(a) Show that $O(n)$ is a manifold of dimension $\frac{1}{2}n(n-1)$ and class $C^{(\infty)}$. [*Hints*: It suffices to show that $\mathbf{I}$ has a neighborhood U with the properties required in the definition of manifold. Use Proposition 4.1. Here $\mathbf{I}$ is the identity transformation.
(b) Show that $\mathbf{L}$ is a tangent vector to $O(n)$ at $\mathbf{I}$ if and only if $\mathbf{L}^t = -\mathbf{L}$. (Such a linear transformation is called *skew symmetric*.)

(c) Let $SO(n)$ be the set of all rotations of E^n about $\mathbf{0}$. Show that $SO(n)$ is a relatively open subset of $O(n)$, and hence $SO(n)$ is also a manifold of dimension $\frac{1}{2}n(n-1)$. Show that $SO(n)$ is the largest connected subset of $O(n)$ which contains $\mathbf{I}$. [*Hint*: Using induction on n, show that any $\mathbf{L} \in SO(n)$ can be joined with $\mathbf{I}$ be a path in $SO(n)$.]

4.8 The multiplier rule

Let M be a manifold and f be a real valued function of class $C^{(1)}$ on some open set containing M. Let us consider the problem of finding the extrema of the function $f \mid M$. This is called a problem of *constrained extrema.*

If $\mathbf{x}_0$ is a point of M at which f has a constrained relative maximum, then $\mathbf{x}_0$ has a neighborhood U_0 such that

$$f(\mathbf{x}) \leq f(\mathbf{x}_0) \quad \text{for every } \mathbf{x} \in M \cap U_0.$$

(recall the definition in Section 3.5). If f has a constrained relative minimum at $\mathbf{x}_0$, then the inequality sign is reversed.

Since M is a manifold, there exists a neighborhood U of $\mathbf{x}_0$ and $\mathbf{\Phi}$ of class $C^{(1)}$ on U such that

$$M \cap U = \{\mathbf{x} \in U : \mathbf{\Phi}(\mathbf{x}) = \mathbf{0}\}$$

and $D\mathbf{\Phi}(\mathbf{x})$ has maximum rank m for every $\mathbf{x} \in U$. We may assume that $U \subset U_0$.

Roughly speaking, the multiplier rule states that by introducing suitable multipliers $\sigma_1, \ldots, \sigma_m$ the constrained extremum problem can be treated as one of ordinary (unconstrained) extremum. More precisely:

Lagrange multiplier rule. *Let f have a constrained relative extremum at $\mathbf{x}_0$. Then there exist real numbers $\sigma_1, \ldots, \sigma_m$ such that $\mathbf{x}_0$ is a critical point of the function*

$$F = f + \sigma_1 \Phi^1 + \cdots + \sigma_m \Phi^m.$$

PROOF. It suffices to consider the case of a constrained relative maximum, Let $\mathbf{h}$ be any tangent vector to M at $\mathbf{x}_0$. Let $\phi(t) = f[\psi(t)]$, where ψ is the same as in the definition of tangent vector. Since $\psi(t) \in M$ and $f \mid M$ has a relative maximum at $\mathbf{x}_0$, ϕ has a relative maximum at 0. Therefore $\phi'(0) = 0$. By the chain rule,

$$\phi'(0) = \operatorname{grad} f[\psi(0)] \cdot \psi'(0) = \operatorname{grad} f(\mathbf{x}_0) \cdot \mathbf{h}.$$

This shows that $\operatorname{grad} f(\mathbf{x}_0)$ is a normal vector to M at $\mathbf{x}_0$. However, $\operatorname{grad} \Phi^1(\mathbf{x}_0), \ldots, \operatorname{grad} \Phi^m(\mathbf{x}_0)$ form a basis for the space of normal vectors to M at $\mathbf{x}_0$ (Corollary 1, Section 4.7). Therefore, $\operatorname{grad} f(\mathbf{x}_0)$ is a linear combination of these vectors:

$$\operatorname{grad} f(\mathbf{x}_0) = a_1 \operatorname{grad} \Phi^1(\mathbf{x}_0) + \cdots + a_m \operatorname{grad} \Phi^m(\mathbf{x}_0).$$

Let $\sigma_l = -a_l$ for $l = 1, \ldots, m$. Then $\operatorname{grad} F(\mathbf{x}_0) = \mathbf{0}$. □

EXAMPLE 1. Let $f(x, y, z) = x - y + 2z$. Let us find the maximum and minimum values of f on the ellipsoid

$$M = \{(x, y, z) : x^2 + y^2 + 2z^2 = 2\}.$$

Let $\Phi(x, y, z) = 2 - (x^2 + y^2 + 2z^2)$ and $F = f + \sigma\Phi$. The multiplier σ is yet to be determined. From the multiplier rule we get three equations

$$\begin{aligned} F_1 &= \quad 1 - 2\sigma x_0 = 0, \\ F_2 &= -1 - 2\sigma y_0 = 0, \\ F_3 &= \quad 2 - 4\sigma z_0 = 0. \end{aligned}$$

From these and the fourth equation $\Phi = 0$, we get

$$x_0 = \frac{1}{2\sigma}, \qquad y_0 = -\frac{1}{2\sigma}, \qquad z_0 = \frac{1}{2\sigma}, \qquad \frac{1}{\sigma} = \pm\sqrt{2}.$$

Therefore $\mathbf{x}_0 = \pm(\sqrt{2}/2)(\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)$, depending on which of the two possible values for σ is used. Since f is continuous and M is a compact set, f has a maximum and a minimum value on M. One of the two critical points obtained by the multiplier rule must give the maximum and the other the minimum. Since

$$f[(\sqrt{2}/2)(\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)] = 2\sqrt{2}$$

and

$$f[-(\sqrt{2}/2)(\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)] = -2\sqrt{2},$$

these numbers are the maximum and minimum values, respectively.

EXAMPLE 2. Let $f(x) = \sum_{i=1}^{n} b_i(x^i)^2$, where $b_i \neq 0$ for each $i = 1, \ldots, n$. Let M be the hyperplane $\{\mathbf{x} : \mathbf{y} \cdot \mathbf{x} = 1\}$, and $F(\mathbf{x}) = f(\mathbf{x}) + \sigma(1 - \mathbf{y} \cdot \mathbf{x})$. If $\mathbf{x}_0$ is a critical point of F, then

$$\mathbf{0} = \operatorname{grad} F(\mathbf{x}_0) = \operatorname{grad} f(\mathbf{x}_0) - \sigma\mathbf{y}.$$

Thus, $\operatorname{grad} f(\mathbf{x}_0) = \sigma\mathbf{y}$, or

$$f_i(\mathbf{x}_0) = 2b_i x_0^i = \sigma y^i, \qquad i = 1, \ldots, n.$$

From this and the equation $\mathbf{y} \cdot \mathbf{x}_0 = 1$,

$$x_0^i = \frac{\sigma y^i}{2b_i}, \qquad \frac{1}{\sigma} = \frac{1}{2}\sum_{i=1}^{n} \frac{(y^i)^2}{b_i},$$

provided the sum is not zero. To determine whether $\mathbf{x}_0$ gives an extremum, we use the formula

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \operatorname{grad} f(\mathbf{x}_0) \cdot \mathbf{h} + f(\mathbf{h}),$$

which is valid for homogeneous quadratic polynomials. Points of the hyperplane M are of the form $\mathbf{x}_0 + \mathbf{h}$, where $\mathbf{y} \cdot \mathbf{h} = 0$. Since $\operatorname{grad} f(\mathbf{x}_0) = \sigma\mathbf{y}$,

the above formula simplifies to

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + f(\mathbf{h}).$$

If $f(\mathbf{h}) \geq 0$ for every $\mathbf{h}$ satisfying $\mathbf{y} \cdot \mathbf{h} = 0$, then f has an absolute constrained minimum at $\mathbf{x}_0$.

The characteristic values of a symmetric matrix

Let (c_j^i) be an $n \times n$ matrix and $\mathbf{L}$ the corresponding linear transformation. A number λ is a *characteristic value of* $\mathbf{L}$ if the linear transformation $\mathbf{L} - \lambda\mathbf{I}$ is singular. There are n characteristic values $\lambda_1, \ldots, \lambda_n$, counting multiplicities. If $\mathbf{L}(\mathbf{x}) = \lambda_i \mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}$ is a *characteristic vector* corresponding to the characteristic value λ_i. The numbers $\lambda_1, \ldots, \lambda_n$ may be complex, and the characteristic vectors may have complex components [12, p. 164].

Let us suppose, however, that (c_j^i) is a symmetric matrix, $c_j^i = c_i^j$ for $i, j = 1, \ldots, n$. Let us show that the characteristic values are real. In fact, λ_i can be characterized as the value of a certain constrained maximum.

Consider the homogeneous quadratic polynomial

$$f(\mathbf{x}) = \mathbf{L}(\mathbf{x}) \cdot \mathbf{x} = \sum_{i,j=1}^{n} c_j^i x^i x^j,$$

and let M_1 be the unit $(n-1)$-sphere in E^n, $M_1 = \{\mathbf{x} : |\mathbf{x}| = 1\}$. Let

$$\lambda_1 = \max\{f(\mathbf{x}) : \mathbf{x} \in M_1\}, \tag{4.27}$$

and let $\mathbf{v}_1$ be a point of M_1 at which the maximum is attained. By the multiplier rule, with

$$F(\mathbf{x}) = f(\mathbf{x}) + \sigma(1 - \mathbf{x} \cdot \mathbf{x}),$$

there is a multiplier σ such that $\mathbf{v}_1$ is a critical point of F. Then $\operatorname{grad} f(\mathbf{x}) = 2\mathbf{L}(\mathbf{x})$ since the matrix (c_j^i) is symmetric. Hence

$$\mathbf{0} = \operatorname{grad} F(\mathbf{v}_1) = 2\mathbf{L}(\mathbf{v}_1) - 2\sigma\mathbf{v}_1.$$

Hence $\mathbf{L}(\mathbf{v}_1) = \sigma\mathbf{v}_1$, which shows that σ is a characteristic value and $\mathbf{v}_1$ a characteristic vector. Since $f(\mathbf{x}) = \mathbf{L}(\mathbf{x}) \cdot \mathbf{x}$,

$$\lambda_1 = f(\mathbf{v}_1) = \mathbf{L}(\mathbf{v}_1) \cdot \mathbf{v}_1 = \sigma\mathbf{v}_1 \cdot \mathbf{v}_1,$$

and since $\mathbf{v}_1 \cdot \mathbf{v}_1 = 1$, $\sigma = \lambda_1$.

We next let

$$M_2 = \{\mathbf{x} : |\mathbf{x}| = 1, \mathbf{x} \cdot \mathbf{v}_1 = 0\},$$
$$\lambda_2 = \max\{f(\mathbf{x}) : \mathbf{x} \in M_2\},$$

and $\mathbf{v}_2 \in M_2$ be a point such that $f(\mathbf{v}_2) = \lambda_2$. We have added another constraint. Hence $M_2 \subset M_1$ and $\lambda_2 \leq \lambda_1$. Obviously $\mathbf{v}_2 \cdot \mathbf{v}_1 = 0$. For $k = 3, \ldots, n$, let

$$M_k = \{\mathbf{x} : |\mathbf{x}| = 1, \mathbf{x} \cdot \mathbf{v}_i = 0 \quad \text{for } i = 1, \ldots, k-1\},$$
$$\lambda_k = \max\{f(\mathbf{x}) : \mathbf{x} \in M_k\}, \tag{4.28}$$

and $\mathbf{v}_k \in M_k$ be such that $\lambda_k = f(\mathbf{v}_k)$. Then

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$$

and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for E^n. Let us show by induction on k that λ_k is a characteristic value and $\mathbf{v}_k$ a characteristic vector. This is true if $k = 1$. Let $k \geq 2$. Applying the multiplier rule with

$$F(\mathbf{x}) = f(\mathbf{x}) + \sigma_1(1 - \mathbf{x} \cdot \mathbf{x}) + \sum_{i=1}^{k-1} \sigma_{i+1} \mathbf{x} \cdot \mathbf{v}_i,$$

we get

$$\mathbf{0} = 2\mathbf{L}(\mathbf{v}_k) - 2\sigma_1 \mathbf{v}_k + \sum_{i=1}^{k-1} \sigma_{i+1} \mathbf{v}_i.$$

Since $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$, we get upon taking the inner product with $\mathbf{v}_j$,

$$0 = 2\mathbf{L}(\mathbf{v}_k) \cdot \mathbf{v}_j + \sigma_{j+1}, \quad \text{if } j < k,$$
$$\sigma_1 = \mathbf{L}(\mathbf{v}_k) \cdot \mathbf{v}_k = f(\mathbf{v}_k),$$

or $\sigma_1 = \lambda_k$. Since (c_j^i) is symmetric, $\mathbf{L}(\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{L}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$. In particular,

$$\mathbf{L}(\mathbf{v}_k) \cdot \mathbf{v}_j = \mathbf{v}_k \cdot \mathbf{L}(\mathbf{v}_j).$$

Using the induction hypothesis, we get

$$\mathbf{L}(\mathbf{v}_k) \cdot \mathbf{v}_j = \mathbf{v}_k \cdot (\lambda_j \mathbf{v}_j) = 0 \quad \text{if } j < k.$$

Hence $\sigma_2 = \cdots = \sigma_k = 0$, and $\mathbf{L}(\mathbf{v}_k) = \lambda_k \mathbf{v}_k$.

If $k = n$, the multiplier rule does not apply. However, we used the multiplier rule only to show that $\mathbf{L}(\mathbf{v}_k)$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. If $k = n$, this is clear from the fact that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for E^n.

Theorem 4.4. *The characteristic values of the symmetric matrix (c_j^i) are $\lambda_1, \ldots, \lambda_n$. For each $i = 1, \ldots, n$, $\mathbf{v}_i$ is a characteristic vector corresponding to λ_i. If $\xi^1, \ldots, \xi^n$ denote the components of $\mathbf{x}$ with respect to the orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then*

$$f(\mathbf{x}) = \lambda_1(\xi^1)^2 + \cdots + \lambda_n(\xi^n)^2 \tag{4.29}$$

for every $\mathbf{x} \in E^n$.

PROOF. The first two statements have already been proved. To prove the third, we have

$$\mathbf{x} = \sum_{i=1}^{n} \xi^i \mathbf{v}_i, \qquad \xi^i = \mathbf{v}_i \cdot \mathbf{x},$$

$$\mathbf{L}(\mathbf{x}) = \sum_{i=1}^{n} \xi^i \mathbf{L}(\mathbf{v}_i) = \sum_{i=1}^{n} \lambda_i \xi^i \mathbf{v}_i,$$

$$f(\mathbf{x}) = \mathbf{L}(\mathbf{x}) \cdot \mathbf{x} = \sum_{i=1}^{n} \lambda_i \xi^i \mathbf{v}_i \cdot \mathbf{x},$$

which is just (4.29). □

Corollary. *f is positive definite if and only if $\lambda_i > 0$ for each $i = 1, \ldots, n$.*

PROOF. If each λ_i is positive, then by (4.29), $f(\mathbf{x}) > 0$ whenever $\mathbf{x} \neq \mathbf{0}$. Conversely, if $f(\mathbf{x}) > 0$ for every $\mathbf{x} \neq \mathbf{0}$, then $\lambda_i = f(\mathbf{v}_i) > 0$ for each i. □

These results have a geometric interpretation. Let

$$B = \{\mathbf{x} : f(\mathbf{x}) = 1\}.$$

If f is positive definite, then B is called an $(n-1)$-*dimensional ellipsoid.* Setting $\mu_i = 1/\sqrt{\lambda_i}$, we have (Figure 4.10)

$$B = \left\{\mathbf{x} : \sum_{i=1}^{n} (\xi^i/\mu_i)^2 = 1\right\},$$

if $\lambda_i > 0$ for each $i = 1, \ldots, n$. If $n = 2$ and $\lambda_1 > 0$, $\lambda_2 < 0$, then B is a hyperbola. If $n = 3$ and $\lambda_1 > 0$, $\lambda_3 < 0$, then B is a hyperboloid. It has one sheet if $\lambda_2 > 0$ and two sheets if $\lambda_2 < 0$.

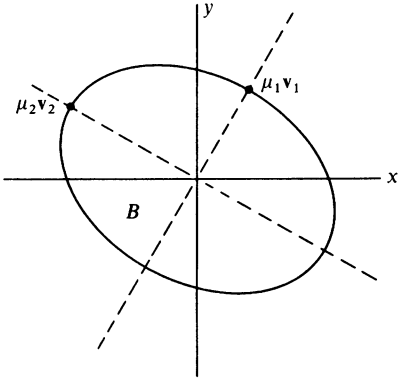


Figure 4.10

PROBLEMS

1. Set $M = \{(x, y, z) : x + y + z = 1\}$ and $f(x, y, z) = 3x^2 + 3y^2 + z^2$. Show that there is a constrained absolute minimum, and find the minimum value of f on M.
2. Use the multiplier rule to find the distance to the parabola $y^2 = x$ from the point $c\mathbf{e}_1$, $c \neq 0$. [*Hint*: Let $f(x, y) = (x - c)^2 + y^2$, which is the square of the distance.]
3. Find the distance from the point $\mathbf{e}_1 - 2\mathbf{e}_2 - \mathbf{e}_3$ to the line $\{(x, y, z) : x = y = z\}$.
4. (a) Use the multiplier rule to show that $|\mathbf{a}| = \max\{\mathbf{a} \cdot \mathbf{x} : |\mathbf{x}| = 1\}$.
   (b) Deduce the same result from Cauchy's inequality.
5. Let M be a manifold, $\mathbf{x}_1 \notin M$, and suppose that $\mathbf{x}_0$ is a point of M nearest $\mathbf{x}_1$. Using the multiplier rule, show that $\mathbf{x}_1 - \mathbf{x}_0$ is a normal vector to M at $\mathbf{x}_0$.
6. Show that the distance from a point $\mathbf{x}_1$ to the hyperplane $\{\mathbf{x} : \mathbf{y} \cdot \mathbf{x} = b\}$ is $|\mathbf{y} \cdot \mathbf{x}_1 - b|/|\mathbf{y}|$.

7. Let $f(\mathbf{x}) = x^1x^2 \cdots x^n$ and $M = \{\mathbf{x} : x^1 + \cdots + x^n = 1, x^i > 0 \text{ for } i = 1, \ldots, n\}$.
(a) Show that $f(\mathbf{x}) \leq n^{-n}$ for every $\mathbf{x} \in M$, with equality if $x^1 = \cdots = x^n = n^{-1}$. [*Hint*: First show that f has an absolute maximum on M. Apply the multiplier rule to $\log f$, which has a maximum at the same point where f has one.]
(b) Using (a), prove that the geometric mean of n positive numbers is no more than their arithmetic mean. See Problem 9(b), Section 3.6.

8. Let $p > 1$ and p' be the number such that $p^{-1} + (p')^{-1} = 1$. Let $\|\mathbf{x}\|$ be as in Problem 4, Section 2.11, and for each covector $\mathbf{a}$ let $\|\mathbf{a}\| = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{x}\| = 1\}$. Show that

$$\|\mathbf{a}\| = \left(\sum_{i=1}^{n} |a_i|^{p'}\right)^{1/p'}.$$

[*Note*: For these norms the inequality $|\mathbf{a} \cdot \mathbf{x}| \leq \|\mathbf{a}\| \, \|\mathbf{x}\|$ is called *Hölder's inequality*. A related inequality for integrals is given in Section 5.13].

9. Let $f(x, y, z) = 2xz + y^2$.
(a) Find the characteristic values $\lambda_1, \lambda_2, \lambda_3$.
(b) Sketch the surface with equation $2xz + y^2 = 1$. With equation $2xz + y^2 = 0$.

10. Let $\|\mathbf{L}\|$ be defined as in Section 4.3. Show that $\|\mathbf{L}\|^2$ is the largest characteristic value of $\mathbf{L}^t \circ \mathbf{L}$. [*Hint*: Use (4.10) with $\mathbf{y} = \mathbf{L}(\mathbf{t})$].

11. (*Second derivative test for constrained relative maxima.*) Let f and Φ be of class $C^{(2)}$, and let $Q(\mathbf{x}, \mathbf{h}) = \sum_{i,j=1}^{n} F_{ij}(\mathbf{x})h^ih^j$ where F is as in the multiplier rule. Show that:
(a) If $f \mid M$ has a relative maximum at $\mathbf{x}_0$, then $Q(\mathbf{x}_0, \mathbf{h}) \geq \mathbf{0}$ for every $\mathbf{h} \in T(\mathbf{x}_0)$.
(b) If $Q(\mathbf{x}_0, \mathbf{h}) > 0$ for every $\mathbf{h} \in T(\mathbf{x}_0)$, $\mathbf{h} \neq \mathbf{0}$, then $f \mid M$ has a strict relative maximum at $\mathbf{x}_0$. [*Hints*: See the proof of Theorem 3.4. Set $\mathbf{h}_t = t^{-1}(\mathbf{y}_t - \mathbf{x}_0)$, and show that $\lim_{t \to 0} Q(\mathbf{x}_0, \mathbf{h}_t) = Q(\mathbf{x}_0, \mathbf{h})$.]

Integration 5

The integral of a real valued function over a set is a generalization of the notion of sum. It is defined by approximating in a suitable way by certain finite sums. The first careful definition was due to Riemann (1854). Riemann defined the integral of a function over an interval $[a, b]$ of the real line E^1. In the succeeding years Riemann's idea was extended in several ways. However, the Riemann integral has several intrinsic drawbacks, and for a truly satisfactory treatment of integration a different approach had to be found.

About 1900 Lebesgue discovered a more sophisticated and flexible theory of integrals. In this chapter the elements of the Lebesgue theory are given. The first step is to define the measure of a set $A \subset E^n$. For $n = 1, 2$, or 3, the measure is respectively the length, area, or volume of A. An important property of Lebesgue measure is its countable additivity [Formula (5.9)]. While not every set A is assigned a measure, countable additivity insures that the class of measurable sets is large enough for all applications encountered in mathematical analysis.

After measure, the integral of a bounded function f over a bounded set A is defined using upper and lower integrals. The integral exists under the very mild assumptions that A is measurable and f is measurable on A. Later (Sections 5.6 and 5.11), the integral is studied without those boundedness assumptions.

The definition of an integral does not furnish an effective procedure for the actual evaluation of integrals. However, the theorems on iterated integrals and transformation of integrals (Section 5.5 and 5.8), together with the fundamental theorem of calculus, provide a useful technique for this purpose.

Among the important features of the Lebesgue theory are the theorems about integration term by term in sequences of functions. Such questions are treated in Section 5.11.

Notation. The n-dimensional measure of a set A is denoted by $V_n(A)$. If the dimension n is clear from the context, then we write simply "measure" rather than "n-dimensional measure" and $V(A)$ rather than $V_n(A)$. The integral of f over A is denoted by

$$\int_A f\,dV_n \qquad \text{or} \qquad \int_A f(\mathbf{x})dV_n(\mathbf{x}).$$

If $A = E^n$, we write simply $\int f\,dV_n$. The symbols dV_n and $dV_n(\mathbf{x})$ are used after an integral sign merely for convenience and for traditional reasons. They will have no significance by themselves.

If $n = 1$, then we write, as is customary, $\int_A f(x)dx$ instead of $\int_A f(x)dV_1(x)$, and if $A = [a, b]$, we write $\int_a^b f(x)dx$.

5.1 Intervals

What is the n-dimensional measure of a subset A of E^n? To start with, let us consider the simplest possible case—where A is an n-dimensional interval.

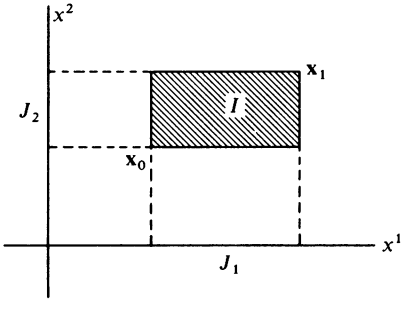


Figure 5.1

A 2-dimensional interval is a rectangle with sides parallel to the coordinate axes (Figure 5.1). Its area is the product of the lengths of its sides. Since E^2 is the cartesian product $E^1 \times E^1$, a 2-dimensional interval is just the cartesian product of 1-dimensional intervals. Similarly, a set $I \subset E^n$ is called an *n-dimensional interval* if I is the cartesian product of 1-dimensional intervals:

$$I = J_1 \times \cdots \times J_n,$$

where each J_i is a finite interval of E^1. The interval I is *closed* if each J_i is closed, and *open* if each J_i is open. For instance, if I is closed, then there exist

$$\mathbf{x}_0 = (x_0^1, \ldots, x_0^n), \qquad \mathbf{x}_1 = (x_1^1, \ldots, x_1^n)$$

with $x_0^i < x_1^i$ for each $i = 1, \ldots, n$, such that $J_i = [x_0^i, x_1^i]$ and

$$I = \{\mathbf{x} : x_0^i \leq x^i \leq x_1^i, i = 1, \ldots, n\}.$$

The *n-dimensional measure* $V(I)$ of I is the product of the lengths of the intervals $J_1, \ldots, J_n$.

We next define the measure of a set that is a finite union of n-dimensional intervals. For this purpose the idea of grid of hyperplanes is introduced.

Grids

For each $i = 1, \ldots, n$ let us take a finite set of real numbers; let the elements of these sets be denoted by x_j^i, where

$$x_1^i < x_2^i < \cdots < x_{m_i+1}^i$$

and $m_i + 1$ is the number of elements of the ith set. Let P_j^i be the hyperplane with equation $x^i = x_j^i$, and let Π be the union of all these hyperplanes. Such a set Π is called a *grid* of hyperplanes. A grid divides E^n into a finite number of n-dimensional intervals, called *intervals of* Π, and a finite number of unbounded sets. The latter could be called *semiinfinite intervals* of Π, but we have no occasion to do so. The intervals of Π have the form $I = J_1 \times \cdots \times J_n$ where $J_i = [x_{j_i}^i, x_{j_i+1}^i]$ and the integers $j_1, \ldots, j_n$ may be chosen arbitrarily subject to $1 \leq j_i \leq m_i$. There are $m_1 \cdots m_n$ intervals of the grid, and for convenience we have taken them to be closed (see Figure 5.2).

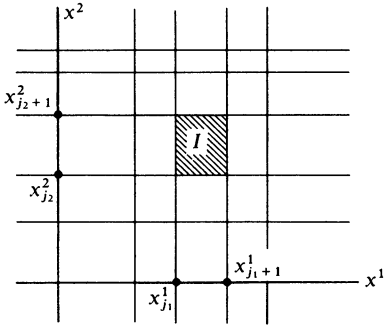


Figure 5.2

Let us call a set Y a *figure* if Y is the union of certain intervals $I_1, \ldots, I_p$ of some grid Π. The *measure* of Y is

$$V(Y) = V(I_1) + \cdots + V(I_p). \tag{5.1}$$

There are many possible choices for Π. Consequently, we must show that $V(Y)$ depends only on Y and not on the particular grid chosen. Let us call Π' a refinement of Π if $\Pi \subset \Pi'$. It is easy to show that $V(Y)$ is unchanged if Π' is obtained by adding one hyperplane to Π, and hence by induction if Π is replaced by any refinement of it. Now let Π and Π' be any two grids such that Y is the union of intervals of Π and also the union of intervals Π'. Then $\Pi \cup \Pi'$ is a refinement of both. Consequently, $V(Y)$ is the same whether Π or Π' is used.

This same reasoning shows that if Y and Z are figures, then Y and Z can be written as unions of intervals of the same grid Π. Therefore $Y \cup Z$ is a figure. Moreover,

$$V(Y \cup Z) \leq V(Y) + V(Z). \tag{5.2}$$

If $Y \cap Z$ is empty, then equality holds in (5.2).

Problems

1. Let $n = 1$ and $Y = [0, 1] \cup [2, 3]$, $Z = [1, 3] \cup [4, 5]$. Verify Formula (5.2) in this example.

2. Let $n = 2$ and $Y = [0, 2] \times [0, 1] \cup [1, 3] \times [1, 2]$, $Z = [-1, 2] \times [-1, 3]$. Find a grid Π such that both Y and Z are unions of intervals of Π. Find the areas of Y, Z, $Y \cup Z$, and $Y \cap Z$, and verify that $V(Y) + V(Z) - V(Y \cup Z) = V(Y \cap Z)$.

3. Let $I_1 = [0, 1] \times [0, 1] \times [0, 1]$ and $I_2 = [\frac{1}{2}, 2] \times [0, 2] \times [-1, 2]$. Find the volume of $I_1 \cup I_2$ and of $I_1 \cap I_2$.

4. Let m be a positive integer, $f(x) = \exp x$, and $Y = I_1 \cup \cdots \cup I_m$, where for $k = 1, \ldots, m$

$$I_k = [(k - 1)/m, k/m] \times [0, f(k/m)].$$

Find the area $V_2(Y)$. Show that it is approximately $e - 1$ if m is large.

5. (a) Let I_1 and I_2 be n-dimensional intervals. Show that $I_1 \cap I_2$ is also an interval provided that it has nonempty interior.
   (b) When is $I_1 \cup I_2$ an interval?

5.2 Measure

We now define measure in the Lebesgue sense for a large class of subsets of E^n. A set A in this class is called *measurable*, and the measure of A is denoted by $V(A)$. Some main properties of measurable sets and of measure are stated in Theorems 5.1 through 5.3. In studying the present section, the reader should first concentrate on understanding the definitions and the statements of these main theorems. A careful study of the various lemmas and propositions in the section might be postponed.

We begin by defining the measure of a bounded set A. This is done in two stages. First, the measure of an open set G is defined by approximating G from within by figures, and that of a compact set K by approximating K from without by figures. In the second stage, A is approximated from within by compact sets and from without by open sets. This two-stage approximation process is an important feature of the Lebesgue theory of measure.

There is an older theory of measure due to Jordan. In this theory A is approximated simultaneously from within and without by figures. The Jordan theory is unsatisfactory for several reasons. Among them is the fact

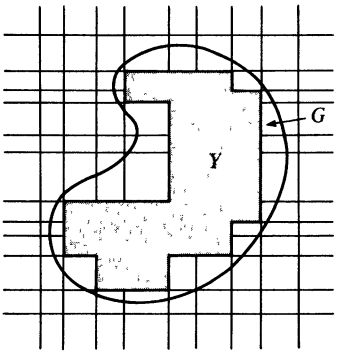


Figure 5.3

that the class of sets to which it applies is too small. For instance, there are compact sets to which the Jordan theory does not assign any measure.

Let G be an open set. If Y is a figure contained in G, then the measure of G must be more than $V(Y)$. It is defined to be the least upper bound of the set of all such numbers $V(Y)$ (see Figure 5.3).

Definition. The *measure* of an open set G is $V(G) = \sup\{V(Y): Y \subset G\}$.

If the set $S = \{V(Y): Y \subset G\}$ has no upper bound, then we set $V(G) = +\infty$. For instance, $V(E^n) = +\infty$. If G is bounded, then G is contained in some interval I, and $V(I)$ is an upper bound for S. In this case $V(G)$ is finite.

By definition (Section 1.1), the least upper bound $\sup S$ has the property that $s \leq \sup S$ for every $s \in S$, and given $\varepsilon > 0$ there exists an $s \in S$ such that $\sup S < s + \varepsilon$.

EXAMPLE 1. Let $G = \text{int } Z$ where Z is a figure (recall that any figure is a compact set). Let us show that $V(\text{int } Z) = V(Z)$. For any figure $Y \subset \text{int } Z$, $V(Y) \leq V(Z)$. Given $\varepsilon > 0$ one can find such a figure Y with $V(Z) < V(Y) + \varepsilon$. To see this, write $Z = I_1 \cup \cdots \cup I_p$, where $I_1, \ldots, I_p$ are intervals. Let $Y = I_1' \cup \cdots \cup I_p'$, where each I_j' is an interval such that $I_j' \subset \text{int } I_j$ and $V(I_j) < V(I_j') + p^{-1}\varepsilon$. Then $Y \subset \text{int } Z$ and $V(Z) < V(Y) + \varepsilon$ as required. Thus, $V(Z) = \sup\{V(Y): Y \subset \text{int } Z\}$.

Now let K be a compact set. If Z is any figure whose interior $\text{int } Z$ contains K, then $V(Z)$ must exceed the measure of K.

Definition. The *measure* of a compact set K is $V(K) = \inf\{V(Z): K \subset \text{int } Z\}$.

EXAMPLE 1 (*continued*). Let $K = Y$, where Y is a figure. Then $V(Y)$ as defined immediately above agrees with the definition of $V(Y)$ in Section 5.1. Moreover, $V(Y) = V(\text{int } Y)$ as was proved above.

Lemma 1. *Let K be a compact set and G an open set such that $K \subset G$. Then there is a figure Y such that $K \subset \operatorname{int} Y$ and $Y \subset G$.*

The proof is left to the reader (Problem 6).

Now let A be any bounded set. Its *outer measure*, denoted by $\overline{V}(A)$, is defined by approximating from without by open sets:

$$\overline{V}(A) = \inf\{V(G) : A \subset G\}.$$

Similarly, the *inner measure* $\underline{V}(A)$ is defined by approximating from within by compact sets:

$$\underline{V}(A) = \sup\{V(K) : K \subset A\}.$$

If $K \subset A$, then by Lemma 1, $V(K) \leq V(G)$ whenever $A \subset G$. Hence $V(K)$ is a lower bound for $\{V(G) : A \subset G\}$, and $V(K) \leq \overline{V}(A)$. Thus $\overline{V}(A)$ is an upper bound for $\{V(K) : K \subset A\}$, and

$$\underline{V}(A) \leq \overline{V}(A).$$

It is easy to show that if $B \subset A$, then

$$\underline{V}(B) \leq \underline{V}(A), \qquad \overline{V}(B) \leq \overline{V}(A).$$

We are interested in those bounded sets A whose outer and inner measures are equal.

Definition. A bounded set A is called *measurable* if its outer and inner measures are equal. If A is measurable, then the number

$$V(A) = \underline{V}(A) = \overline{V}(A)$$

is called the *n-dimensional measure of* A.

Let us show that bounded open sets and compact sets are measurable and that the new definition of their measures agrees with the previous one. Let H be a bounded open set. If G is open and $H \subset G$, then any figure Y contained in H is also contained in G. By the definition of measure for open sets, $V(H) \leq V(G)$. When $G = H$, equality holds. From the definition of outer measure, $\overline{V}(H) = V(H)$. Given $\varepsilon > 0$, there is a figure $Y \subset H$ with $V(H) - \varepsilon < V(Y)$. Since Y is a compact set, $V(Y) \leq \underline{V}(H)$. Since this is true for every $\varepsilon > 0$, $V(H) \leq \underline{V}(H)$. But $\underline{V}(H) \leq \overline{V}(H)$, and therefore

$$\underline{V}(H) = \overline{V}(H) = V(H).$$

Similarly, if L is a compact set, then

$$\underline{V}(L) = \overline{V}(L) = V(L).$$

In addition to open sets and compact sets, many other sets are measurable. In fact, the only examples of nonmeasurable sets are obtained in a quite nonconstructive way using the "axiom of choice" of set theory [18, p. 157].

Let us next show that the union, intersection, and difference of two bounded measurable sets is measurable (Proposition 5.2). For this a series of preparatory lemmas is needed.

Lemma 2. *Let G and H be open sets and K be a compact subset of $G \cup H$. Then there exists $d > 0$ such that the d-neighborhood of any $\mathbf{x} \in K$ is either contained in G or in H.*

PROOF. The sets $E^n - G$ and $E^n - H$ are closed. Let

$$f(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, E^n - G), \qquad g(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, E^n - H),$$

where $\operatorname{dist}(\mathbf{x}, A)$ is the distance from $\mathbf{x}$ to the set A (Section 2.8). The functions f and g are continuous and $f(\mathbf{x}) + g(\mathbf{x}) > 0$ for every $\mathbf{x} \in G \cup H$. The continuous function $f + g$ has a positive minimum value c on the compact set K. Let $d = c/2$. For every $\mathbf{x} \in K$, either $f(\mathbf{x}) \geq d$ or $g(\mathbf{x}) \geq d$. □

Lemma 3a. *Let G and H be open sets of finite measure. Then*

$$V(G \cup H) \leq V(G) + V(H). \tag{5.3}$$

PROOF. Let W be any figure such that $W \subset G \cup H$. Let d be as in Lemma 2 with $K = W$. The figure W is the union of intervals of a grid Π. By refining Π if necessary we may suppose that each interval of Π has diameter less than d. Let Y be the union of those intervals of Π which are contained in G, and Z the union of those contained in H. By Lemma 2, $W \subset Y \cup Z$. Consequently,

$$V(W) \leq V(Y \cup Z) \leq V(Y) + V(Z).$$

Since $Y \subset G$, $V(Y) \leq V(G)$; similarly, $V(Z) \leq V(H)$. Hence

$$V(W) \leq V(G) + V(H).$$

The number $V(G) + V(H)$ is an upper bound for $\{V(W) : W \subset G \cup H\}$; and hence it is no less than the least upper bound $V(G \cup H)$. This proves (5.3). □

Lemma 4. *Let K and L be compact sets such that $K \cap L$ is empty. Then*

$$V(K \cup L) \geq V(K) + V(L). \tag{5.4}$$

PROOF. Let $f(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, L)$. Since $K \cap L$ is empty and L is closed, $f(\mathbf{x}) > 0$ for every $\mathbf{x} \in K$. Since K is compact and f is continuous, f has a positive minimum value d on K.

Let W be any figure such that $K \cup L \subset \operatorname{int} W$. Then W is the union of intervals $I_1, \ldots, I_p$ with diameters less than $d/2$. Let Y be the union of those intervals I_j such that $I_j \cap K$ is not empty, and Z the union of those such that $I_j \cap L$ is not empty. Then $Y \cup Z \subset W$, $Y \cap Z$ is empty, and $K \subset \operatorname{int} Y$, $L \subset \operatorname{int} Z$. Hence

$$V(K) + V(L) \leq V(Y) + V(Z) \leq V(W).$$

This shows that $V(K) + V(L)$ is a lower bound for $\{V(W): K \cup L \subset W\}$, and hence is no more than the greatest lower bound $V(K \cup L)$. □

Lemma 5. *Let A and B be bounded sets. Then*

$$\overline{V}(A \cup B) \leq \overline{V}(A) + \overline{V}(B). \tag{5.5}$$

If $A \cap B$ is empty, then

$$\underline{V}(A \cup B) \geq \underline{V}(A) + \underline{V}(B). \tag{5.6}$$

PROOF. Given $\varepsilon > 0$, there are open sets $G \supset A$, $H \supset B$ such that

$$V(G) < \overline{V}(A) + \varepsilon/2, \qquad V(H) < \overline{V}(B) + \varepsilon/2.$$

Then $G \cup H$ is an open set containing $A \cup B$, and from Lemma 3a

$$\overline{V}(A \cup B) \leq V(G \cup H) \leq V(G) + V(H),$$
$$\overline{V}(A \cup B) < \overline{V}(A) + \overline{V}(B) + \varepsilon.$$

Since the last inequality is true for every $\varepsilon > 0$, we must have (5.5).

Similar reasoning, using Lemma 4, gives (5.6). □

Proposition 5.1a. *Let A and B be bounded measurable sets such that $A \cap B$ is empty. Then $A \cup B$ is measurable and*

$$V(A \cup B) = V(A) + V(B). \tag{5.7}$$

PROOF. By Lemma 5,

$$V(A) + V(B) \leq \underline{V}(A \cup B) \leq \overline{V}(A \cup B) \leq V(A) + V(B).$$

Since the extreme left-hand and right-hand sides are the same, both $\underline{V}(A \cup B)$ and $\overline{V}(A \cup B)$ must equal $V(A) + V(B)$. □

Corollary 1. *Let A be a bounded set. Then A is measurable if and only if for every $\varepsilon > 0$ there exist a compact set K and an open set G such that $K \subset A \subset G$ and $V(G - K) < \varepsilon$.*

The proof is left to the reader (Problem 7).

Proposition 5.2. *Let A and B be bounded measurable sets. Then $A - B$, $A \cap B$, and $A \cup B$ are also measurable.*

PROOF. Let us first prove that $A - B$ is measurable. Given $\varepsilon > 0$, let G, G' be open sets and K, K' compact sets such that $K \subset A \subset G$, $K' \subset B \subset G'$, and

$$V(G - K) < \frac{\varepsilon}{2}, \qquad V(G' - K') < \frac{\varepsilon}{2}.$$

Let $H = G - K$, $L = K - G'$. Then H is open, L is compact,

$$L \subset A - B \subset H.$$

Moreover, $H - L$ is open and

$$H - L \subset (G - K) \cup (G' - K').$$

By Lemma 3a,

$$V(H - L) \leq V(G - K) + V(G' - K') < \varepsilon.$$

This shows that $A - B$ is measurable.

Now $A \cap B = A - (A - B)$. Since A and $A - B$ are bounded measurable sets, by the first part of the proposition their difference $A \cap B$ is measurable. Finally,

$$A \cup B = (A - B) \cup B,$$

and the two sets on the right-hand side are measurable and do not intersect. By Proposition 5.1a, $A \cup B$ is measurable. □

Countable additivity of measure

If $A_1, A_2, \ldots, A_m$ are bounded measurable sets and $A_k \cap A_l$ is empty for $k \neq l$, then

$$V(A_1 \cup \cdots \cup A_m) = \sum_{k=1}^{m} V(A_k). \tag{5.8}$$

This follows from Proposition 5.1a and induction on m. Formula (5.8) expresses a property called *finite additivity of measure*. Let us now prove a stronger result.

A series $\sum_{k=1}^{\infty} a_k$ of nonnegative numbers converges if the partial sums $s_m = \sum_{k=1}^{m} a_k$ are bounded, since the partial sums then form a bounded nondecreasing sequence (see Section 2.3). If the sequence of partial sums is unbounded, the series is said to diverge to $+\infty$. A sequence $A_1, A_2, \ldots$ of sets is *disjoint* if $A_k \cap A_l$ is empty whenever $k \neq l$.

Proposition 5.1b. *If $A_1, A_2, \ldots$ is a disjoint sequence of measurable sets and if $A = A_1 \cup A_2 \cup \cdots$ is bounded, then A is measurable and*

$$V(A) = \sum_{k=1}^{\infty} V(A_k). \tag{5.9}$$

This property is called *countable additivity of measure*. If we let A_k be empty for $k > m$, then (5.8) is a special case of (5.9).

To prove Proposition 5.1b, we first state:

Lemma 3b. *Let $G_1, G_2, \ldots$ be open sets each of which has finite measure. If $G = G_1 \cup G_2 \cup \cdots$, then*

$$V(G) \leq \sum_{k=1}^{\infty} V(G_k).$$

PROOF. Let $Y \subset G$. The figure Y is a compact set and $G_1, G_2, \ldots$ form an open covering of Y. Hence $Y \subset G_1 \cup \cdots \cup G_m$ for some m. By Lemma 3a and induction on m,

$$V(G_1 \cup \cdots \cup G_m) \leq \sum_{k=1}^{m} V(G_k).$$

Then

$$V(Y) \leq \sum_{k=1}^{m} V(G_k) \leq \sum_{k=1}^{\infty} V(G_k).$$

Since this is true for every such Y, the lemma follows. □

PROOF OF PROPOSITION 5.1b. We have $A_1 \cup \cdots \cup A_m \subset A$ for every m. Using (5.8)

$$\sum_{k=1}^{m} V(A_k) \leq \underline{V}(A).$$

Since this is true for each m,

$$\sum_{k=1}^{\infty} V(A_k) \leq \underline{V}(A).$$

On the other hand, given $\varepsilon > 0$ let G_k be an open set such that $A_k \subset G_k$ and $V(G_k) < V(A_k) + \varepsilon 2^{-k}$, $k = 1, 2, \ldots$, and let $G = G_1 \cup G_2 \cup \cdots$. Then $A \subset G$ and therefore $\overline{V}(A) \leq V(G)$. By Lemma 3b,

$$\overline{V}(A) < \sum_{k=1}^{\infty} V(A_k) + \varepsilon \sum_{k=1}^{\infty} 2^{-k}$$

and $\sum 2^{-k} = 1$. Since this is true for any $\varepsilon > 0$, A is measurable and (5.9) holds. □

If in Proposition 5.1b the sets $A_1, A_2, \ldots$ are not disjoint, then the equality (5.9) becomes an inequality. This is part of Theorem 5.2 below.

Unbounded sets

For a possibly unbounded set A the concepts of A being measurable and measure of A are defined as follows. Let $U_r = \{\mathbf{x} : |\mathbf{x}| < r\}$. A set A is called *measurable* if $A \cap U_r$ is measurable for every $r > 0$. The *measure* of A is

$$V(A) = \lim_{r \to +\infty} V(A \cap U_r).$$

If we set $\phi(r) = V(A \cap U_r)$, then ϕ is a nondecreasing function. Therefore the limit exists. It may be finite or $+\infty$. If A is a bounded set, then $A \subset U_{r_0}$ for some r_0. For $r \geq r_0$, $A = A \cap U_r$. For bounded sets this definition agrees with the previous one.

We summarize several important facts about measurable sets and measure in the following theorems.

Theorem 5.1

(a) *Every open set G is measurable.*
(b) *If A is any measurable set, then its complement A^c is measurable.*
(c) *Let $A_1, A_2, \ldots$ be measurable sets. Then the union $A_1 \cup A_2 \cup \cdots$ and the intersection $A_1 \cap A_2 \cap \cdots$ are measurable sets.*

Theorem 5.2

(a) *If Y is a figure, then its measure $V(Y)$ is given by the elementary formula* (5.1).
(b) *If G is open, then $V(G) = \sup\{V(Y): Y \subset G\}$, where Y denotes a figure.*
(c) *If K is compact, then $V(K) = \inf\{V(Z): K \subset \operatorname{int} Z\}$, where $\operatorname{int} Z$ denotes the interior of a figure Z.*
(d) *If $A = A_1 \cup A_2 \cup \cdots$, where A_k is measurable for each $k = 1, 2, \ldots$, then*

$$V(A) \leq \sum_{k=1}^{\infty} V(A_k). \tag{5.10}$$

(e) *Equality holds in* (5.10) *if $A_1, A_2, \ldots$ are disjoint.*

PROOF OF THEOREM 5.1. If G is open, then $G \cap U_r$ is open and bounded for each $r > 0$. Hence $G \cap U_r$ is measurable. Since this is true for each r, G is measurable; this proves (a). To prove (b), let A be measurable. We note that $A^c \cap U_r = U_r - (A \cap U_r)$. By Proposition 5.2, $A^c \cap U_r$ is measurable since U_r and $A \cap U_r$ are measurable sets. Since this is true for each $r > 0$, A^c is measurable. To prove (c), first consider the case when $A = A_1 \cup A_2 \cup \cdots$ is bounded. Let

$$B_1 = A_1, B_2 = A_2 - A_1, \ldots, B_k = A_k - (A_1 \cup \cdots \cup A_{k-1}), \ldots.$$

By Proposition 5.2, each B_k is measurable. Moreover, $B_1, B_2, \ldots$ are disjoint and their union is A. By Proposition 5.1b, A is measurable in case A is a bounded set. The case when A is unbounded reduces to this one, by observing that

$$A \cap U_r = (A_1 \cap U_r) \cup (A_2 \cap U_r) \cup \cdots.$$

Finally, the measurability of $A_1 \cap A_2 \cap \cdots$ follows from what has already been proved and

$$\bigcap_{k=1}^{\infty} A_k = \left(\bigcup_{k=1}^{\infty} A_k^c\right)^c. \qquad \square$$

PROOF OF THEOREM 5.2. Part (a) of Theorem 5.2 was verified earlier (see Example 1), and parts (b), (c) are just the definitions. Let us prove (d) when A is bounded and postpone the proof for unbounded A to Section 5.10.

Define the disjoint sets $B_1, B_2, \ldots$ as in the proof of Theorem 5.1. By Proposition 5.1b,

$$V(A) = \sum_{k=1}^{\infty} V(B_k).$$

Since $B_k \subset A_k$, we have $V(B_k) \leq V(A_k)$. This implies (5.10), in case A is bounded. If $A_1, A_2, \ldots$ are disjoint, then $B_k = A_k$ and (5.9) holds. This is (e). □

EXAMPLE 2. Let G be an open subset of E^1. If G is the union of a finite number of disjoint open intervals $I_1, \ldots, I_m$, then $V(G) = V(I_1) + \cdots + V(I_m)$. It can also happen that G is the union of a disjoint infinite sequence $I_1, I_2, \ldots$ of open intervals, in which case

$$V(G) = \sum_{k=1}^{\infty} V(I_k).$$

The third possibility is that G contains some half-line. In that case $V(G) = +\infty$.

The sets of measure 0 play a special role. They turn out to be negligible in integration theory, and for that reason we shall call them null sets.

Definition. If $V(A) = 0$, then A is a *null set*.

Corollary 2. *If $A_1, A_2, \ldots$ are null sets, then $A_1 \cup A_2 \cup \cdots$ is a null set. If $B \subset A$ and A is a null set, then B is a null set.*

PROOF. By (5.10),

$$0 \leq V(A_1 \cup A_2 \cup \cdots) \leq \sum_{k=1}^{\infty} 0 = 0,$$

which proves the first assertion. If A is a bounded null set and $B \subset A$, then

$$0 \leq \underline{V}(B) \leq \overline{V}(B) \leq V(A) = 0.$$

Hence B is measurable and is a null set. If A is any null set, then $A \cap U_r$ is a null set for each r and $B \cap U_r \subset A \cap U_r$. Hence $B \cap U_r$ is a null set, which implies that B is measurable and is a null set. □

EXAMPLE 3. A set A is *countable* if either A is a finite set or its elements can be arranged in an infinite sequence, $A = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ where $\mathbf{x}_k \neq \mathbf{x}_l$ for $k \neq l$. Any one point set is a null set. Hence, taking $A_k = \{\mathbf{x}_k\}$, we find that *any countable set is a null set.*

EXAMPLE 4. Let A be the set of rational numbers in the interval (0, 1). Then A is countable. For instance, one can write

$$A = \{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \ldots\}.$$

Hence A is a null set. Since $V_1(A) + V_1[(0, 1) - A] = V_1[(0, 1)] = 1$, the set of irrational numbers in (0, 1) has measure 1 and therefore must be an uncountable set.

EXAMPLE 5. Let $A \subset M$, where M is an $(n-1)$-manifold. It is plausible that the n-dimensional measure of A is 0, and this fact is proved in Section 5.8. Hence any such set A is a null set.

A sequence of sets $A_1, A_2, \ldots$ is called *monotone* if either $A_1 \subset A_2 \subset \cdots$ or $A_1 \supset A_2 \supset \cdots$. In the first instance the sequence is called *nondecreasing*, and in the second instance *nonincreasing*.

Theorem 5.3

(a) *Let $A_1, A_2, \ldots$ be a nondecreasing sequence of measurable sets. Then*

$$V\left(\bigcup_{v=1}^{\infty} A_v\right) = \lim_{v \to \infty} V(A_v) \tag{5.11}$$

(b) *Let $A_1, A_2, \ldots$ be a nonincreasing sequence of measurable sets, such that $V(A_1) < \infty$. Then*

$$V\left(\bigcap_{v=1}^{\infty} A_v\right) = \lim_{v \to \infty} V(A_v) \tag{5.12}$$

PROOF. Let us prove the theorem under the assumption that there is a spherical ball U such that $A_v \subset U$ for each $v = 1, 2, \ldots$. This restriction will be removed in Section 5.10. To prove (a), define $B_1, B_2, \ldots$ as in the proofs of Theorems 5.1 and 5.2. Then

$$V\left(\bigcup_{v=1}^{\infty} A_v\right) = \sum_{k=1}^{\infty} V(B_k).$$

Since $A_1 \subset A_2 \subset \cdots$, we have $B_1 \cup \cdots \cup B_v = A_v$. Therefore

$$V(A_v) = \sum_{k=1}^{v} V(B_k).$$

We get (5.11) by taking the limit as $v \to \infty$. To get (5.12), we apply (5.11) to the nondecreasing sequence of sets $C_v = U - A_v$, and note that $V(C_v) = V(U) - V(A_v)$ since $A_v \subset U$. □

EXAMPLE 6. To see the need for the assumption $V(A_1) < \infty$ in Theorem 5.3(b), let $n = 1$ and $A_v = [v, \infty)$. Then $A_1 \supset A_2 \supset \cdots$ and $V(A_v) = +\infty$ for each $v = 1, 2, \ldots$. However, $A_1 \cap A_2 \cap \cdots$ is empty, and hence $V(A_1 \cap A_2 \cap \cdots) = 0$.

Problems

In 1, 2, and 3 assume that the sets are bounded.

1. Let A and B be measurable. Show that
(a) $V(A - B) = V(A) - V(A \cap B)$.
(b) $V(A \cup B) + V(A \cap B) = V(A) + V(B)$.

2. Show that if A, B, and C are measurable, then

$$V(A \cup B \cup C) = V(A) + V(B) + V(C) - V(A \cap B) - V(A \cap C) - V(B \cap C) + V(A \cap B \cap C).$$

3. Show that if A is measurable and B is a null set, then

$$V(A \cup B) = V(A - B) = V(A).$$

4. Let $A = A_1 \cup A_2 \cup \cdots$, where $A_k = \{(x, y) : x = 1/k, 0 \leq y \leq 1\}$ for $k = 1, 2, \ldots$ Show that $V_2(A) = 0$.

5. Let A_0 be the circular disk with center $(0, 0)$ and radius 1. For $k = 1, 2, \ldots$, let A_k be the circular disk with center $(1 - 4^{-k})\mathbf{e}_1$ and radius 4^{-k-1}. Let $A = A_0 - (A_1 \cup A_2 \cup \cdots)$. Find $V_2(A)$.

6. Prove Lemma 1. [*Hint*: Consider the collection of all intervals I such that $I \subset G$. The interiors $\operatorname{int} I$ of these intervals form an open covering of K.]

7. Prove Corollary 1 to Proposition 5.1a. [*Hint*: If $K \subset G$, then $G = K \cup (G - K)$. By Proposition 5.1a, $V(G) = V(K) + V(G - K)$.]

8. (a) Show that if A and B are countable sets, then $A \cup B$ is countable.
(b) Show that if $B \subset A$ and A is countable, then B is countable.
(c) Show that if $A_1, A_2, \ldots$ are countable sets, then $A_1 \cup A_2 \cup \cdots$ is countable.

9. Let $A = \{x_1, x_2, \ldots\}$ be a countable subset of $(0, 1)$. Given $0 < \varepsilon < 1$, let $\varepsilon_k = \varepsilon 2^{-k-1}$, $I_k = (x_k - \varepsilon_k, x_k + \varepsilon_k)$, and $G = I_1 \cup I_2 \cup \cdots$.
(a) Show that $V_1(G) \leq \varepsilon$.
(b) In particular, let A be the set of rational numbers in $(0, 1)$. Let $K = [0, 1] - G$. Then K is a compact subset of the irrational numbers. Show that $V_1(K) \geq 1 - \varepsilon$.
(c) Show that $K = \operatorname{fr} K$.

10. Let C be the Cantor set, defined in Problem 5, Section 2.4.
(a) Show that C is a null set ($V_1(C) = 0$).
(b) Show that $x \in C$ if and only if $x = \sum_{i=1}^{\infty} a_i 3^{-i}$ where $a_i = 0$ or 2, $i = 1, 2, \ldots$.
(c) Let $f(x) = \sum_{i=1}^{\infty} a_i 2^{-i-1}$ for $x \in C$. Show that $f(C) = [0, 1]$. Hence C is uncountable.
(d) For x in the kth interval of A_j (Problem 5, Section 2.4) let $f(x)$ have the constant value $(2k - 1)2^{-j}$, $k = 1, 2, \ldots, 2^{j-1}$, $j = 1, 2, \ldots$. Show that f is continuous and nondecreasing on $[0, 1]$. [*Note*: f is called the *Cantor function*.]

11. Show that any straight line in E^2 has area 0.

12. Show that if A is an unbounded measurable set, then

$$V(A) = \sup\{V(K) : K \subset A\}.$$

(If $V(A) = +\infty$, this means that for every $C > 0$ there is a compact set $K \subset A$ with $V(K) \geq C$.)

5.3 Integrals over E^n

We now begin the theory of integration. It is convenient, for technical reasons, to begin with a rather artificial case. This concerns the integral over all of E^n of a bounded function f, such that $f(\mathbf{x}) = 0$ for all $\mathbf{x}$ outside some compact set. The integral is defined by approximating f from above and below by functions (called *step functions*), which take only a finite number of values. We denote the integral of f over all of E^n by $\int f\,dV$.

Consider first functions taking only a finite number of values. In that case the integral is just a certain finite sum. A function ϕ is called a *step function* if there exists a disjoint collection $\{A_1, \ldots, A_m\}$ of bounded measurable sets such that $\phi(\mathbf{x})$ is constant on each A_k and $\phi(\mathbf{x}) = 0$ for $\mathbf{x} \notin A_1 \cup \cdots \cup A_m$. If $\phi(\mathbf{x}) = c_k$ for every $\mathbf{x} \in A_k$, then the *integral of ϕ over E^n* is

$$\int \phi\, dV = \sum_{k=1}^{m} c_k V(A_k). \tag{5.13}$$

EXAMPLE 1. Let $\phi(x) = k/m$ for $x \in [(k-1)/m, k/m)$, $k = 1, 2, \ldots, m$, and $\phi(x) = 0$ for $x \notin [0, 1)$. Then $A_k = [(k-1)/m, k/m)$, $c_k = k/m$, and

$$\int \phi\, dx = \frac{1}{m^2} + \frac{2}{m^2} + \cdots + \frac{m}{m^2} = \frac{m+1}{2m}.$$

We write dx instead of dV_1 in case $n = 1$. Note that for large m the integral is approximately $\frac{1}{2}$, which is the area of the triangle bounded by the lines $y = x$, $y = 0$, and $x = 1$ in E^2.

EXAMPLE 2. Let $\phi(x, y) = k/m$ for $(x, y) \in A_k$, $k = 1, \ldots, m$, where $A_k = \{(x, y) : r \in [(k-1)/m, k/m)\}$, $r = (x^2 + y^2)^{1/2}$ (Figure 5.4). For $r \geq 1$, we set $\phi(x, y) = 0$. Then

$$\int \phi\, dV = \sum_{k=1}^{m} \frac{k}{m} V(A_k) = \pi \sum_{k=1}^{m} \frac{k}{m}\left[\left(\frac{k}{m}\right)^2 - \left(\frac{k-1}{m}\right)^2\right]$$

$$\int \phi\, dV = \frac{\pi}{m^3}\left[2\sum_{k=1}^{m} k^2 - \sum_{k=1}^{m} k\right] = \frac{\pi}{m^3}\left[\frac{2m(m+1)(2m+1)}{6} - \frac{m(m+1)}{2}\right]$$

$$\int \phi\, dV = \frac{\pi(m+1)(4m-1)}{6m^2}.$$

For large m the integral is approximately $2\pi/3$, which is the volume of the solid in E^3 bounded above by the cone $z = (x^2 + y^2)^{1/2}$, below by the plane $z = 0$, and with lateral boundary the cylinder $x^2 + y^2 = 1$.

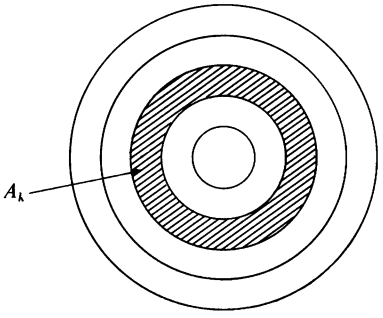


Figure 5.4

Proposition 5.3. *Let ϕ and ψ be step functions, and c any real number. Then $\phi + \psi$ and $c\phi$ are step functions. Moreover,*

$$\int(\phi + \psi)dV = \int \phi\, dV + \int \psi\, dV. \tag{5.14a}$$

$$\int(c\phi)dV = c\int \phi\, dV \text{ for any scalar } c. \tag{5.14b}$$

$$\int \psi\, dV \leq \int \phi\, dV \text{ if } \psi \leq \phi. \tag{5.14c}$$

The notation $\psi \leq \phi$ means that $\psi(\mathbf{x}) \leq \phi(\mathbf{x})$ for every $\mathbf{x} \in E^n$.

PROOF. Let A_k, c_k be as in the definition of $\int \phi\, dV$ above, $k = 1, \ldots, m$. Let $A_0 = (A_1 \cup \cdots \cup A_m)^c$ and $c_0 = 0$. Similarly, there are disjoint bounded measurable sets $B_1, \ldots, B_p$ and $d_1, \ldots, d_p$ such that $\psi(\mathbf{x}) = d_l$ for $\mathbf{x} \in B_l$. Let $B_0 = (B_1 \cup \cdots \cup B_p)^c$ and $d_0 = 0$. Then

$$\int \phi\, dV = \sum_{k=0}^{m} c_k V(A_k) = \sum_{k=0}^{m}\sum_{l=0}^{p} c_k V(A_k \cap B_l)$$

$$\int \psi\, dV = \sum_{l=0}^{p} d_l V(B_l) = \sum_{l=0}^{p}\sum_{k=0}^{m} d_l V(A_k \cap B_l).$$

The order of summation on the right-hand side can be reversed. Moreover, the sets $A_k \cap B_l$ form a disjoint collection; and both $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ are constant on each set $A_k \cap B_l$. We add to get (5.14a). To get (5.14c) we note that $c_k \leq d_l$ on $A_k \cap B_l$, and (5.14b) is immediate. □

Definition. Let f be real valued with domain E^n. The *support* of f is the smallest closed set K such that $f(\mathbf{x}) = 0$ for every $\mathbf{x} \notin K$.

EXAMPLE 3. Let $f(x) = x + 1$ if $x \in (0, 1)$ and $f(x) = 0$ if $x \notin (0, 1)$. The support of f is the closed interval $[0, 1]$.

We are now ready to define upper and lower integrals for a bounded function f that has compact support. The upper integral of f is denoted

by $\overline{\int} f\,dV$. If ϕ is any step function such that $\phi \geq f$, then $\int \phi\,dV$ is an upper estimate for it. We take the greatest lower bound of the set of all such numbers $\int \phi\,dV$.

Definition. The upper integral over E^n of a bounded function f with compact support is

$$\overline{\int} f\,dV = \inf\left\{\int \phi\,dV : \phi \geq f\right\}. \tag{5.15a}$$

We must check that there is at least one such step function, to ensure that the set on the right-hand side is not empty. However, since f is bounded, there is a number C such that $|f(\mathbf{x})| \leq C$ for every $\mathbf{x}$. Since its support is compact there is an interval I such that $f(\mathbf{x}) = 0$ for every $\mathbf{x} \notin I$. Let $\phi_0(\mathbf{x}) = C$ if $\mathbf{x} \in I$ and $\phi_0(\mathbf{x}) = 0$ if $\mathbf{x} \notin I$. Then $\phi_0 \geq f$ and ϕ_0 is a step function.

In the same way, the *lower integral of f over E^n* is denoted by $\underline{\int} f\,dV$. It is the least upper bound of the set of all numbers $\int \psi\,dV$, where ψ is a step function and $\psi \leq f$:

$$\underline{\int} f\,dV = \sup\left\{\int \psi\,dV : \psi \leq f\right\}. \tag{5.15b}$$

If $\psi \leq f \leq \phi$, then by (5.14c), $\int \psi\,dV \leq \int \phi\,dV$.

This implies that

$$\underline{\int} f\,dV \leq \overline{\int} f\,dV.$$

If ϕ is a step function, then $\int \phi\,dV = \underline{\int} \phi\,dV = \overline{\int} \phi\,dV$.

Definition. A bounded function f with compact support is *integrable* if its upper and lower integrals are equal. Its *integral over E^n* is

$$\int f\,dV = \underline{\int} f\,dV = \overline{\int} f\,dV. \tag{5.16}$$

We just observed that any step function is integrable. In Section 5.4 it is shown that every function in a much larger class is integrable.

Proposition 5.4. *Let f and g be integrable functions. Then $f + g$ is integrable and cf is integrable for any scalar c. Moreover,*

$$\int (f + g)dV = \int f\,dV + \int g\,dV. \tag{5.17a}$$

$$\int (cf)dV = c \int f\,dV. \tag{5.17b}$$

$$\int f\,dV = \int g\,dV \text{ if } f \leq g. \tag{5.17c}$$

PROOF. Given $\varepsilon > 0$ there exists a step function $\phi_1 \geq f$ such that

$$\int \phi_1 \, dV < \overline{\int} f \, dV + \frac{\varepsilon}{2},$$

and a step function $\phi_2 \geq g$ such that

$$\int \phi_2 \, dV < \overline{\int} g \, dV + \frac{\varepsilon}{2}.$$

Then $\phi_1 + \phi_2$ is a step function and $\phi_1 + \phi_2 \geq f + g$. Consequently,

$$\overline{\int}(f + g)dV \leq \int (\phi_1 + \phi_2)dV.$$

Using (5.14a), we have

$$\overline{\int}(f + g)dV < \overline{\int} f \, dV + \overline{\int} g \, dV + \varepsilon.$$

Since this is true for every $\varepsilon > 0$,

$$\overline{\int}(f + g)dV \leq \overline{\int} f \, dV + \overline{\int} g \, dV. \tag{5.18a}$$

Similarly,

$$\underline{\int}(f + g)dV \geq \underline{\int} f \, dV + \underline{\int} g \, dV. \tag{5.18b}$$

If f and g are integrable, then the right-hand sides of (5.18a) and (5.18b) are equal. The left-hand side of (5.18b) is not greater than the left-hand side of (5.18a). Hence both upper and lower integrals of $f + g$ equal $\int f \, dV + \int g \, dV$. This proves that $f + g$ is integrable and (5.17a). The rest of the proof is left to the reader (Problem 4). □

An n-dimensional interval $I = J_1 \times \cdots \times J_n$ is *half-open to the right* if each of the 1-dimensional intervals J_i is half-open *to the right*. (In the definition of figure we could equally well have used intervals half-open to the right instead of closed intervals.) Let us call a function ϕ an *elementary step function* if ϕ is constant on each interval of some grid Π and ϕ has the value 0 outside the intervals of Π. To avoid ambiguity about the values of ϕ on the bounding faces of intervals we take the intervals of Π half-open to the right.

**Riemann integral*

If in (5.15a) only elementary step functions ϕ are allowed, then the *upper Riemann integral* of f is obtained. The *lower Riemann integral* of f is obtained by allowing only elementary step functions ψ in (5.15b). Let us denote upper

and lower Riemann integrals by $\bar{S}(f)$ and $\underline{S}(f)$. Then

$$\underline{S}(f) \leq \underline{\int} f\,dV \leq \overline{\int} f\,dV \leq \bar{S}(f). \tag{5.19}$$

If $\underline{S}(f) = \bar{S}(f)$, then f is called *Riemann integrable*. Their common value $S(f)$ is the *Riemann integral* of f. From (5.19), if f is Riemann integrable, then f is integrable [in the sense of (5.16)] and

$$S(f) = \int f\,dV. \tag{5.20}$$

It can be proved that a bounded function f with compact support is Riemann integrable if and only if $V(\{\mathbf{x}: f \text{ is discontinuous at } \mathbf{x}\}) = 0$ [1, pp. 230 and 260].

Problems

1. Determine whether f is bounded. Find its support.
   (a) $f(x) = x - |x|$.
   (b) $f(x, y) = x \exp(-x^2 - y^2)$.
   (c) $f(x, y) = 1$ if either x or y is a rational number, $f(x, y) = 0$ if both x and y are irrational.
   (d) $f(x, y) = (x - y)|x + y| - (x + y)|x - y|$ if $|x| + |y| < 1$, $f(x, y) = 0$ if $|x| + |y| \geq 1$. Illustrate with a sketch.

2. Let $[a]$ denote the largest integer which is no greater than a (for instance, $[\pi] = 3$). Let $\phi(x, y) = [x + y]$ if $0 \leq x < r, 0 \leq y < s$, where r and s are positive integers. For all other (x, y) let $\phi(x, y) = 0$. Show that
$$\int \phi\,dV_2 = \frac{rs(r + s - 1)}{2}.$$

3. Let a unit square be divided into a square of side $(4m + 1)^{-1}$ in the center and $2m$ annular figures of equal width $(4m + 1)^{-1}$ surrounding it, as shown in Figure 5.5.

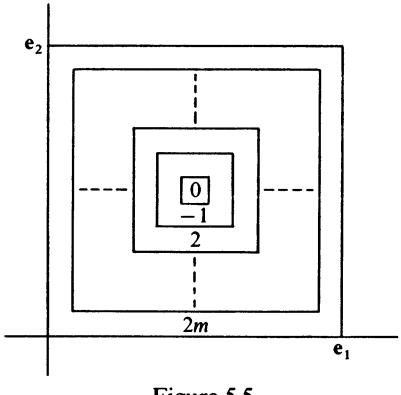


Figure 5.5

Let $\phi(x, y) = 0$ for (x, y) in the small square or outside the large square. Let $\phi(x, y) = (-1)^k k$ in the kth annular figure, $k = 1, \ldots, 2m$. Show that

$$\int \phi \, dV_2 = \frac{8m(2m + 1)}{(4m + 1)^2}.$$

What is this approximately when m is large?

4. (a) Show that if f is integrable, then $\int (cf)dV = c \int f \, dV$. [*Hint*: Show that this is true if $c \geq 0$, and that $-\underline{\int} g \, dV = \overline{\int}(-g)dV$ for every g. If $c < 0$, set $g = cf$ and $g = -cf$.]

(b) Show that $\overline{\int} f \, dV \leq \overline{\int} g \, dV$ if $f \leq g$.

5.4 Integrals over bounded sets

Let A be a bounded measurable set and f be a function that is bounded on A. More precisely, the domain of f contains A and there is a number C such that $|f(\mathbf{x})| \leq C$ for every $\mathbf{x} \in A$. Let us consider a new function with the same values as f on A and the value 0 otherwise. This function is denoted by f_A. Thus

$$f_A(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in A \\ 0 & \text{if } \mathbf{x} \notin A. \end{cases}$$

The function f_A is bounded and has compact support. The values of f outside A should contribute nothing to the integral of f over A.

Definition. The function f is *integrable over* A if f_A is an integrable function. The *integral of* f *over* A is the number

$$\int_A f \, dV = \int f_A \, dV. \tag{5.21}$$

In later sections it is sometimes convenient to use the notation $\int_A f(\mathbf{x})dV(\mathbf{x})$ for the integral $\int_A f \, dV$. Moreover, we sometimes emphasize the role of the dimension n by writing dV_n instead of dV. When $n = 1$, we usually write $\int_A f(x)dx$ instead of $\int_A f(x)dV_1(x)$.

Proposition 5.4 implies that sums and scalar multiples of functions integrable over A are also integrable over A. Theorem 5.5 gives a widely applicable condition for integrability of f. In the meantime, we summarize a number of properties of the integral in the following theorem.

Theorem 5.4. *If all the integrals involved exist, then:*

(1) $\int_A (f + g)dV = \int_A f \, dV + \int_A g \, dV$.
(2) $\int_A (cf)dV = c \int_A f \, dV$.
(3) $\int_A 1 \, dV = V(A)$.
(4) *If* $f(\mathbf{x}) \leq g(\mathbf{x})$ *for every* $\mathbf{x} \in A$, *then* $\int_A f \, dV \leq \int_A g \, dV$.
(5) *If* $|f(\mathbf{x})| \leq C$ *for every* $\mathbf{x} \in A$, *then* $|\int_A f \, dV| \leq \int_A |f| \, dV \leq CV(A)$.
(6) *If* A *is a null set, then* $\int_A f \, dV = 0$.
(7) *If* $A \cap B$ *is a null set, then* $\int_{A \cup B} f \, dV = \int_A f \, dV + \int_B f \, dV$.

PROOF. First of all,

$$(f + g)_A = f_A + g_A, \qquad (cf)_A = cf_A.$$

Therefore (1) and (2) follow from Proposition 5.4. Let 1_A denote the function with the value 1 on A and otherwise 0. It is a step function, called the *characteristic function of* A, and by (5.13) with $m = 1, c_1 = 1, \int 1_A\, dV = V(A)$. This establishes (3). For (4) we have $f_A \leq g_A$ and apply Proposition 5.4. To prove (5) we have

$$f(\mathbf{x}) \leq |f(\mathbf{x})| \leq C \quad \text{for every } \mathbf{x} \in A.$$

Hence from (4)

$$\int_A f\, dV \leq \int_A |f|\, dV \leq \int_A C\, dV,$$

and the right-hand side is $CV(A)$. Similarly, $-f(\mathbf{x}) \leq |f(\mathbf{x})|$ and

$$-\int_A f\, dV \leq \int_A |f|\, dV \leq CV(A).$$

Since $|\int_A f\, dV|$ is either $\int_A f\, dV$ or its negative, this proves (5). Part (6) follows from (5). To prove (7), $f_{A\cup B} = f_A + f_B - f_{A\cap B}$. By Proposition 5.4

$$\int f_{A\cup B}\, dV = \int f_A\, dV + \int f_B\, dV - \int f_{A\cap B}\, dV,$$

and the last term is 0 by (6). □

From (6) and (7), the integral is unchanged if A is replaced by $A \cup N$ or $A - N$, where N is any null set. Similarly, if $f(\mathbf{x}) = g(\mathbf{x})$ except for $\mathbf{x}$ in some null set, then f and g have the same integral.

EXAMPLE 1. If fr A is a null set, then

$$\int_A f\, dV = \int_{\text{int } A} f\, dV = \int_{\text{cl } A} f\, dV.$$

In elementary examples fr A is always a null set. If A is the set of rational numbers in $[0, 1]$, then fr $A = [0, 1]$, which is not a null set. Problem 9, Section 5.2, furnishes an example of a compact subset of E^1 with frontier of positive length. There are open, connected subsets of E^2 with frontiers that have positive area.

Let us next show that under quite mild assumptions about f, the integral exists. It is for this purpose that the idea of measurable function is introduced. Let f have domain E^n.

Definition. If $\{\mathbf{x} : f(\mathbf{x}) > c\}$ is a measurable set for every scalar c, then f is a *measurable function.*

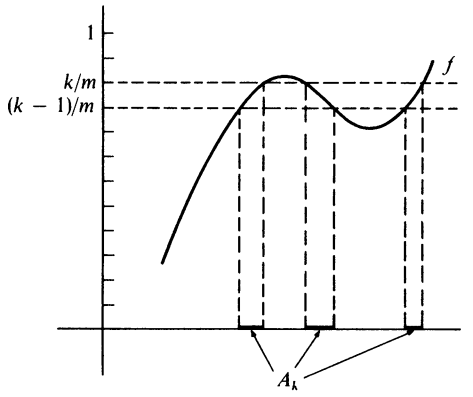


Figure 5.6

It is shown in Section 5.10 that such operations as taking the sum of two measurable functions or the limit of a sequence of measurable functions lead again to measurable functions. Just as for nonmeasurable sets, the only examples of nonmeasurable functions are obtained in a nonconstructive way using the axiom of choice.

Lemma 1. *If f is a bounded, nonnegative, measurable function with compact support, then f is integrable.*

PROOF. By replacing f by f/C, where C is an upper bound for $f(\mathbf{x})$ we may assume that $0 \leq f(\mathbf{x}) \leq 1$ for every $\mathbf{x}$. Let I be an interval containing the support of f. Given $\varepsilon > 0$, let m be a positive integer such that $V(I) < \varepsilon m$. For each $k = 1, \ldots, m$ let

$$E_k = \left\{\mathbf{x} : f(\mathbf{x}) > \frac{k-1}{m}\right\},$$

$$A_k = \left\{\mathbf{x} : \frac{k-1}{m} < f(\mathbf{x}) \leq \frac{k}{m}\right\}$$

(see Figure 5.6). Since f is measurable, each E_k is a measurable subset of I, hence $A_k = E_k - E_{k+1}$ is also measurable. Let ϕ and ψ be step functions defined by

$$\phi(\mathbf{x}) = \frac{k}{m}, \qquad \psi(\mathbf{x}) = \frac{k-1}{m}$$

for $\mathbf{x} \in A_k$, and $\phi(\mathbf{x}) = \psi(\mathbf{x}) = 0$ if $f(\mathbf{x}) = 0$. Then $\phi(\mathbf{x}) - \psi(\mathbf{x}) = 1/m$ on $E_1 = \{\mathbf{x} : f(\mathbf{x}) > 0\}$ and is 0 otherwise. Hence

$$\int \phi \, dV - \int \psi \, dV = \frac{V(E_1)}{m},$$

$$\frac{V(E_1)}{m} \leq \frac{V(I)}{m} < \varepsilon.$$

Moreover, $\psi \leq f \leq \phi$, from which

$$\int \psi \, dV \leq \underline{\int} f \, dV \leq \overline{\int} f \, dV \leq \int \phi \, dV.$$

Hence the upper and lower integrals differ by less than ε. Since this is true for every positive ε, f is integrable. □

The construction of the step functions ϕ, ψ in the proof just completed is a key to understanding the power of the Lebesgue theory of integrals. In Figure 5.6 we first subdivided the vertical axis, and then obtained the sets A_k on which ϕ and ψ are defined. From their definition, $\phi(\mathbf{x}) - \psi(\mathbf{x})$ is uniformly small for large m. In fact, $\phi(\mathbf{x}) - \psi(\mathbf{x}) \leq 1/m$ for all $\mathbf{x} \in E^n$. This construction is quite different from that used in defining the Riemann integral. As mentioned at the end of Section 5.3, the latter begins with subdividing E^n into intervals, on each of which elementary step functions approximating f from above and below must be constant. Thus, the Lebesgue theory subdivides the 1-dimensional range of f, while the Riemann theory subdivides the n-dimensional domain of f.

To apply this key lemma, we let A be a bounded measurable set, and f be a function whose domain contains A. If $\{\mathbf{x} \in A : f(\mathbf{x}) > c\}$ is measurable for every scalar c, then we call f *measurable on* A.

Theorem 5.5. *If f is bounded and measurable on A, then f is integrable over A.*

PROOF. Let us first assume that $f \geq 0$. If $c \geq 0$, then the set

$$\{\mathbf{x} : f_A(\mathbf{x}) > c\} = \{\mathbf{x} \in A : f(\mathbf{x}) > c\}$$

is measurable since f is measurable on A. If $c < 0$, the left-hand side is E^n which is measurable. Hence f_A is measurable. By Lemma 1, f_A is integrable.

If f has negative values on A, let $g(\mathbf{x}) = f(\mathbf{x}) + C$ where C is an upper bound for $|f(\mathbf{x})|$ on A. Then $g \geq 0$, and g is bounded and measurable on A. Hence g is integrable over A, and so is f. □

Corollary. *If f is bounded and continuous on A, then f is integrable over A.*

PROOF. For every c, $\{\mathbf{x} \in A : f(\mathbf{x}) > c\}$ is open relative to A. In other words, it is the intersection of A with an open set, and hence is measurable. □

In particular, if A is compact, then any f continuous on A is bounded and therefore integrable over A.

Theorem 5.5 has a sort of converse. If a bounded function f is integrable over A, then f is measurable on A. We do not prove this.

*Relation to the Riemann integral

A function f is *Riemann integrable over* A if f_A is Riemann integrable according to the definition in Section 5.3. Its *Riemann integral over* A is $S(f_A)$. If f is Riemann integrable over A, then from (5.20), $S(f_A) = \int_A f \, dV$.

A bounded set A is *Jordan measurable* if its characteristic function 1_A is Riemann integrable. It can be shown that A is Jordan measurable if and only if fr A is a null set [1, p.256].

If $A = [a, b]$, a closed interval of E^1, then the definition of Riemann integral given above can rather easily be shown to agree with Riemann's original definition of integral as the limit of sums.

Problems

1. Let $f(x) = 2x - x^2$ if $0 \leq x \leq 2$, $f(x) = 0$ otherwise. Using the notation in the proof of Lemma 1, describe the sets $A_1, \ldots, A_m$. Sketch the step functions ϕ and ψ in the case $m = 4$.

2. In each case show that f is integrable over A.
   (a) $f(x) = x^2 \exp x$, $A = [0, a]$.
   (b) $f(x) = \sin(1/x)$ if $x \neq 0$, $f(0) = 5$, $A = [-1, 1]$.
   (c) $f(x, y) = (x^4 - y^2)/(x^2 - y)$, $A = \{(x, y) : |x| \leq 1, |y| \leq 1, x^2 \neq y\}$.
   (d) $f(x) = 0$ if x is irrational, $f(x) = 1/q$ if $x = p/q$ where p and q are integers with no common factor, $A = (0, 1]$.
   (e) $f(x) = 1$ if x is irrational, $f(x) = 0$ if x is rational, $A = [a, b]$.

3. For each part of Problem 2 describe the sets $\{\mathbf{x} \in A : f(\mathbf{x}) > c\}$.

4. Show that if f, g, and h are integrable over A and $|f(\mathbf{x}) - g(\mathbf{x})| \leq h(\mathbf{x})$ for every $\mathbf{x} \in A$, then $|\int_A f\, dV - \int_A g\, dV| \leq \int_A h\, dV$.

5. Let f be of class $C^{(2)}$ on $[0, a]$ and $b = \max\{f''(x)| : 0 \leq x \leq a\}$. Let $g(x) = f(0) + f'(0)x$. Using Problem 4, show that $|\int_0^a f\, dx - af(0) - a^2 f'(0)/2| \leq a^3 b/6$. Use this result to estimate $\int_0^{1/2} \exp(-x^2/2) dx$.

6. (*Mean value theorem for integrals.*) Let A be compact and connected. Let f be continuous on A and g be integrable over A with $g(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in A$. Prove that there exists $\mathbf{x}^* \in A$ such that

$$\int_A fg\, dV = f(\mathbf{x}^*) \int_A g\, dV.$$

   [*Hint*: Let C and c be the maximum and minimum values of f on A. Then $cg \leq fg \leq Cg$. Use (2) and (4) of Theorem 5.4 and the intermediate value theorem.]

5.5 Iterated integrals

Thus far we have given no effective procedure for the actual evaluation of integrals. One method for doing this is by writing the integral as an iterated integral and applying the fundamental theorem of calculus. Let $1 \leq s < n$. In most cases we shall take $s = 1$ or $s = n - 1$. Then E^n can be regarded as the cartesian product $E^s \times E^{n-s}$. Let us write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, where

$$\mathbf{x}' = (x^1, \ldots, x^s) \in E^s, \qquad \mathbf{x}'' = (x^{s+1}, \ldots, x^n) \in E^{n-s}.$$

Let A be a set and f a function whose domain contains A. In the present section we assume that A is compact, and that f is continuous on A. These

assumptions are relaxed in Section 5.11. We show that $\int_A f\,dV$ can be expressed as an iterated integral as follows. First for fixed $\mathbf{x}'$, integrate over a set $A(\mathbf{x}') \subset E^{n-s}$. Then integrate the result of the first integration over a set $R \subset E^s$. The sets $A(\mathbf{x}')$ and R are defined as follows:

$$A(\mathbf{x}') = \{\mathbf{x}'' : (\mathbf{x}', \mathbf{x}'') \in A\} \tag{5.22}$$

$$R = \{\mathbf{x}' : A(\mathbf{x}') \text{ is not empty}\}.$$

The set R is simply the projection of A onto E^s (see Figure 5.7).

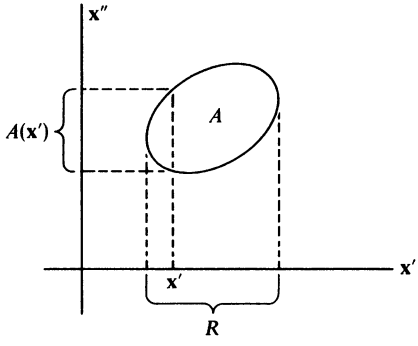


Figure 5.7

Theorem 5.6. *Let f be continuous on a compact set A. Then*

$$\int_A f(\mathbf{x})dV_n(\mathbf{x}) = \int_R \left\{\int_{A(\mathbf{x}')} f(\mathbf{x}', \mathbf{x}'')dV_{n-s}(\mathbf{x}'')\right\}dV_s(\mathbf{x}'). \tag{5.23}$$

As mentioned earlier, the left-hand side of (5.23) is another notation for $\int_A f\,dV$. A precise interpretation of the right-hand side is as follows. For fixed $\mathbf{x}' \in R$, let $f(\mathbf{x}', \ )$ denote the function whose value at each $\mathbf{x}'' \in A(\mathbf{x}')$ is $f(\mathbf{x}', \mathbf{x}'')$. The inner integral is the integral over $A(\mathbf{x}')$ of the function $f(\mathbf{x}', \ )$. Let

$$g(\mathbf{x}') = \int_{A(\mathbf{x}')} f(\mathbf{x}', \mathbf{x}'')dV_{n-s}(\mathbf{x}'').$$

The outer integral on the right-hand side of (5.23) is the integral of g over R. The set R is compact, since R is the projection of the compact set A on E^s. It is easily shown that $A(\mathbf{x}')$ is compact for each $\mathbf{x}'$, and that $f(\mathbf{x}', \ )$ is continuous on $A(\mathbf{x}')$. However, we do not need to use these facts in the proof of Theorem 5.6. The function g need not be continuous. However, it is shown in the course of proving Theorem 5.6 that g is integrable over R. In the proof of Theorem 5.6, the following fact about monotone sequences of functions is used. For each $\nu = 1, 2, \ldots$ let F_ν be a bounded measurable function with

compact support such that

$$F_1 \geq F_2 \geq \cdots \geq 0.$$

Let

$$F(\mathbf{x}) = \lim_{\nu \to \infty} F_\nu(\mathbf{x}) \quad \text{for every } \mathbf{x}.$$

Then F is measurable (and of course bounded), and

$$\int F \, dV_n = \lim_{\nu \to \infty} \int F_\nu \, dV_n.$$

A proof of this fact is given in the section on convergence theorems (Corollary 3, Section 5.11).

PROOF. The proof of Theorem 5.6 proceeds by observing that the theorem is true for elementary step functions, and then by constructing a monotone sequence of elementary step functions tending to f_A.

Let us first show that

$$(*) \qquad \int \Phi(\mathbf{x}) dV_n(\mathbf{x}) = \int \left\{ \int \Phi(\mathbf{x}', \mathbf{x}'') dV_{n-s}(\mathbf{x}'') \right\} dV_s(\mathbf{x}')$$

if Φ is any elementary step function. If I is any n-dimensional interval, then $I = I' \times I''$, where I' and I'' are s- and $(n - s)$-dimensional intervals. If Φ is the characteristic function 1_I of I, then (*) becomes $V_n(I) = V_s(I')V_{n-s}(I'')$, which is true by definition of measure for intervals. But any elementary step function can be written as a linear combination $\Phi = c_1\Phi_1 + \cdots + c_p\Phi_p$, where each Φ_k is the characteristic function of an n-dimensional interval. Then

$$\int \Phi \, dV_n = \sum_{k=1}^{p} c_k \int \left\{ \int \Phi_k \, dV_{n-s} \right\} dV_s = \int \left\{ \int \left(\sum_{k=1}^{p} c_k \Phi_k \right) dV_{n-s} \right\} dV_s,$$

which is just (*).

Now let A be a compact set and f be continuous on A. For the moment, assume that $f \geq 0$. Let $F = f_A$. Let us define a monotone sequence of elementary step functions $F_1 \geq F_2 \geq \ldots$ as follows. Let I_0 be some interval containing A and let C be the maximum value of f on A. Let $F_1(\mathbf{x}) = C$ for $\mathbf{x} \in I_0$, and $F_1(\mathbf{x}) = 0$ otherwise. Divide I_0 into 2^n congruent subintervals $I_1, \ldots, I_m$, $m = 2^n$, and let $A_k = A \cap \operatorname{cl} I_k$. If $\mathbf{x} \in I_k$ and A_k is not empty, let $F_2(\mathbf{x})$ be the maximum value of f on A_k. Otherwise, let $F_2(\mathbf{x}) = 0$. Then $F_1 \geq F_2 \geq F$. The function F_3 is defined similarly by dividing each interval I_k into 2^n congruent subintervals, and so on (Figure 5.8).

Let $d_0 = \operatorname{diam} I_0$. At the νth step the diameter d_ν of each interval I is $2^{-\nu+1} d_0$, and $F_\nu(\mathbf{x}) = 0$ except on those intervals of diameter d_ν whose closures meet A. Let us show that for every $\mathbf{x}$

$$(**) \qquad F(\mathbf{x}) = \lim_{\nu \to \infty} F_\nu(\mathbf{x}).$$

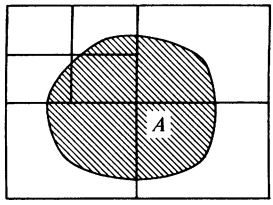


Figure 5.8

If $\mathbf{x} \notin A$, then since A is closed there exists $\nu(\mathbf{x})$ such that $d_\nu < \operatorname{dist}(\mathbf{x}, A)$ for every $\nu \geq \nu(\mathbf{x})$. In this case $0 = F(\mathbf{x}) = F_\nu(\mathbf{x})$ when $\nu \geq \nu(\mathbf{x})$. Thus (**) holds for every $\mathbf{x} \notin A$. Suppose that $\mathbf{x} \in A$. Since f is continuous, given $\varepsilon > 0$ there exists $\delta(\mathbf{x}) > 0$ such that $|f(\mathbf{y}) - f(\mathbf{x})| < \varepsilon$ for every $\mathbf{y} \in A$ such that $|\mathbf{y} - \mathbf{x}| < \delta(\mathbf{x})$. Choose $\nu(\mathbf{x})$ such that $d_\nu < \delta(\mathbf{x})$ for every $\nu \geq \nu(\mathbf{x})$. Then $F(\mathbf{x}) \leq F_\nu(\mathbf{x}) < F(\mathbf{x}) + \varepsilon$ for every $\nu \geq \nu(\mathbf{x})$. Therefore (**) also holds when $\mathbf{x} \in A$.

By the monotone sequences theorem

$$\int F \, dV_n = \lim_{\nu \to \infty} \int F_\nu \, dV_n.$$

For each $\mathbf{x}'$, the functions $F_\nu(\mathbf{x}', \ )$ form a monotone sequence tending to $F(\mathbf{x}', \ )$. Let

$$G_\nu(\mathbf{x}') = \int F_\nu(\mathbf{x}', \ ) dV_{n-s}, \qquad G(\mathbf{x}') = \int F(\mathbf{x}', \ ) dV_{n-s}.$$

Applying the monotone sequences theorem to the sequence $[F_\nu(\mathbf{x}', \ )]$,

$$\lim_{\nu \to \infty} G_\nu(\mathbf{x}') = G(\mathbf{x}')$$

for every $\mathbf{x}'$. Moreover, $G_1 \geq G_2 \geq \cdots$. Applying the monotone sequences theorem to this sequence,

$$\lim_{\nu \to \infty} \int G_\nu \, dV_s = \int G \, dV_s.$$

Since F_ν is an elementary step function, we have by (*) with $\Phi = F_\nu$,

$$\int F_\nu \, dV_n = \int G_\nu \, dV_s.$$

Therefore

$$\int F \, dV_n = \int G \, dV_s. \tag{5.23a}$$

Since $F = f_A$, $\int F \, dV_n = \int_A f \, dV_n$. For $\mathbf{x}' \in R$, the inner integral on the right-hand side of (5.23) is $G(\mathbf{x}')$; and $G(\mathbf{x}') = 0$ for $\mathbf{x} \notin R$. Thus, $\int G \, dV_s$ equals the

outer integral on the right-hand side of (5.23). From (5.23a), Theorem 5.6 is proved in case $f \geq 0$.

To remove the assumption $f \geq 0$, write $f = (f + C) - C$, where C is the maximum value of $|f|$ on A. Then f is the difference of nonnegative continuous functions for each of which (5.23) holds. By subtraction, (5.23) holds for f. □

It is not essential in Theorem 5.6 that $\mathbf{x}' = (x^1, \ldots, x^s)$. One can equally well take integers $i_1 < \cdots < i_s$ and

$$\mathbf{x}' = (x^{i_1}, \ldots, x^{i_s}), \qquad \mathbf{x}'' = (x^{j_1}, \ldots, x^{j_{n-s}}),$$

where $j_1 < \cdots < j_{n-s}$ are those integers between 1 and n not included among $i_1, \ldots, i_s$.

In particular, let $n = 2$. Then $s = 1$, and we can take either $x' = x$ or $x' = y$. To avoid writing parentheses we write $\int dx \int f\, dy$ instead of $\int \{\int f\, dy\} dx$. Then

$$\int_A f\, dV_2 = \int dx \int f\, dy = \int dy \int f\, dx,$$

where the iterated integrals are taken over the appropriate subsets of E^1. Many authors write the iterated integral as $\iint f\, dy\, dx$, but this notation would lead to confusion when we come to the exterior differential calculus in later chapters.

The iterated integral is usually easier to evaluate when taken in one of the two possible orders than in the other order.

EXAMPLE 1. Consider the iterated integral

$$\int_{-1}^{1} dx \int_{x^2}^{1} (x^2 + y) dy.$$

Then $f(x, y) = x^2 + y$ and A is as shown (Figure 5.9). Evaluating the inner integral first, we get

$$\int_{-1}^{1} \left(x^2 y + \frac{y^2}{2}\right)\Bigg|_{x^2}^{1} dx = \int_{-1}^{1} \left[\left(x^2 + \frac{1}{2}\right) - \left(x^4 + \frac{x^4}{2}\right)\right] dx = \frac{16}{15}.$$

Writing $\int_A f\, dV_2$ as an iterated integral in the opposite order, we get

$$\int_0^1 dy \int_{-\sqrt{y}}^{\sqrt{y}} (x^2 + y) dx = \int_0^1 \left(\frac{x^3}{3} + xy\right)\Bigg|_{-\sqrt{y}}^{\sqrt{y}} dy = \frac{8}{3} \int_0^1 y^{3/2}\, dy = \frac{16}{15}.$$

In evaluating these integrals we have, of course, used the fundamental theorem of calculus (Appendix A.3). If $n = 3$, then the integral can be written as an iterated triple integral.

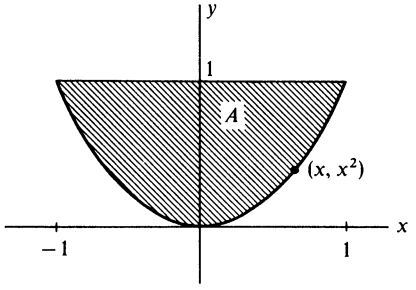


Figure 5.9

EXAMPLE 2. Let $A = \{(x, y, z) : x \geq 0, z \geq 0, 0 \leq y \leq 4 - x^2 - z^2\}$. Then

$$\int_A f\,dV_3 = \int_R \left\{ \int_{A(x,y)} f\,dz \right\} dV_2(x, y),$$

and $A(x, y)$ is the interval $[0, \sqrt{4 - x^2 - y}]$ (see Figure 5.10.) Writing the integral over R as an iterated integral, we get

$$\int_A f\,dV_3 = \int_0^2 dx \int_0^{4-x^2} dy \int_0^{\sqrt{4-x^2-y}} f\,dz.$$

There are five other possible ways of writing $\int_A f\,dV_3$ as an iterated triple integral. For instance,

$$\begin{aligned} \int_A f\,dV_3 &= \int_S \left\{ \int_0^{4-x^2-z^2} f\,dy \right\} dV_2(x, z) \\ &= \int_0^2 dz \int_0^{\sqrt{4-z^2}} dx \int_0^{4-x^2-z^2} f\,dy, \end{aligned}$$

where $S = \{(x, y) : x \geq 0, z \geq 0, x^2 + z^2 \leq 4\}$.

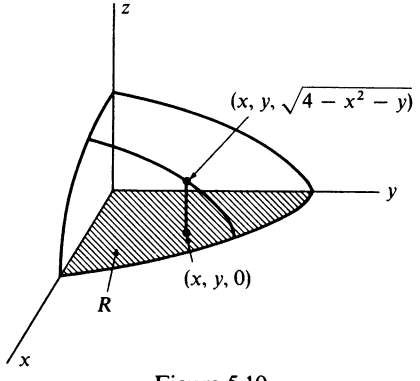


Figure 5.10

In the same way, for any n the integral can be written in $n!$ possible ways as an n-fold iterated integral.

As a special case of Theorem 5.6, one can take $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in A$. One gets the following formula for the measure of A.

Corollary

$$V_n(A) = \int_R V_{n-s}[A(\mathbf{x}')]dV_s(\mathbf{x}'). \tag{5.24}$$

In particular, let $s = 1$. Writing $x' = x^1 = u$, (5.24) becomes

$$V_n(A) = \int_R V_{n-1}[A(u)]du. \tag{5.25}$$

The set $A(u)$ is congruent to the intersection of A and the hyperplane $x^1 = u$. In effect, (5.25) states that $V_n(A)$ is the integral of the $(n-1)$-dimensional measures of these intersections. For $n = 3$, this is the method of "volumes by slices" of elementary calculus.

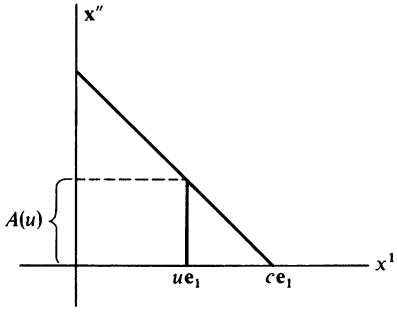


Figure 5.11

EXAMPLE 3. Let A be the n-simplex with vertices $\mathbf{0}, c\mathbf{e}_1, \ldots, c\mathbf{e}_n$, where $c > 0$. For $c = 1$ this is the standard n-simplex (Section 1.5). Let us show by induction on n that $V_n(A) = c^n/n!$. If $n = 1$, then $A = [0, c]$ and $V_1(A) = c$. Assuming the result in dimension $n-1$, we apply (5.25) (see Figure 5.11). Now

$$A = \{\mathbf{x} : x^1 + \cdots + x^n \leq c, x^i \geq 0 \text{ for } i = 1, \ldots, n\},$$
$$A(u) = \{\mathbf{x}'' : x^2 + \cdots + x^n \leq c - u, x^i \geq 0 \text{ for } i = 2, \ldots, n\},$$

and $R = [0, c]$. Therefore

$$V_n(A) = \int_0^c \frac{(c-u)^{n-1}}{(n-1)!}\,du = -\frac{(c-u)^n}{n(n-1)!}\bigg|_0^c = \frac{c^n}{n!}.$$

EXAMPLE 4. Let $s = n - 1$. Suppose that A has the particular form

$$A = \{\mathbf{x} : h(\mathbf{x}') \leq x^n \leq H(\mathbf{x}'), \mathbf{x}' \in R\}.$$

Then

$$A(\mathbf{x}') = [h(\mathbf{x}'), H(\mathbf{x}')]$$

and

$$V_n(A) = \int_R [H(\mathbf{x}') - h(\mathbf{x}')] dV_{n-1}(\mathbf{x}').$$

For instance, if $n = 3$ then A is a solid bounded above by the surface with equation $z = H(x, y)$ and below by the one with equation $z = h(x, y)$, $(x, y) \in R$.

Moments about hyperplanes, centroids

Let P be a hyperplane (Section 1.3). Then $P = \{\mathbf{x} : \mathbf{y} \cdot \mathbf{x} = c\}$. We may suppose that $|\mathbf{y}| = 1$. This determines the vector $\mathbf{y}$ up to a change in sign. We make a particular choice for $\mathbf{y}$, which amounts to choosing one of the two half spaces into which P divides E^n as "positive" and the other "negative." The moment of a point $\mathbf{x}$ about P is defined to be $\mathbf{y} \cdot \mathbf{x} - c$. By Problem 6, Section 4.8, the moment is in absolute value equal to the distance from $\mathbf{x}$ to P. The *moment* of a bounded measurable set A about P is defined as $\int_A (\mathbf{y} \cdot \mathbf{x} - c) dV_n(\mathbf{x})$.

In particular, let us take $\mathbf{y} = \mathbf{e}_i$ a standard basis vector and $c = 0$. Then $P = \{\mathbf{x} : x^i = 0\}$. We denote the moment of A about P by m^i in this case. Thus

$$m^i = \int_A x^i \, dV_n(\mathbf{x}), \qquad i = 1, \ldots, n.$$

Let $\mathbf{m} = (m^1, \ldots, m^n)$. The *centroid* of A is the point $\bar{\mathbf{x}}$ such that $\mathbf{m} = V_n(A)\bar{\mathbf{x}}$, provided $V_n(A) > 0$. The components of $\bar{\mathbf{x}}$ satisfy

$$\bar{x}^i = \frac{m^i}{V_n(A)}, \qquad i = 1, \ldots, n.$$

The centroid has the following simple interpretation in elementary mechanics. Suppose that A is a body made of some material of density $\rho(\mathbf{x})$. The *mass* of A is $\int_A \rho(\mathbf{x}) dV_n(\mathbf{x})$, and the *moment of mass* about hyperplane P is $\int_A (\mathbf{y} \cdot \mathbf{x} - c)\rho(\mathbf{x}) dV_n(\mathbf{x})$. The *center of mass* is the point $\mathbf{x}^*$ such that

$$(x^i)^* = \frac{\int_A x^i \rho(\mathbf{x}) dV_n(\mathbf{x})}{\int_A \rho(\mathbf{x}) dV_n(\mathbf{x})}, \qquad i = 1, \ldots, n.$$

If ρ is constant, then $\mathbf{x}^* = \bar{\mathbf{x}}$.

EXAMPLE 5. Find the centroid of the hemispherical ball

$$H = \{\mathbf{x} : |\mathbf{x}| \leq 1, x^1 \geq 0\}.$$

Let α_n denote the measure of an n-ball of radius 1. Then $V_n(H) = \frac{1}{2}\alpha_n$. From elementary geometry α_n is known for $n \leq 3$. Problem 7 gives α_4; in Section 5.9 a general formula for α_n is given. Let us use the intuitively evident fact that the measure of an n-ball of radius r is $\alpha_n r^n$. This follows from Theorem 5.7, Section 5.7.

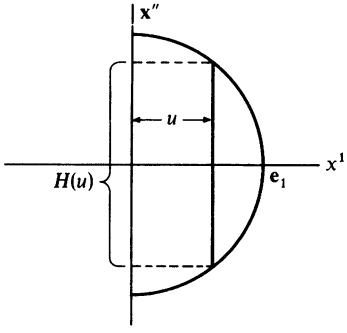


Figure 5.12

The components of the centroid $\bar{\mathbf{x}}$ are $\bar{x}^i = m^i/V_n(H)$. By symmetry considerations, $m^i = 0$ if $i > 1$. By the same method as in Example 3, using the fact that $H(u)$ is an $(n-1)$ ball of radius $(1-u^2)^{1/2}$ (Figure 5.12).

$$m^1 = \int_H x^1 \, dV_n(\mathbf{x}) = \int_0^1 u \, du \int_{H(u)} 1 \, dV_{n-1}(\mathbf{x}'')$$

$$= \alpha_{n-1} \int_0^1 u(1-u^2)^{(n-1)/2} \, du = \frac{\alpha_{n-1}}{n+1}.$$

Thus

$$\bar{\mathbf{x}} = \frac{2\alpha_{n-1}}{(n+1)\alpha_n} \mathbf{e}_1.$$

Even-order moments about a closed set

Let B be a closed set, and k an even positive integer ($k = 2, 4, \ldots$). The kth order moment of a point $\mathbf{x}$ about B is $[\text{dist}(\mathbf{x}, B)]^k$, where $\text{dist}(\mathbf{x}, B)$ is the distance from $\mathbf{x}$ to B. The kth order moment of a bounded measurable set A about B is

$$\int_A [\text{dist}(\mathbf{x}, B)]^k \, dV_n(\mathbf{x}).$$

The case $k = 2$ is of interest in mechanics.

Problems

1. Find the area and also the centroid of:
   (a) $\{(x, y): x^2 \le y \le x + 2\}$. (b) $\{(x, y): |y| - 1 \le x \le \sqrt{1 - y^2}\}$.

2. Express the iterated integral

$$\int_0^1 dy \int_0^{f(y)} xy\, dx, \text{ where } f(y) = \min[1, \log(1/y)],$$

   as an integral over a set $A \subset E^2$, and then as an iterated integral in the opposite order. Evaluate it.

3. Express as an iterated triple integral:

$$\int_A f\, dV_3, \text{ where } A = \{(x, y, z): x^2 + z^2 \le y^2 \le 8 - (x^2 + z^2)\}.$$

4. Find the volume of

$$\{(x, y, z): |x| + |y| + |z| \le 2, |x| \le 1, |y| \le 1\}.$$

5. Find the volume of

$$\{(x, y, z): |x| + |y| + |z| \le 2, z^2 \le y\}.$$

6. (a) Suppose that $f(x, y) = g(x)h(y)$ for every $(x, y) \in A$ and that $A = R \times S$. Show that

$$\int_A f\, dV_2 = \left(\int_R g\, dx\right)\left(\int_S h\, dy\right).$$

   (b) Evaluate $\int_0^1 dx \int_0^1 \exp(x + y)dy$.
   (c) Evaluate $\int_0^{\pi} dy \int_0^{\pi/2} xy \cos(x + y)dx$.

7. Let α_n be the measure of the unit n-ball $\{\mathbf{x}: |\mathbf{x}| \le 1\}$. Show that

$$\alpha_n = 2\alpha_{n-1} \int_0^1 (1 - u^2)^{(n-1)/2} du.$$

   Show that $\alpha_4 = \pi^2/2$. (In Section 5.9 we give a general formula for α_n.)

8. Write points of E^{n+1} as $(\mathbf{x}, z)$, where $\mathbf{x} = (x^1, \ldots, x^n)$. Let

$$A = \{(\mathbf{x}, z): 0 \le z \le 1 - |\mathbf{x}|^2\}.$$

   Show that $V_{n+1}(A) = 2\alpha(n)/(n + 2)$.

9. Let Σ be the standard n-simplex.
   (a) Show that the centroid of Σ is at the barycenter.
   (b) Show that the second moment of Σ about the $(n - r)$-dimensional plane spanned by $\mathbf{e}_{r+1}, \ldots, \mathbf{e}_n$ is $2r/(n + 2)!$

10. Recall the definition of uniform convergence of a sequence of functions (Section 2.10). Show that the sequence $F_1, F_2, \ldots$ constructed in the proof of Theorem 5.6 converges uniformly to f_A if and only if $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in \text{fr } A$. [*Hint*: f is uniformly continuous on the compact set A (see Problem 8, Section 2.5).]

5.6 Integrals of continuous functions

In the previous sections the integral of a bounded function f over a bounded set A was considered. However, in some instances the integral can be defined without these boundedness assumptions. In the present section we suppose that f is continuous on A. In Section 5.11 we make the weaker assumption that f is a measurable function.

Let us first assume that $f \geq 0$. Let A be a measurable set. For any compact set $K \subset A$ the integral $\int_K f\,dV$ exists in the sense of Section 5.4. The integral of f over A is defined as the least upper bound of such integrals:

$$\int_A f\,dV = \sup\left\{\int_K f\,dV : K \subset A\right\}. \tag{5.26}$$

If the set of numbers on the right-hand side has an upper bound, then f is *integrable* over A. Otherwise, the integral *diverges to* $+\infty$. The definition has two immediate consequences, which we state in the following form.

Lemma 1. *Let g and h be continuous on A, with $g \geq 0$, $h \geq 0$.*

(a) *If $h \leq g$ and g is integrable over A, then h is integrable over A.*

(b) *If g and h are both integrable over A, then $g + h$ is integrable over A.*

PROOF. To prove (a), for every compact set $K \subset A$,

$$\int_K h\,dV \leq \int_K g\,dV \leq \int_A g\,dV.$$

Hence $\int_A g\,dV$ is an upper bound for $\{\int_K h\,dV : K \subset A\}$.

To prove (b), for every compact set $K \subset A$,

$$\int_K (g + h)dV = \int_K g\,dV + \int_K h\,dV \leq \int_A g\,dV + \int_A h\,dV.$$

Hence the right-hand side is an upper bound for $\{\int_K (g + h)dV : K \subset A\}$. □

Let us consider some important special cases of (5.26).

Case 1. A is closed, f is continuous on A, and $f \geq 0$. For every $r \geq 0$ let $A(r) = \{\mathbf{x} \in A : |\mathbf{x}| \leq r\}$, and let

$$\psi(r) = \int_{A(r)} f\,dV.$$

Since $f \geq 0$, $\psi(r)$ is a nondecreasing function. Hence, as $r \to \infty$ $\psi(r)$ tends to a limit l, finite or $+\infty$. Since $A(r)$ is a compact subset of A, by (5.26) $\psi(r) \leq \int_A f\,dV$ for each r. Hence, $l \leq \int_A f\,dV$. On the other hand, if $K \subset A$ and K is compact, then $K \subset A(r)$ for large enough r. Hence

$$\int_K f\,dV \leq \int_{A(r)} f\,dV \leq l.$$

Since this is true for each such K, $\int_A f\,dV \leq l$. We have shown that

$$\int_A f\,dV = \lim_{r \to +\infty} \int_{A(r)} f\,dV. \tag{5.27}$$

In particular, if $n = 1$ and $A = [a, \infty)$, then

$$\int_a^\infty f\,dx = \lim_{r \to +\infty} \int_a^r f\,dx.$$

EXAMPLE 1. Let $f(x) = x^{-p}$. Then

$$\int_1^\infty x^{-p}\,dx = \frac{1}{p-1}$$

if $p > 1$. If $p \leq 1$, then the integral diverges to $+\infty$.

EXAMPLE 2. Let $f(x) = \exp(-bx)$, $b > 0$. Then f is integrable over $[a, \infty)$ for any a.

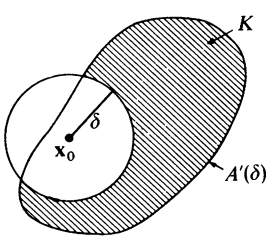


Figure 5.13

Case 2. Let $A = K - \{\mathbf{x}_0\}$, where K is compact. The function f is continuous on A, $f \geq 0$, but f may be unbounded on any neighborhood of $\mathbf{x}_0$. For each $\delta > 0$ let $A'(\delta) = \{\mathbf{x} \in K : |\mathbf{x} - \mathbf{x}_0| \geq \delta\}$ (Figure 5.13). Each of the sets $A'(\delta)$ is compact, and

$$\int_A f\,dV = \lim_{\delta \to 0^+} \int_{A'(\delta)} f\,dV. \tag{5.28}$$

The proof of (5.28) is similar to that for (5.27). The same formula holds if f is continuous on $A = K - L$, where L is any closed set. In this case $A'(\delta)$ is the set of points of K distant at least δ from L.

EXAMPLE 3. Let $A = (0, 1]$ and $f(x) = x^{-p}$. Then $\int_0^1 x^{-p}\,dx = 1/(1-p)$ if $p < 1$, and the integral diverges to $+\infty$ if $p \geq 1$. [*Note*: If $A \subset [a, b]$ and $[a, b] - A$ is a null set, then we still use the notation $\int_a^b$ for $\int_A$.]

Case 3. Let f be continuous on $A = K - \{\mathbf{x}_0\}$, where K is closed but not compact. Let $A_1 = \{\mathbf{x} \in K : |\mathbf{x} - \mathbf{x}_0| \geq 1\}$, $A_2 = \{\mathbf{x} \in K : 0 < |\mathbf{x} - \mathbf{x}_0| \leq 1\}$.

Then $\int_{A_1} f\,dV$ and $\int_{A_2} f\,dV$ can be treated respectively as in Cases 1 and 2. Since $A_1 \cap A_2$ is a null set, if both of these integrals exist their sum is $\int_A f\,dV$. If either the integral of f over A_1 or the integral over A_2 does not exist, then f is not integrable over A.

EXAMPLE 3 (*continued*). Let $A = (0, \infty)$ and $f(x) = x^{-p}$. Taking $A_1 = [1, \infty)$, $A_2 = (0, 1]$, the integral over A_1 exists only if $p > 1$, and over A_2 only if $p < 1$. Hence $\int_0^\infty x^{-p}\,dx$ diverges to $+\infty$ for every p.

Let us now consider a function f continuous on A, such that f may have both positive and negative values. Its integral is defined by (5.30), after writing f as the difference of two nonnegative functions. Let

$$f^+(\mathbf{x}) = \max\{f(\mathbf{x}), 0\},$$
$$f^-(\mathbf{x}) = \max\{-f(\mathbf{x}), 0\}.$$

Then f^+ and f^- are continuous on A (Problem 10) and

$$f(\mathbf{x}) = f^+(\mathbf{x}) - f^-(\mathbf{x}), \qquad |f(\mathbf{x})| = f^+(\mathbf{x}) + f^-(\mathbf{x}), \tag{5.29}$$

for every $\mathbf{x} \in A$ (see Figure 5.14). The function f is called *integrable over A* if f^+ and f^- are integrable over A. Its integral is

$$\int_A f\,dV = \int_A f^+\,dV - \int_A f^-\,dV. \tag{5.30}$$

If both f and A are bounded, the new definition of integral agrees with that in Section 5.4 (Problem 11).

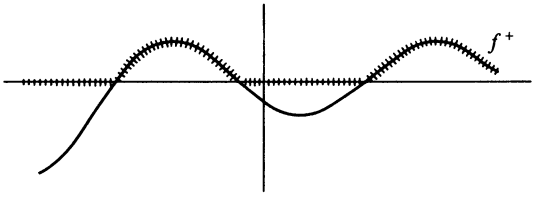


Figure 5.14

Some authors assign a value $+\infty$ or $-\infty$ to the integral in case at most one of the functions f^+ and f^- has a divergent integral. However, in no case should one try to evaluate the meaningless expression $\infty - \infty$ if both f^+ and f^- have divergent integrals.

Lemma 1 above has two important corollaries.

Corollary 1. *Let f be continuous on A. Then f is integrable over A if and only if $|f|$ is integrable over A.*

PROOF. Let f be integrable over A. Then f^+ and f^- are integrable over A, and by (b) of Lemma 1 so is their sum $|f| = f^+ + f^-$. Conversely, let $|f|$ be integrable over A. Since $0 \leq f^+ \leq |f|, 0 \leq f^- \leq |f|$, by (a) of Lemma 1 f^+ and f^- are integrable over A. Hence f is integrable over A. □

Corollary 2 (*Comparison test*). *Let f and g be continuous on A. If $|f| \leq g$ and g is integrable over A, then f is integrable over A.*

PROOF. In (a) of Lemma 1, let $h = |f|$. □

In many instances it can be shown that f is integrable by comparing $|f|$ with a function g known to be integrable (Corollary 2).

EXAMPLE 4. Let $A = K - \{\mathbf{x}_0\}$, where K is an n-ball with center $\mathbf{x}_0$. Then

$$\int_A |\mathbf{x} - \mathbf{x}_0|^{-p}\, dV$$

exists if $p < n$ and diverges to $+\infty$ if $p \geq n$. This can be proved by introducing generalized spherical coordinates (Section 5.9). Let $f(\mathbf{x}) = \phi(\mathbf{x})|\mathbf{x} - \mathbf{x}_0|^{-p}$, where ϕ is continuous and $|\phi(\mathbf{x})| \leq C$ for every $\mathbf{x} \in A$. Let $g(\mathbf{x}) = C|\mathbf{x} - \mathbf{x}_0|^{-p}$. By the comparison test, f is integrable over A if $p < n$. If $\phi(\mathbf{x}) \geq c > 0$ for every $\mathbf{x} \in A$, let $h(\mathbf{x}) = c|\mathbf{x} - \mathbf{x}_0|^{-p}$. If $p \geq n$, then $f \geq h$. The integral of h over A diverges to $+\infty$, and hence so does the integral of f.

In the same way

$$\int_{E^n - K} |\mathbf{x} - \mathbf{x}_0|^{-p}\, dV$$

exists if $p > n$ and diverges to $+\infty$ if $p \leq n$. A similar discussion applies.

EXAMPLE 5. Let

$$\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx, \qquad u > 0. \tag{5.31}$$

The function Γ is called the *gamma function*. Let $f(x) = x^{u-1} \exp(-x)$. Since $f(x) \leq x^{u-1}$ and $p = 1 - u < 1$, f is integrable over $(0, 1]$ by comparison with x^{-p}. For any b, $x^b \exp(-x) \to 0$ as $x \to +\infty$ and therefore is bounded on $[1, \infty)$. Letting $b = u + 1$ we see that there is a number C such that $f(x) \leq C/x^2$ for every $x \in [1, \infty)$. Thus f is integrable over $[1, \infty)$ and over $(0, 1]$, therefore over $(0, \infty)$.

The gamma function generalizes the factorial. First of all,

$$\Gamma(1) = \int_0^\infty \exp(-x) dx = 1. \tag{5.32}$$

Integrating by parts,

$$\Gamma(u+1) = \int_0^\infty x^u \exp(-x)dx$$
$$= -x^u \exp(-x)\Big|_0^\infty + u\int_0^\infty x^{u-1}\exp(-x)dx,$$

which gives

$$\Gamma(u+1) = u\Gamma(u). \tag{5 33}$$

The integration by parts over $(0, \infty)$ is justified by taking it first over intervals $[\delta, 1]$ and $[1, r]$ and letting $\delta \to 0^+, r \to +\infty$. In particular, if m is an integer, then $\Gamma(m+1) = m\Gamma(m)$. Since $\Gamma(1) = 1$,

$$\Gamma(m+1) = m!. \tag{5.34}$$

Conditionally convergent integrals

Let us return to the special case when A is closed (Case 1, above). If f is integrable on A, then Formula (5.27) remains correct. This follows by applying (5.27) to f^+, f^-, and subtraction. On the other hand, if f has both positive and negative values on A, then there may be a finite limit in (5.27) while the integral of $|f|$ diverges to $+\infty$. In that case, the right-hand side of (5.27) defines a conditionally convergent integral. While conditionally convergent integrals are important in some parts of mathematical analysis (e.g., in the treatment of Fourier integrals), they are not within the scope of the Lebesgue theory. We treat them only in Problems 5, 8, and 9.

In a similar way, one can define in Case 2 a conditionally convergent integral by (5.28), if the limit there exists while $\int_A |f|dV$ diverges to $+\infty$.

Iterated integrals

Theorem 5.6 regarding iterated integrals has the following extension (we defer the proof to the end of Section 5.11). Let f be continuous on A. If f is integrable over A, then Formula (5.23) holds. In case $f \geq 0$. then a stronger statement can be made. If either side of (5.23) diverges to $+\infty$, then so does the other. Therefore, if $f \geq 0$, one way to show that f is integrable over A is to show that the iterated integral exists.

EXAMPLE 6. Let $f(x, y) = |x|/(1 + x^2 + y^2)^2$, $A = E^2$. By symmetry

$$\int f\,dV_2 = 4\int_0^\infty dy \int_0^\infty \frac{x\,dx}{(1+x^2+y^2)^2} = 2\int_0^\infty \frac{dy}{1+y^2} = \pi.$$

The reader is cautioned that if f has both positive and negative values on A, then one cannot conclude that f is integrable from the fact that the iterated integral exists.

EXAMPLE 7. Let $f(x, y) = y^{-1} \cos x$, $A = [0, \pi] \times (0, 1]$. Then

$$\int_0^1 dy \int_0^\pi y^{-1} \cos x \, dx = \int_0^1 0 \, dy = 0.$$

The iterated integral in the opposite order does not exist, hence f is not integrable over A.

PROBLEMS

1. Determine whether the integral exists or is divergent to $+\infty$.

   (a) $\int_1^2 \frac{dx}{\sqrt{x^2 - 1}}$. [*Hint*: Let $\phi(x) = 1/\sqrt{x + 1}$ as in Example 4.]

   (b) $\int_{-\pi/2}^{\pi/2} |\sin x|^{-p} dx$. [*Hint*: $x/\sin x \to 1$ as $x \to 0$.]

   (c) $\int_0^\infty P(x)\exp(-cx)dx$, P a polynomial, $c > 0$.

   (d) $\int_{-\infty}^\infty \frac{dx}{\sqrt[3]{x}(|x| + 1)}$.

   (e) $\int_1^\infty \frac{x^{-q}}{(x - 1)^p} dx$.

   (f) $\int_A \frac{\sqrt[3]{x}}{x^2 + y^2} dV_2(x, y)$, $A = E^2 - \{(0, 0)\}$

   (g) $\int_{E^2} \frac{\exp(-|x - y|)}{1 + |x - y|^2} dV_2(x, y)$.

2. Show that $\{(x, y, z): 0 \le z \le (x^2 + y^2)/xy, \ 0 < x \le 1, \ 0 < y \le 1\}$ has infinite volume.

3. Show that the volume of $\{(x, y, z): 0 \le z \le |xy|\exp(-x^2 - y^2)\}$ is 1.

4. Show that $\int f(x, y)dV_2(x, y)$ exists if f is bounded, continuous, and $|f(x, y)| \le C(1 + x^2 + y^2)^{-p/2}$, $p > 2$.

5. If $\lim_{r\to\infty} \int_{-r}^r f(x)dx$ exists but f is not integrable on E^1, then this limit is called *the Cauchy principal value*. Find the Cauchy principal value:
   (a) $f(x) = x/(1 + x^2)$.
   (b) $f(x) = x + 1/(1 + x^2)$.
   (c) Any odd function f, $f(-x) = -f(x)$ for every x.

6. Let A be open, and f a function continuous and integrable on A. For $r > 0$, let $A(r) = \{\mathbf{x} \in A: |\mathbf{x}| \le r, \text{dist}(\mathbf{x}, A^c) \ge r^{-1}\}$. Show that:
   (a) $A(r)$ is compact.
   (b) $\int_A f dV = \lim_{r\to\infty} \int_{A(r)} f dV$.

7. Show that each of the following integrals over E^n converges.
   (a) $\int (|\mathbf{x}|^2 + 1)^{-p/2} dV(\mathbf{x})$, $p > n$.
   (b) $\int |\mathbf{x}|^{-|\mathbf{x}|} dV(\mathbf{x})$.

8. Let $f(x) = x^{-1} \sin x$.
(a) Show that $\lim_{r\to\infty} \int_1^r f(x)dx$ exists. [*Hint*: Integrate by parts.]
(b) Show that $\int_1^\infty |f(x)|dx$ diverges to $+\infty$.

9. Let $f(x) = (-1)^m/m$ if $x \in [m, m+1)$, $m = 1, 2, \ldots$
(a) Show that $\lim_{r\to\infty} \int_1^r f(x)dx$ exists and equals $-1 + \frac{1}{2} - \frac{1}{3} + \ldots$
(b) Let $K_\nu = (\bigcup_{k=1}^{\nu} [2k-1, 2k]) \cup (\bigcup_{l=1}^{2\nu} [2l, 2l+1])$. Show that $K_1 \cup K_2 \cup \cdots = [1, \infty)$, but

$$\lim_{\nu\to\infty} \int_{K_\nu} f(x)dx \neq \lim_{r\to\infty} \int_1^r f(x)dx.$$

10. (a) Assume that for every real number c, both of the sets $\{\mathbf{x} \in A : f(\mathbf{x}) < c\}$ and $\{\mathbf{x} \in A : f(\mathbf{x}) > c\}$ are open relative to A. Show that f is continuous on A. [*Hint*: Show that $\{\mathbf{x} \in A : c < f(\mathbf{x}) < d\}$ is open relative to A.]
(b) Using (a) and Corollary 2, Section 2.6, prove: if f is continuous on A, then f^+ is continuous on A.

11. Let A be a bounded, measurable set, and let f be bounded and continuous on A. Show that $\int_A f\,dV$, as defined by (5.30), is the same as in Section 5.4. [*Hint*: Suppose first that $f \geq 0$. For any compact set $K \subset A$, we have, taking the integral over A in the previous sense, $0 \leq \int_A f\,dV - \int_K f\,dV \leq C[V(A-K)]$ provided $f(\mathbf{x}) \leq C$ for every $\mathbf{x} \in A$. But $V(A-K) = V(A) - V(K)$ is arbitrarily small. Apply this to f^+, f^-, and subtract.]

12. (*Difficult.*) Let f be continuous on an open set D. Assume that the integrals of f^+ and f^- over D both diverge to $+\infty$. Show that given any number l there is a sequence of compact sets $K_1 \subset K_2 \subset \cdots$ such that $D = K_1 \cup K_2 \cup \cdots$ and $\lim_{\nu\to\infty} \int_{K_\nu} f\,dV = l$.

5.7 Change of measure under affine transformations

Our next objective is to give formulas which describe how measure and integrals change under a regular transformation $\mathbf{g}$. In this section we consider the special case when $\mathbf{g}$ is affine and prove the formula for the measure of $\mathbf{g}(A)$ when A is a compact set. This special result is then used in the proof of the general transformation formula (5.38).

We begin with the following.

Lemma 1. *Let* $\mathbf{L}$ *be a linear transformation from* E^n *into* E^n, *and* I *an* n*-dimensional interval. Then* $V[\mathbf{L}(I)] = |\det \mathbf{L}|\, V(I)$.

PROOF. First of all, Lemma 1 is valid for the following particular kinds of linear transformations. Points in the domain of $\mathbf{L}$ are denoted by $\mathbf{t}$.

(1) For some k and l, the transformation $\mathbf{L}$ merely interchanges the components t^k and t^l of $\mathbf{t}$. Then $V[\mathbf{L}(I)] = V(I)$ and $\det \mathbf{L} = -1$.

(2) The matrix of $\mathbf{L}$ is diagonal. If $\mathbf{x} = \mathbf{L}(\mathbf{t})$, then $x^i = c_i^i t^i$, where $c_1^1, \ldots, c_n^n$ are the diagonal elements. We have $I = J_1 \times \cdots \times J_n$, where each J_i is a 1-dimensional interval. Then $\mathbf{L}(I) = J_1' \times \cdots \times J_n'$, where J_i' is a 1-dimensional interval of length $|c_i^i|$ times the length of J_i. Since $\det \mathbf{L} = c_1^1 \cdots c_n^n$, we have $V[\mathbf{L}(I)] = |\det \mathbf{L}| V(I)$.

(3) There exist k and l, $k \neq l$, such that $x^k = t^k + ct^l$, $x^i = t^i$ for $i \neq k$.

For notational simplicity we take $k = 1$, $l = 2$, and regard E^n as $E^2 \times E^{n-2}$. Then $I = I' \times I''$, where $I' = J_1 \times J_2$, $I'' = J_3 \times \cdots \times J_n$. Let $\mathbf{L}'$ be the transformation from E^2 into E^2, such that

$$\mathbf{L}'(t^1, t^2) = (t^1 + ct^2, t^2).$$

Then $\mathbf{L}'(I') = P'$, where P' is a parallelogram of the same area as I', and $\mathbf{L}(I) = P' \times I''$. By (5.24),

$$V_n[\mathbf{L}(I)] = V_2(P')V_{n-2}(I'') = V_2(I')V_{n-2}(I'') = V_n(I).$$

Since $\det \mathbf{L} = 1$, Lemma 1 is valid for $\mathbf{L}$ of type (3).

Next, we observe that if $\mathbf{M}$ and $\mathbf{N}$ are any two linear transformations for which the theorem holds, then

$$\begin{aligned} V[(\mathbf{M} \circ \mathbf{N})(K)] &= |\det \mathbf{M}| V[\mathbf{N}(K)] \\ &= |\det \mathbf{M}||\det \mathbf{N}| V(K), \end{aligned}$$

and $(\det \mathbf{M})(\det \mathbf{N}) = \det \mathbf{M} \circ \mathbf{N}$. Hence the theorem also holds for their composite $\mathbf{M} \circ \mathbf{N}$.

If $\mathbf{N}$ has row covectors $\mathbf{w}^1, \ldots, \mathbf{w}^n$ and $\mathbf{M}$ is of Type (3), then the kth row covector of $\mathbf{M} \circ \mathbf{N}$ is $\mathbf{w}^k + c\mathbf{w}^l$ and the others are unchanged. The kth column vector of $\mathbf{N} \circ \mathbf{M}$ is $\mathbf{v}_k + c\mathbf{v}_l$ and the others are unchanged, where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the column vectors of $\mathbf{N}$. A transformation $\mathbf{M}$ of type (1) when applied on the left interchanges the kth and lth row covectors of $\mathbf{N}$, and when applied on the right interchanges the kth and lth column vectors. Moreover, the inverse of a transformation of Type (1) or (3) is of the same type.

Now let $\mathbf{L}$ be any linear transformation. Then

$$\mathbf{L} = \mathbf{M}_p \circ \cdots \circ \mathbf{M}_1,$$

where $\mathbf{M}_1, \ldots, \mathbf{M}_p$ are of Types (1), (2), and (3). Since Lemma 1 holds for each $\mathbf{M}_j$, it is valid for $\mathbf{L}$. □

Now let $\mathbf{g}$ be an affine transformation from E^n into E^n. According to Section 4.2, there exist $\mathbf{x}_0$ and a linear transformation $\mathbf{L}$ such that

$$\mathbf{g}(\mathbf{t}) = \mathbf{L}(\mathbf{t}) + \mathbf{x}_0$$

for every $\mathbf{t} \in E^n$.

Theorem 5.7. *For every compact set K,*

$$V[\mathbf{g}(K)] = |\det \mathbf{L}| V(K). \tag{5.35}$$

PROOF. First of all, if $\mathbf{g}$ is a translation ($\mathbf{L} = \mathbf{I}$, the identity), then $V[\mathbf{g}(Z)] = V(Z)$ for any figure Z. Hence $V[\mathbf{g}(K)] = V(K)$ for any compact K, if $\mathbf{g}$ is a translation. We may assume from now on that $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{g} = \mathbf{L}$. By Lemma 1 above, (5.35) holds for any n-dimensional interval I. Hence (5.35) also holds for any figure Z. If K is compact, the construction in the proof of Theorem 5.6 gives a sequence of figures $Y_1 \supset Y_2 \supset \cdots$ such that

$$K = Y_1 \cap Y_2 \cap \cdots, \qquad V(K) = \lim_{\nu \to \infty} V(Y_\nu).$$

Then $\mathbf{g}(Y_1) \supset \mathbf{g}(Y_2) \supset \cdots$,

$$\mathbf{g}(K) = \mathbf{g}(Y_1) \cap \mathbf{g}(Y_2) \cap \cdots, \qquad V[\mathbf{g}(K)] = \lim_{\nu \to \infty} V[\mathbf{g}(Y_\nu)].$$

Since $V[\mathbf{g}(Y_\nu)] = |\det \mathbf{L}| V(Y_\nu)$ for $\nu = 1, 2, \ldots$, we get (5.35). □

Corollary. *If $\mathbf{g}$ is an isometry of E^n, then $V[\mathbf{g}(K)] = V(K)$.*

PROOF. For then $\det \mathbf{L} = \pm 1$. □

Let us apply Theorem 5.7 to calculate the measures of simplexes and parallelepipeds.

Measure of an n-simplex

Let S be an n-simplex with vertices $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ (see Section 1.5). Let $\mathbf{v}_i = \mathbf{x}_i - \mathbf{x}_0$. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Let $\mathbf{L}$ be the linear transformation with $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as column vectors, and let $\mathbf{g}(\mathbf{t}) = \mathbf{L}(\mathbf{t}) + \mathbf{x}_0$. As before, let Σ be the standard n-simplex. If $\mathbf{t} = (t^1, \ldots, t^n) \in \Sigma$, let $t^0 = 1 - (t^1 + \cdots + t^n)$ and let $\mathbf{x} = \mathbf{g}(\mathbf{t})$. Then

$$\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{n} t^i(\mathbf{x}_i - \mathbf{x}_0) = \sum_{i=0}^{n} t^i \mathbf{x}_i.$$

Hence $\mathbf{x} \in S$ and $t^0, t^1, \ldots, t^n$ are its barycentric coordinates. Conversely, every $\mathbf{x} \in S$ is obtained in this way. Thus $S = \mathbf{g}(\Sigma)$. By Example 3, Section 5.5, $V(\Sigma) = 1/n!$. Hence, from (5.35),

$$V(S) = |\det \mathbf{L}|/n! \tag{5.36}$$

Measure of an n-parallelepiped

Given $\mathbf{x}_0$ and linearly independent vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, let

$$P = \left\{ \mathbf{x} : \mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{n} t^i \mathbf{v}_i,\ 0 \le t^i \le 1,\ i = 1, \ldots, n \right\}.$$

Then P is the *n-parallelepiped* spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with $\mathbf{x}_0$ as vertex (Figure 5.15). Let I_0 be the unit n-cube. Then $P = \mathbf{g}(I_0)$, where $\mathbf{g}$ is as before. Since $V(I_0) = 1$,

$$V(P) = |\det \mathbf{L}|. \tag{5.37}$$

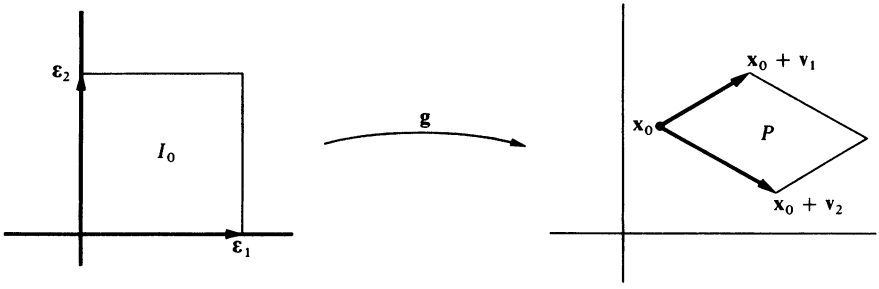


Figure 5.15

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent, then P is called a *degenerate* n-parallelepiped. In that case $V(P) = 0$.

PROBLEMS

1. Find the volume of the tetrahedron with vertices $\mathbf{e}_1$, $-\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$.

2. Find the area of the parallelogram with vertices $\mathbf{e}_1 - \mathbf{e}_2, 2\mathbf{e}_1 + \mathbf{e}_2, -2\mathbf{e}_1, -\mathbf{e}_1 + 2\mathbf{e}_2$.

3. Let S be an n-simplex, with vertices $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$. Show that

$$V(S) = \frac{1}{n!}\left|\det\begin{pmatrix} x_0^1 & x_1^1 & \cdots & x_n^1 \\ \vdots & \vdots & & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \\ 1 & 1 & \cdots & 1 \end{pmatrix}\right|.$$

   [*Hint*: Subtract the first column from each of the other columns. This does not change the determinant.]

4. Let (c_j^i) be a positive definite symmetric $n \times n$ matrix, and let $\lambda_1, \ldots, \lambda_n$ be its characteristic values. Show that

$$V\left(\left\{\mathbf{x} : \sum_{i,j=1}^{n} c_j^i x^i x^j \leq 1\right\}\right) = \alpha_n / \sqrt{\lambda_1 \cdots \lambda_n},$$

   where α_n is the measure of the unit n-ball. [*Hint*: See Theorem 4.4.]

5. Show that if $|\mathbf{v}_i| \leq C$ for each $i = 1, \ldots, n$ and P is a parallelepiped spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then $V(P) \leq C^n$. [*Hint*: Use induction, a suitable rotation of E^n, and the method of slices.]

5.8 Transformation of integrals

Let $\mathbf{g}$ be a regular transformation with domain an open set $\Delta \subset E^n$. Regularity means that $\mathbf{g}$ is of class $C^{(1)}$ and has an inverse $\mathbf{g}^{-1}$ of class $C^{(1)}$ (Section 4.5). Let f be continuous on the open set $D = \mathbf{g}(\Delta)$. The object of this section is to prove the formula (5.38) which expresses the integral of f over a set $A \subset D$ as an integral over the corresponding subset $B = \mathbf{g}^{-1}(A)$ of Δ.

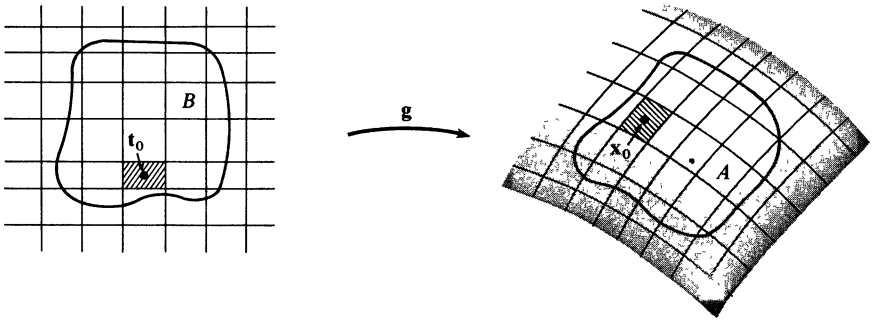


Figure 5.16

Let us first give an imprecise derivation of the transformation formula (5.38) (see Figure 5.16). For the moment, let B be compact, and let Z be a figure approaching B from without. Let $I \subset Z$ and $\mathbf{t}_0$ be a point of I. Let

$$\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0), \qquad \mathbf{L} = D\mathbf{g}(\mathbf{t}_0).$$

The affine approximation $\mathbf{G}$ to $\mathbf{g}$ at $\mathbf{t}_0$ is given by

$$\mathbf{G}(\mathbf{t}) = \mathbf{x}_0 + \mathbf{L}(\mathbf{t} - \mathbf{t}_0).$$

If I is small, then $\mathbf{G}(I)$ and $\mathbf{g}(I)$ nearly coincide. Hence $V[\mathbf{g}(I)]$ is nearly $V[\mathbf{G}(I)]$, which by Theorem 5.7 equals $|\det \mathbf{L}|\, V(I)$. Since f is continuous, its integral over $\mathbf{g}(I)$ is approximately $f(\mathbf{x}_0)V[\mathbf{g}(I)]$, and since $\det \mathbf{L}$ is the Jacobian $J\mathbf{g}(\mathbf{t}_0)$,

$$\int_{\mathbf{g}(I)} f\, dV \sim f[\mathbf{g}(\mathbf{t}_0)]\,|J\mathbf{g}(\mathbf{t}_0)|\, V(I).$$

Let

$$\phi(\mathbf{t}) = f[\mathbf{g}(\mathbf{t})]\,|J\mathbf{g}(\mathbf{t})|.$$

Since f is continuous and $\mathbf{g}$ is of class $C^{(1)}$, the composite $f \circ \mathbf{g}$ is continuous and $|J\mathbf{g}|$ is continuous. Hence ϕ is continuous. The integral of ϕ over I is approximately $\phi(\mathbf{t}_0)V(I)$, which is the right-hand side of the above expression.

The figure Z is the union of small intervals $I_1, \ldots, I_p$. Let $\mathbf{t}_k \in I_k$. We should have approximately

$$\int_{\mathbf{g}(Z)} f\, dV \sim \sum_{k=1}^{p} \phi(\mathbf{t}_k)V(I_k) \sim \int_Z \phi\, dV.$$

This suggests the formula

$$\int_A f\, dV = \int_B \phi\, dV.$$

The following theorem states that the formula is correct, and not merely for compact sets.

Theorem 5.8. *Let* $\mathbf{g}$ *be a regular transformation from* Δ *onto* D. *Let* f *be continuous on* D *and* A *be any measurable subset of* D. *Then*

$$\int_A f(\mathbf{x})dV(\mathbf{x}) = \int_{\mathbf{g}^{-1}(A)} f[\mathbf{g}(\mathbf{t})]|J\mathbf{g}(\mathbf{t})|dV(\mathbf{t}), \tag{5.38}$$

provided either integral exists.

More generally, Theorem 5.8 is valid if f is any function measurable on D. We do not prove this.

To prove Theorem 5.8 let us first prove three lemmas. In these lemmas I denotes an n-dimensional cube. It is convenient to take I half-open to the right. Then $I = \text{cl } I - N$, where N is a certain compact set composed of $(n-1)$-dimensional faces of I. Hence $\mathbf{g}(I) = \mathbf{g}(\text{cl } I) - \mathbf{g}(N)$ is the difference of compact sets, and so it is measurable. Let l be the side length of I, d its diameter, and I' the concentric closed n-cube of side length $(1+\tau)l$. Let $\mathbf{G}$, $\mathbf{L}$, $\mathbf{x}_0$ be as defined at the beginning of the section.

Lemma 1. *Let* $\mathbf{t}_0 \in \Delta$ *and* $\tau > 0$ *be given. Then* $\mathbf{t}_0$ *has a neighborhood* $\Omega \subset \Delta$ *such that* $\mathbf{g}(I) \subset \mathbf{G}(I')$ *for every* n*-cube* $I \subset \Omega$ *with* $\mathbf{t}_0 \in \text{cl } I$ *(see Figure* 5.17*).*

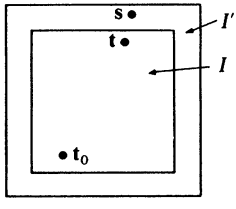


Figure 5.17

PROOF. If $\mathbf{x} = \mathbf{G}(\mathbf{s})$, $\mathbf{y} = \mathbf{G}(\mathbf{t})$, then $\mathbf{x} - \mathbf{y} = \mathbf{L}(\mathbf{s} - \mathbf{t}_0) - \mathbf{L}(\mathbf{t} - \mathbf{t}_0)$. Since $\mathbf{L}$ is linear, $\mathbf{x} - \mathbf{y} = \mathbf{L}(\mathbf{s} - \mathbf{t})$. Let $C = \|\mathbf{L}^{-1}\|$ (Section 4.3). Then $\mathbf{s} - \mathbf{t} = \mathbf{L}^{-1}(\mathbf{x} - \mathbf{y})$, and

$$|\mathbf{s} - \mathbf{t}| \leq C|\mathbf{x} - \mathbf{y}|. \tag{*}$$

Let $\sigma = \tau/(2\sqrt{n}C)$. Since $\mathbf{g}$ is differentiable, $\mathbf{t}_0$ has a neighborhood $\Omega \subset \Delta$ such that

$$|\mathbf{g}(\mathbf{t}) - \mathbf{G}(\mathbf{t})| \leq \sigma|\mathbf{t} - \mathbf{t}_0| \tag{**}$$

for every $\mathbf{t} \in \Omega$.

Let $\mathbf{x} \in \mathbf{g}(I)$ and $\mathbf{s} = \mathbf{G}^{-1}(\mathbf{x})$. Then $\mathbf{x} = \mathbf{G}(\mathbf{s}) = \mathbf{g}(\mathbf{t})$ for some $\mathbf{t} \in I$. By (*) and (**)

$$|\mathbf{s} - \mathbf{t}| \leq C\sigma|\mathbf{t} - \mathbf{t}_0|.$$

Since $\mathbf{t}_0 \in I$, $|\mathbf{t} - \mathbf{t}_0| \leq d$. Since $d = \sqrt{n}l$,

$$|\mathbf{s} - \mathbf{t}| \leq C\sigma\sqrt{n}l \leq \tau l/2.$$

This implies that $\mathbf{s} \in I'$ and $\mathbf{x} \in \mathbf{G}(I')$. □

In Lemmas 2 and 3 we assume that $f \geq 0$. As above, denote the integrand on the right-hand side of (5.38) by $\phi(\mathbf{t}) = f[\mathbf{g}(\mathbf{t})]|J\mathbf{g}(\mathbf{t})|$.

Lemma 2. *Let $\mathbf{t}_0 \in \Delta$ and $\varepsilon > 0$ be given. Then $\mathbf{t}_0$ has a neighborhood $\Omega_1 \subset \Delta$ such that*

$$\int_{\mathbf{g}(I)} f\,dV < \int_I \phi\,dV + \varepsilon V(I)$$

for every n-cube $I \subset \Omega_1$ with $\mathbf{t}_0 \in \operatorname{cl} I$.

PROOF. Let

$$a = f(\mathbf{x}_0) = f[\mathbf{g}(\mathbf{t}_0)], \qquad b = |J\mathbf{g}(\mathbf{t}_0)|.$$

Then $a \geq 0$ since $f \geq 0$, and $b > 0$ since $\mathbf{g}$ is regular. Let $\psi(\xi, \tau) = (a - \xi)(b - \xi) - (a + \xi)b(1 + \tau)^n$. Then $\psi(0, 0) = 0$ and ψ is continuous. Hence $|\psi(\xi, \tau)| < \varepsilon$ for every (ξ, τ) in some neighborhood V of $(0, 0)$. Choose some $\xi > 0$, $\tau > 0$ small enough that $(\xi, \tau) \in V$ and $\xi < b$. Then

$$(*) \qquad (a + \xi)(1 + \tau)^n b < (a - \xi)(b - \xi) + \varepsilon.$$

Since f is continuous, there is a neighborhood U of $\mathbf{x}_0$ such that $a - \xi < f(\mathbf{x}) < a + \xi$ for every $\mathbf{x} \in U$. Since $\mathbf{g}$ and $|J\mathbf{g}|$ are continuous, there is a neighborhood Ω_1 of $\mathbf{t}_0$ such that $\mathbf{g}(\mathbf{t}) \in U$ and $b - \xi < |J\mathbf{g}(\mathbf{t})|$ for every $\mathbf{t} \in \Omega_1$. We may assume that $\Omega_1 \subset \Omega$, where Ω is as in Lemma 1.

Then

$$\int_{\mathbf{g}(I)} f\,dV \leq (a + \xi)V[\mathbf{g}(I)],$$

and by Lemma 1, $V[\mathbf{g}(I)] \leq V[\mathbf{G}(I')]$. By Theorem 5.7,

$$V[\mathbf{G}(I')] = bV(I') = b(1 + \tau)^n V(I).$$

Therefore

$$(**) \qquad \int_{\mathbf{g}(I)} f\,dV \leq (a + \xi)b(1 + \tau)^n V(I).$$

On the other hand,

$$(***) \qquad \int_I \phi\,dV \geq (a - \xi)(b - \xi)V(I).$$

This follows from the inequality

$$\phi(\mathbf{t}) = f[\mathbf{g}(\mathbf{t})]|J\mathbf{g}(\mathbf{t})| \geq (a - \xi)(b - \xi)$$

which holds for $\mathbf{t} \in \Omega_1$ provided $a - \xi \geq 0$. If $a - \xi < 0$, (***) holds trivially since its left side is nonnegative and the right side is negative. Inequalities (*), (**), and (***) give Lemma 2. □

Lemma 3. *Let $I \subset \Delta$ be an n-cube. Then*

$$\int_{\mathbf{g}(I)} f\,dV \leq \int_I \phi\,dV.$$

PROOF. Suppose this is false for some n-cube I^0. Then

$$c = \int_{\mathbf{g}(I^0)} f\,dV - \int_{I^0} \phi\,dV$$

is positive. Divide I^0 into $m = 2^n$ congruent n-cubes $I_1, \ldots, I_m$, half-open to the right. Since $I_1, \ldots, I_m$ are disjoint and $\mathbf{g}$ is univalent, the sets $\mathbf{g}(I_1), \ldots, \mathbf{g}(I_m)$ are disjoint. Hence

$$\int_{I^0} \phi\,dV = \sum_{j=1}^{m} \int_{I_j} \phi\,dV,$$

$$\int_{\mathbf{g}(I^0)} f\,dV = \sum_{j=1}^{m} \int_{\mathbf{g}(I_j)} f\,dV.$$

For at least one j we must have

$$\int_{\mathbf{g}(I_j)} f\,dV - \int_{I_j} \phi\,dV \geq c2^{-n}.$$

Choose some such j and let $I^1 = I_j$. By dividing I^1 into 2^n congruent n-cubes and repeating the argument, we obtain I^2. Continuing, we obtain a sequence of n-cubes $I^1 \supset I^2 \supset \cdots$ such that

$$\int_{\mathbf{g}(I^l)} f\,dV - \int_{I^l} \phi\,dV \geq c2^{-ln}, \qquad l = 1, 2, \ldots,$$

and the diameter of I^l tends to 0 as $l \to \infty$. By Theorem 2.3, $(\mathrm{cl}\, I^1) \cap (\mathrm{cl}\, I^2) \cap \cdots$ contains just one point $\mathbf{t}_0$. Let $\varepsilon < c/V(I^0)$. Then

$$\varepsilon V(I^l) = \varepsilon 2^{-ln} V(I^0) < c2^{-ln}.$$

If Ω_1 is as in Lemma 2, then $I^l \subset \Omega_1$ for sufficiently large l and we obtain a contradiction. □

Let us now prove Theorem 5.8 in case A is compact and $f \geq 0$. Let $B = \mathbf{g}^{-1}(A)$. The inverse $\mathbf{g}^{-1}$ is a regular transformation, since $\mathbf{g}$ is regular. In particular, $\mathbf{g}$ is continuous, and therefore B is compact since A is compact. Let Z_0 be some n-cube, half-open to the right, containing B. Divide Z_0 into 2^n congruent n-cubes and let Z_1 be the union of those which meet B. Then divide each n-cube of Z_1 into 2^n congruent n-cubes, and let Z_2 be the union

of those meeting B, and so on. Then $Z_1 \supset Z_2 \supset \cdots$ and their intersection is B. There exists v_0 such that $Z_v \subset \Delta$ for all $v \geq v_0$.

Applying Lemma 3 to each of these congruent disjoint n-cubes comprising Z_v, and adding, we get

$$\int_A f\,dV \leq \int_{\mathbf{g}(Z_v)} f\,dV \leq \int_{Z_v} \phi\,dV$$

for each $v \geq v_0$. Moreover,

$$\int_B \phi\,dV = \lim_{v\to\infty} \int_{Z_v} \phi\,dV.$$

This fact will be proved in the section on convergence theorems [Corollary 5(b), Section 5.11]. Thus

$$\int_A f\,dV \leq \int_B \phi\,dV.$$

But $\mathbf{g}^{-1}$ is also a regular transformation. Interchanging the roles of A and B,

$$\int_B \phi\,dV \leq \int_A \phi \circ \mathbf{g}^{-1} |J\mathbf{g}^{-1}|\,dV.$$

But $|J\mathbf{g}^{-1}(\mathbf{x})| = |J\mathbf{g}(\mathbf{t})|^{-1}$ if $\mathbf{x} = \mathbf{g}(\mathbf{t})$, and

$$\phi[\mathbf{g}^{-1}(\mathbf{x})]\,|J\mathbf{g}^{-1}(\mathbf{x})| = f(\mathbf{x}).$$

Therefore

$$\int_B \phi\,dV \leq \int_A f\,dV,$$

which proves (5.38) for the case of a compact set A.

Now let A be measurable, and again assume that $f \geq 0$. Then $B = \mathbf{g}^{-1}(A)$ is also measurable (Problem 8). For any compact set $K \subset A$, we have $\mathbf{g}^{-1}(K) \subset B$ and

$$\int_K f\,dV = \int_{\mathbf{g}^{-1}(K)} \phi\,dV \leq \int_B \phi\,dV.$$

By (5.26),

$$(\#) \qquad \int_A f\,dV \leq \int_B \phi\,dV.$$

Note, in particular, that f is integrable provided ϕ is integrable. By considering $\mathbf{g}^{-1}$ as above, we find that ϕ is integrable provided that f is integrable, and get the inequality opposite to $(\#)$ with $\geq$ rather than $\leq$. This proves Theorem 5.8 if $f \geq 0$.

Finally, write any continuous f as $f = f^+ - f^-$, as in Section 5.6. Then $\phi = \phi^+ - \phi^-$. Since $|J\mathbf{g}| > 0$, we have

$$\phi^+ = (f^+ \circ \mathbf{g})|J\mathbf{g}|, \qquad \phi^- = (f^- \circ \mathbf{g})|J\mathbf{g}|.$$

If either f or ϕ is integrable, then so is the other, and (5.38) holds. This completes the proof of Theorem 5.8. □

In particular, taking $f(\mathbf{x}) \equiv 1$ in (5.38) we have:

Corollary 1

$$V(A) = \int_{\mathbf{g}^{-1}(A)} |J\mathbf{g}(\mathbf{t})| dV(\mathbf{t}). \tag{5.39}$$

Corollary 2. *If B is a null set, then $\mathbf{g}(B)$ is a null set.*

Corollary 3. *If A is a measurable subset of an r-manifold M, where $r \leq n - 1$, then A is a null set.*

PROOF. By the implicit function theorem (Section 4.6), any $\mathbf{x}_0 \in M$ has a neighborhood U such that $U \cap M = \mathbf{g}(R)$, where R is a relatively open subset of an r-dimensional vector subspace. Then R is a null set. By Corollary 2, $U \cap M$ is a null set.

If $K \subset M$ and K is compact, then there exist a finite number of such neighborhoods $U_1, \ldots, U_m$ such that $K \subset (U_1 \cup \cdots \cup U_m) \cap M$. Hence K is a null set. If A is any measurable set, then $V(A) = \sup\{V(K) : K \subset A\}$. If $A \subset M$, then A is a null set. □

EXAMPLE 1. Let $A = \{(x, y) : x > 0, y > 0, 0 < xy < 3, x < y < 2x\}$, $f(x, y) = y^2$, $\mathbf{g}(s, t) = \sqrt{s/t}\,\mathbf{e}_1 + \sqrt{st}\,\mathbf{e}_2$ for $s > 0, t > 0$. We show that $\mathbf{g}$ is univalent by solving the equations

$$x = \sqrt{\frac{s}{t}}, \qquad y = \sqrt{st}$$

explicitly, and find that $s = xy, t = y/x$. Since $J\mathbf{g}(s, t) = 1/2t \neq 0$, $\mathbf{g}$ is regular. Moreover, the part in A of the hyperbola $xy = c$, $0 < c < 3$, corresponds to the segment $s = c$, $1 < t < 2$ in B. Hence B is as shown in Figure 5.18, and

$$\int_A y^2 \, dV_2(x, y) = \frac{1}{2} \int_1^2 dt \int_0^3 s \, ds = \frac{9}{4}.$$

EXAMPLE 2. Let P be an n-parallelepiped. Then $P = \mathbf{g}(I_0)$ where, as in Section 5.7, $\mathbf{g}$ is affine and I_0 is the unit n-cube. Then

$$\int_P f(\mathbf{x}) dV(\mathbf{x}) = |\det \mathbf{L}| \int_{I_0} f[\mathbf{g}(\mathbf{t})] dV(\mathbf{t}).$$

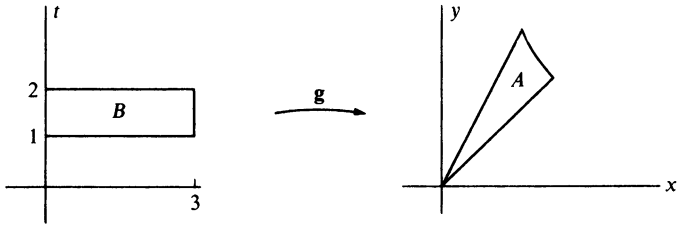


Figure 5.18

PROBLEMS

1. Let $n = 1$, $g(t) = t^2 - 2t + 3$, $B = (0, 1)$. What does (5.38) become?

2. Let $\mathbf{g}(s, t) = (s^2 + t^2)\mathbf{e}_1 + (s^2 - t^2)\mathbf{e}_2$, $s > 0$, $t > 0$, and $A = \{(x, y): 2 < x + y < 4, x - y > 0, y > 0\}$. Show that $\mathbf{g}$ is regular and evaluate $\int_A x^{-1} dV_2(x, y)$.

3. Find the second moment about $(0, 0)$ of the parallelogram with vertices $(0, 0)$, $\mathbf{e}_1 + \mathbf{e}_2$, $-2\mathbf{e}_1 + 3\mathbf{e}_2$, $-\mathbf{e}_1 + 4\mathbf{e}_2$. [*Hint*: Let $\mathbf{g}$ be the linear transformation $\mathbf{L}$ with column vectors $\mathbf{e}_1 + \mathbf{e}_2$, $-2\mathbf{e}_1 + 3\mathbf{e}_2$.]

4. Let B be a compact set, $\bar{\mathbf{x}}$ its centroid, and $\mathbf{g}$ an affine transformation. Show that $\mathbf{g}(\bar{\mathbf{x}})$ is the centroid of $\mathbf{g}(B)$.

5. Let A be symmetric about $\mathbf{0}$, that is, $\mathbf{x} \in A$ implies $-\mathbf{x} \in A$. Let f be integrable over A and $f(-\mathbf{x}) = -f(\mathbf{x})$ for every $\mathbf{x} \in A$. Show that $\int_A f\, dV = 0$.

6. Let $\mathbf{g}$ be of class $C^{(1)}$ on Δ and K be a compact subset of Δ. Show that there is a number C such that $|\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})| \leq C|\mathbf{s} - \mathbf{t}|$ for every $\mathbf{s}, \mathbf{t} \in K$. [*Hint*: Proposition 4.5, Section 4.3.]

7. Let $\mathbf{g}$ be regular and K, C as in Problem 6. Show that if $B \subset \operatorname{int} K$, then $V[\mathbf{g}(B)] \leq C^n V(B)$. [*Hint*: For $\mathbf{t} \in B$ the partial derivatives of $\mathbf{g}$ satisfy $|\mathbf{g}_i(\mathbf{t})| \leq C$. Use Problem 5, Section 5.7, to see that $|J\mathbf{g}(\mathbf{t})| \leq C^n$.]

8. Let $\mathbf{g}$ be a regular transformation from an open set Δ onto D. Let $B = \mathbf{g}^{-1}(A)$, where A is a measurable subset of D. Show that B is measurable. [*Hints*: First consider the case when $A \subset L \subset D$, where L is compact. Also, $\mathbf{g}^{-1}$ is regular. See Problem 7. Find compact sets $L_1 \subset L_2 \subset \cdots$ with union D.]

5.9 Coordinate systems in E^n

Let D be an open subset of E^n. Let $f^1, \ldots, f^n$ be functions of class $C^{(1)}$ on D such that the transformation $\mathbf{f} = (f^1, \ldots, f^n)$ is regular. Since a regular transformation $\mathbf{f}$ is univalent, the numbers $f^1(\mathbf{x}), \ldots, f^n(\mathbf{x})$ uniquely specify $\mathbf{x}$ and can be regarded as a set of "coordinates" for $\mathbf{x}$.

Definition. A regular transformation $\mathbf{f}$ from D into E^n is a *coordinate system* for D. The numbers $f^1(\mathbf{x}), \ldots, f^n(\mathbf{x})$ are the *coordinates* of $\mathbf{x}$ in this coordinate system.

Since we have already considered regular transformations, this definition involves nothing new except a change of viewpoint. In many problems it is D that has actual geometric or physical significance. The transformation $\mathbf{f}$ is introduced simply as a device for solving the problem, and the open set $\Delta = \mathbf{f}(D)$ has only an auxiliary status.

In particular, many integrals can be evaluated by introducing a suitable coordinate system. The transformation formula (5.38) is applied with $B = \mathbf{f}(A)$, $\mathbf{g} = \mathbf{f}^{-1}$. The objective usually is to choose a coordinate system for which B is simpler than A (for instance an interval) or ϕ is simpler than f, or both.

Let us consider some particular coordinate systems.

1. The identity transformation $\mathbf{I}$ gives the *standard cartesian coordinate* system for E^n. The components $x^1, \ldots, x^n$ are the standard cartesian coordinates of $\mathbf{x}$.
2. If $\mathbf{f}$ is an affine transformation, then the coordinate system is called *affine*.
3. If M is an r-manifold, $\mathbf{\Phi} = (\Phi^1, \ldots, \Phi^{n-r})$ is as in Section 4.7 and
$$\frac{\partial(\Phi^1, \ldots, \Phi^{n-r})}{\partial(x^{r+1}, \ldots, x^n)} \neq 0$$
at $\mathbf{x}_0$, then $(X^1, \ldots, X^r, \Phi^1, \ldots, \Phi^{n-r})$ is a coordinate system for some neighborhood U of $\mathbf{x}_0$. Here $X^i(\mathbf{x}) = x^i$ for all $\mathbf{x}$. The coordinates of a point $\mathbf{x} \in M \cap U$ in this system are $(x^1, \ldots, x^r, 0, \ldots, 0)$.
4. (*Polar coordinates.*) Let D be E^2 with the positive x-axis N removed. Let $R(x, y) = \sqrt{x^2 + y^2}$ and $\Theta(x, y)$ be the angle from N to the half-line from $(0, 0)$ through (x, y), with $0 < \Theta(x, y) < 2\pi$. Then (R, Θ) is the *polar coordinate system* for D. If $r = R(x, y)$, $\theta = \Theta(x, y)$, then (r, θ) are the coordinates of (x, y) and
$$x = r \cos\theta = g^1(r, \theta), \qquad y = r \sin\theta = g^2(r, \theta),$$
where $\mathbf{g} = (R, \Theta)^{-1}$. Since $J\mathbf{g}(r, \theta) = r$, the transformation formula becomes
$$\int_A f(x, y) dV_2(x, y) = \int_B f[r \cos\theta, r \sin\theta] r \, dV_2(r, \theta),$$
where $\mathbf{g}(B) = A - N$. Since N is a null set, the integral over A is the same as the integral over $A - N$.
5. (*Spherical coordinates in E^n*). For $n = 2$, this is the polar coordinate system. Proceeding inductively, let $r = |\mathbf{x}|$, θ^1 be the angle from the positive x^1-axis to $\mathbf{x}$; more precisely,
$$\theta^1 = \cos^{-1}(x^1/r), \; 0 < \theta^1 < \pi,$$
and
$$(\rho, \theta^2, \ldots, \theta^{n-1})$$

be spherical coordinates for $\mathbf{x}'' = (x^2, \ldots, x^n)$, where $\rho = |\mathbf{x}''| = r \sin \theta^1$. The coordinates of $\mathbf{x}$ are

$$\begin{aligned} x^1 &= r \cos \theta^1 \\ x^2 &= r \sin \theta^1 \cos \theta^2 \\ &\vdots \\ x^{n-1} &= r \sin \theta^1 \sin \theta^2 \cdots \sin \theta^{n-2} \cos \theta^{n-1} \\ x^n &= r \sin \theta^1 \sin \theta^2 \cdots \sin \theta^{n-2} \sin \theta^{n-1}. \end{aligned}$$

This defines the spherical coordinate system $(R, \Theta^1, \ldots, \Theta^{n-1})$ on $D = E^n - N$, where N is a certain null set. The Jacobian is

$$J\mathbf{g}(r, \theta^1, \ldots, \theta^{n-1}) = r^{n-1} \sin^{n-2} \theta^1 \sin^{n-3} \theta^2 \cdots \sin \theta^{n-2}.$$

EXAMPLE 1. Suppose that $f(\mathbf{x}) = \phi(|\mathbf{x}|)$, where ϕ is continuous on (r_1, r_2). Then

$$\int_{r_1 < |\mathbf{x}| < r_2} f \, dV = \beta_n \int_{r_1}^{r_2} \phi(r) r^{n-1} \, dr,$$

where β_n is a number not depending on ϕ, r_1, or r_2. To find β_n, set $\phi \equiv 1$. Then

$$\alpha_n(r_2^n - r_1^n) = \beta_n \int_{r_1}^{r_2} r^{n-1} \, dr = \frac{\beta_n(r_2^n - r_1^n)}{n},$$

where α_n is the measure of the unit n-ball. Hence $\beta_n = n\alpha_n$, which turns out to be the $(n-1)$-dimensional measure of the unit $(n-1)$-sphere.

6. (*Cylindrical coordinates in E^3.*) Let (R, Θ) be the polar coordinate system. Then (R, Θ, Z) is a coordinate system for $D = E^3 - \{(x, 0, z) : x \geq 0\}$. The equations $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ relate the cylindrical and standard cartesian coordinates of a point $\mathbf{x} \in D$. In a similar way cylindrical coordinates can be introduced in E^n.

7. The idea of barycentric coordinates (Section 1.5) does not agree precisely with the definition in this section. However, let $t^0, t^1, \ldots, t^n$ be the barycentric coordinates of $\mathbf{x}$, with respect to the vertices $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ of an n-simplex S. Let $\mathbf{g}$ be the affine transformation defined at the end of Section 5.7, and let $\mathbf{f} = \mathbf{g}^{-1}$. Then $t^1, \ldots, t^n$ are the coordinates of $\mathbf{x}$ in the affine coordinate system $\mathbf{f}$, and $t^0 = 1 - (t^1 + \cdots + t^n)$.

Gamma and beta functions

The gamma function was defined in Section 5.6. If we let $x = g(s) = s^2/2$, then

$$\int_0^\infty x^{u-1} \exp(-x) dx = 2^{1-u} \int_0^\infty (s^2)^{u-1} \exp\left(-\frac{s^2}{2}\right) s \, ds,$$

and we obtain another expression for $\Gamma(u)$:

$$\Gamma(u) = 2^{1-u} \int_0^\infty s^{2u-1} \exp\left(-\frac{s^2}{2}\right) ds, \qquad u > 0. \tag{5.40}$$

Let us calculate the product $\Gamma(u)\Gamma(v)$. Now

$$\Gamma(u)\Gamma(v) = 2^{2-u-v} \int_0^\infty s^{2u-1} \exp\left(-\frac{s^2}{2}\right) ds \int_0^\infty t^{2v-1} \exp\left(-\frac{t^2}{2}\right) dt.$$

Writing the iterated integral as an integral over the first quadrant Q and introducing polar coordinates,

$$\begin{aligned}\Gamma(u)\Gamma(v) &= 2^{2-u-v} \int_Q s^{2u-1} t^{2v-1} \exp\left(-\frac{s^2+t^2}{2}\right) dV_2(s,t), \\ &= 2^{2-u-v} \int_0^\infty r^{2(u+v)-1} \exp\left(-\frac{r^2}{2}\right) dr \int_0^{\pi/2} (\cos\theta)^{2u-1} (\sin\theta)^{2v-1}\, d\theta.\end{aligned}$$

The first integral on the right-hand side is $2^{u+v-1}\Gamma(u+v)$ by (5.40). Let

$$B(u,v) = 2 \int_0^{\pi/2} (\cos\theta)^{2u-1} (\sin\theta)^{2v-1}\, d\theta, \qquad u > 0, \qquad v > 0. \tag{5.41}$$

The function B is called the *beta function*, and we have just shown that

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{5.42}$$

EXAMPLE 2. Let $u = v = \frac{1}{2}$. Then $B(\frac{1}{2}, \frac{1}{2}) = 2\int_0^{\pi/2} d\theta = \pi$. Hence $[\Gamma(\frac{1}{2})]^2 = \pi\Gamma(1) = \pi$, or

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{5.43}$$

Using the formula

$$\Gamma(u+1) = u\Gamma(u), \tag{5.44}$$

proved earlier, $\Gamma(m + \frac{1}{2})$ can be found explicitly for any positive integer m. For instance,

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{4}.$$

If $u < 0$, then the integral defining $\Gamma(u)$ diverges. However, if u is not an integer $m = 0, -1, -2, \ldots$, one can use (5.44) to define $\Gamma(u)$. For instance, if $-1 < u < 0$, then $0 < u + 1 < 1$ and by definition $\Gamma(u) = \Gamma(u+1)/u$. Next $\Gamma(u)$ is defined for $-2 < u < -1$, and so on. The gamma function can also be defined for complex values of u [23, p. 148].

To obtain another expression for $B(u, v)$, set $\cos^2\theta = g(\theta) = z$ and apply the transformation formula. Then $|g'(\theta)| = 2\cos\theta\sin\theta$, and

$$B(u, v) = \int_0^{\pi/2} (\cos^2\theta)^{u-1}(\sin^2\theta)^{v-1} 2\cos\theta\sin\theta\, d\theta,$$

$$B(u, v) = \int_0^1 z^{u-1}(1-z)^{v-1}\, dz. \tag{5.45}$$

A variety of integrals can be reduced to either (5.41) or (5.45) and hence can be evaluated in terms of the gamma function (see Problem 11).

The measure α_n of the unit n-ball

According to Problem 7, Section 5.5,

$$\frac{\alpha_n}{\alpha_{n-1}} = 2\int_0^1 (1-u^2)^{(n-1)/2}\, du.$$

Setting $u = g(z) = \sqrt{z}$, then $g'(z) = \frac{1}{2}z^{-1/2}$ and

$$\begin{aligned}\frac{\alpha_n}{\alpha_{n-1}} &= \int_0^1 z^{-1/2}(1-z)^{(n-1)/2}\, dz \\ &= B\left(\frac{1}{2}, \frac{n+1}{2}\right),\end{aligned}$$

$$\alpha_n = \alpha_{n-1}\frac{\Gamma(\frac{1}{2})\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\dfrac{n}{2}+1\right)}.$$

Moreover, $\alpha_1 = 2$. By induction on n and Formula (5.43),

$$\alpha_n = \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)}, \qquad n = 1, 2, \ldots \tag{5.46}$$

If n is even, $n = 2l$, then $l\Gamma(l) = l!$ and $\alpha_{2l} = \pi^l/l!$.

Problems

1. Let $A = \{(x, y): x^2 + y^2 \le a^2, x \ge 0\}$. Evaluate $\int_A xy^2 dV_2(x, y)$ by introducing polar coordinates.
2. Find the area of $\{(x, y): x < y < 2x, 1 < x + 4y < 4\}$ by introducing $f^1(x, y) = y/x$, $f^2(x, y) = x + 4y$ as coordinates of (x, y).
3. Let $A = \{(x, y): 0 \le x^2 + y^2 \le 2, x^2 - y^2 \le 1, x \ge 0, y \ge 0\}$. Find $\int_A x dV_2(x, y)$ by introducing the coordinates $f^1(x, y) = x^2 + y^2$, $f^2(x, y) = x^2 - y^2$.
4. Write the iterated integral $\int_0^1 dx \int_0^{1-x} dy \int_0^{g(x,y)} f(x, y, z)dz$ as an iterated integral in cylindrical coordinates.
5. Find $V_3(\{(x, y, z): x^2 + y^2 + z^2 \le a^2, x^2 + y^2 \ge b^2\})$ where $a > b$.

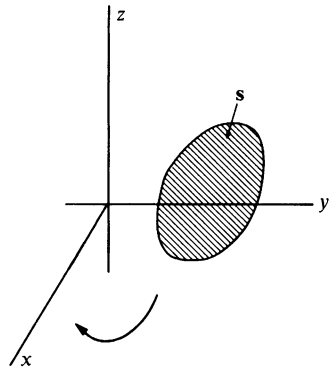


Figure 5.19

6. (*Solids of revolution.*) Let S be a compact subset of the right half-plane and

$$A = \{(x, y, z) : (r, z) \in S\},$$

where $r^2 = x^2 + y^2$. Show that $V_3(A) = 2\pi\bar{y}V_2(S)$, where $(\bar{y}, \bar{z})$ is the centroid of S (Figure 5.19).

7. Find $V_3(\{(x, y, z) : \exp(-2x) > y^2 + z^2, x > 0\})$.

8. Let $f^1(x, y)$ = distance between (x, y) and $(1, 0)$, and $f^2(x, y)$ = distance between (x, y) and $(-1, 0)$. Show that $\mathbf{f} = (f^1, f^2)$ is a coordinate system for the half-plane $D = \{(x, y) : y > 0\}$.

9. Let $(f^1, \ldots, f^n)$ be a coordinate system for $D_1 \subset E^n$, and $(\phi^1, \ldots, \phi^p)$ a coordinate system for $D_2 \subset E^p$. Show that $(f^1, \ldots, f^n, \phi^1, \ldots, \phi^p)$, regarded as functions on $D_1 \times D_2$, form a coordinate system for $D_1 \times D_2$.

10. (*Bipolar coordinates in E^4.*) In this system the coordinates of $\mathbf{x}$ are $r \cos\theta$, $r \sin\theta$, $\rho \cos\alpha$, $\rho \sin\alpha$, where (r, θ) are polar coordinates in the x^1x^2-plane and (ρ, α) polar coordinates in the x^3x^4-plane. Find, using bipolar coordinates,

$$\int_{K \times K} (x^1)^2 \, dV_4(\mathbf{x}),$$

where K is the unit circular disk $x^2 + y^2 \leq 1$.

11. In terms of the gamma-function, find:

(a) $\int_0^1 \sqrt{1 - x^3}\, dx$. [*Hint*: Let $x^3 = z$.]

(b) The area of $\{(x, y) : 0 \leq y \leq \sqrt{\cos x}, -\pi/2 \leq x \leq \pi/2\}$.

(c) $\int_0^\infty x^a \exp(-x^b)\, dx$, $a > -1$, $b > 0$.

(d) $\int_1^\infty (\log x)^c x^d\, dx$, $c > -1$, $d < -1$.

(e) $\int_\Sigma (x^1)^k\, dV_n(\mathbf{x})$, $k = 1, 2, \ldots$, Σ the standard n-simplex.

(f) $\int (x^1)^k \exp(-\sum_{i=1}^n a_i(x^i)^2) dV_n(\mathbf{x})$, $k = 1, 2, \ldots$, $a_i > 0$ for $i = 1, \ldots, n$.

12. Let $Q(\mathbf{x}) = \sum_{i,j=1}^n c_j^i x^i x^j > 0$ for every $\mathbf{x} \neq \mathbf{0}$, where $c_j^i = c_i^j$ for $i, j = 1, \ldots, n$. Show that

$$\int \exp[-Q(\mathbf{x})/2] dV_n(\mathbf{x}) = (2\pi)^{n/2}[\det(c_j^i)]^{-1/2}.$$

[*Hint*: Make a suitable orthogonal transformation.]

5.10 Measurable sets and functions; further properties

The treatment of measurable sets, begun in Section 5.2, is continued here by completing the proofs of Theorems 5.2 and 5.3 for unbounded sets. Then we list and prove several properties of measurable functions.

For the rest of the chapter it is convenient to adjoin to E^1 two "ideal points" $-\infty$ and $+\infty$. We call the ordered number field E^1 with $-\infty$ and $+\infty$ adjoined the *extended* real number system. The points $-\infty$, $+\infty$ are not numbers. However, for present purposes we agree that $-\infty < a < +\infty$ for every number a, and that

$$(+\infty) + (+\infty) = +\infty, \qquad (+\infty) + a = a + (+\infty) = +\infty,$$

$$(+\infty)a = a(+\infty) = \begin{cases} +\infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0. \end{cases}$$

Similar conventions are made regarding $-\infty$. However, $(+\infty) + (-\infty)$ is undefined.

The extended real number system is denoted by $\bar{E}^1$. If S is a nonempty subset of $\bar{E}^1$, let $\sup S$ be the smallest $b \in \bar{E}^1$ such that $x \leq b$ for every $x \in S$. If $+\infty \in S$, then clearly $\sup S = +\infty$. Moreover, if $x < +\infty$ for every $x \in S$ but S has no upper bound in E^1, then $\sup S = +\infty$.

The definition of $\inf S$ is similar. By neighborhood of $+\infty$ let us mean any set $\{x \in \bar{E}^1 : x > c\}$. Then convergence to $+\infty$ of sequences in $\bar{E}^1$ makes sense. Any nondecreasing sequence $x_1, x_2, \ldots$, in $\bar{E}^1$ such that $x_m > -\infty$ for some m has a limit x_0 which is finite (i.e., in E^1) or $+\infty$.

Lemma 1 regarding reversing the order of limits for doubly indexed monotone sequences in $\bar{E}^1$ is used several times.

Lemma 1. *Assume that* $a_{m\nu} \geq 0$, $a_{m\nu} \leq a_{m,\nu+1}$, $a_{m\nu} \leq a_{m+1,\nu}$ *for every* $m = 1, 2, \ldots, \nu = 1, 2, \ldots$ *Then*

$$\lim_{m\to\infty} \lim_{\nu\to\infty} a_{m\nu} = \lim_{\nu\to\infty} \lim_{m\to\infty} \alpha_{m\nu}. \tag{5.47}$$

PROOF. Let

$$b_m = \lim_{\nu\to\infty} a_{m\nu}, \qquad c_\nu = \lim_{m\to\infty} a_{m\nu}.$$

These limits exist since $a_{m1} \leq a_{m2} \leq \cdots$ and $a_{1\nu} \leq a_{2\nu} \leq \cdots$. Moreover, $b_1 \leq b_2 \leq \cdots$ and $c_1 \leq c_2 \leq \cdots$, Let

$$b = \lim_{m\to\infty} b_m, \qquad c = \lim_{\nu\to\infty} c_\nu.$$

For every m and ν, $a_{m\nu} \leq c$. Hence $b_m \leq c$ for every m, and it follows that $b \leq c$. Similarly, $c \leq b$. □

As a first application of Lemma 1, let us complete the proof of Theorem 5.3, which was postponed in Section 5.2 in case of unbounded sets. Using Theorem 5.3 we then complete the proof of Theorem 5.2(d, e). Formulas (5.10) and (5.11) are now to be understood as holding in the extended real number system $\bar{E}^1$.

Completion of proof of Theorem 5.3

First consider a nondecreasing sequence of measurable sets $A_1 \subset A_2 \subset \cdots$. As in Section 5.2, let U_r denote the r-neighborhood of $\mathbf{0}$. Consider integer values $m = 1, 2, \ldots$ for r, and let $a_{m\nu} = V(A_\nu \cap U_m)$. This doubly indexed sequence satisfies the hypotheses of Lemma 1. Let $A = A_1 \cup A_2 \cup \cdots$. For each m,

$$A \cap U_m = (A_1 \cap U_m) \cup (A_2 \cap U_m) \cup \cdots.$$

By definition of measure for possibly unbounded sets (Section 5.2),

$$V(A_\nu) = \lim_{m \to \infty} V(A_\nu \cap U_m), \; V(A) = \lim_{m \to \infty} V(A \cap U_m).$$

By (5.11) for the case of bounded nondecreasing sequences of sets

$$V(A \cap U_m) = \lim_{\nu \to \infty} V(A_\nu \cap U_m).$$

By Lemma 1,

$$V(A) = \lim_{\nu \to \infty} V(A_\nu).$$

This proves (5.11).

Next suppose that $A_1 \supset A_2 \supset \cdots$, with $V(A_1) < \infty$. Let $A = A_1 \cap A_2 \cap \cdots$. For each m,

$$(A_1 - A) \cap U_m = A_1 \cap U_m - A \cap U_m.$$

Since $A \cap U_m \subset A_1 \cap U_m$,

$$V[(A_1 - A) \cap U_m] = V(A_1 \cap U_m) - V(A \cap U_m).$$

Let $m \to \infty$. We get

$$V(A_1 - A) = V(A_1) - V(A).$$

Similarly, $V(A_1 - A_\nu) = V(A_1) - V(A_\nu)$ for each ν. We then get (5.12) by applying (5.11) to the nondecreasing sequence $A_1 - A_1$, $A_1 - A_2, \ldots$, $A_1 - A_\nu, \ldots$. □

Completion of proof of Theorem 5.2(d, e)

First consider two measurable sets A and B (possibly unbounded). From Section 5.2, we know that for each r

$$V[(A \cup B) \cap U_r] \leq V(A \cap U_r) + V(B \cap U_r).$$

Upon letting $r \to \infty$,

$$V(A \cup B) \leq V(A) + V(B).$$

Equality holds if $A \cap B$ is empty. By induction, the same results hold for unions of finitely many measurable sets $A_1, \ldots, A_\nu$.

Now consider any sequence of measurable sets $A_1, A_2, \ldots$, and let $A = A_1 \cup A_2 \cup \cdots$. Let $D_\nu = A_1 \cup \cdots \cup A_\nu$. Then $D_1 \subset D_2 \subset \cdots$ and $A = D_1 \cup D_2 \cup \cdots$. Moreover,

$$V(D_\nu) \leq \sum_{k=1}^{\nu} V(A_k).$$

By (5.11),

$$V(A) = \lim_{\nu \to \infty} V(D_\nu) \leq \sum_{k=1}^{\infty} V(A_k).$$

If $A_1, A_2, \ldots$ are disjoint, then the inequalities become equalities. □

For the discussion of integrals to follow it is convenient to consider functions which may have the values $-\infty$ or $+\infty$.

Properties of measurable functions

Let f be a function with domain E^n and values in the extended real number system $\bar{E}^1$. As in Section 5.4, f is *measurable* if

$$\{\mathbf{x} : f(\mathbf{x}) > c\} \tag{5.48}$$

is a measurable set for every real number c. Let us list and prove some properties of measurable functions.

(1) *If f is measurable, then for every c the sets $\{\mathbf{x} : f(\mathbf{x}) \leq c\}$, $\{\mathbf{x} : f(\mathbf{x}) \geq c\}$, and $\{\mathbf{x} : f(\mathbf{x}) < c\}$ are measurable sets.*

PROOF. The complement of a measurable set is measurable. Since $\{\mathbf{x} : f(\mathbf{x}) \leq c\}$ is the complement of the set in (5.48), it is measurable. Now $\{\mathbf{x} : f(\mathbf{x}) \geq c\} = \bigcap_{m=1}^{\infty} \{\mathbf{x} : f(\mathbf{x}) > c - 1/m\}$. Each set on the right is measurable, and hence their intersection is measurable. Taking complements, the third set is measurable. □

From the definition and (1), $f^{-1}(I)$ is measurable if I is any semiinfinite interval. It can be shown that $f^{-1}(E)$ is measurable if E is any open set or closed set.

In the next statement we agree that $0f = 0$ even when f has extended real values.

(2) *If f is measurable, then af is measurable for any real number a.*

PROOF. This follows from the definition and (1). □

(3a) *If f and g are measurable and*

$$h(\mathbf{x}) = \max\{f(\mathbf{x}), g(\mathbf{x})\} \quad \textit{for every } \mathbf{x},$$

then h is measurable.

PROOF. $\{\mathbf{x} : h(\mathbf{x}) > c\} = \{\mathbf{x} : f(\mathbf{x}) > c\} \cup \{\mathbf{x} : g(\mathbf{x}) > c\}$. □

In particular, if f is measurable and

$$f^+(\mathbf{x}) = \max\{f(\mathbf{x}), 0\}, \qquad f^-(\mathbf{x}) = \max\{-f(\mathbf{x}), 0\},$$

then f^+ and f^- are measurable.

Statement (3a) extends to the maximum of a finite number of measurable functions and, more importantly, to sequences of functions.

(3b) *If $f_1, f_2, \ldots$ are measurable and*

$$g(\mathbf{x}) = \sup\{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots\} \quad \textit{for every } \mathbf{x},$$

then g is measurable.

PROOF

$$\{\mathbf{x} : g(\mathbf{x}) > c\} = \bigcup_{\nu=1}^{\infty} \{\mathbf{x} : f_\nu(\mathbf{x}) > c\}. \qquad \square$$

In particular, if $f_1 \leq f_2 \leq \cdots$ then g is the limit of this nondecreasing sequence.

Similarly, if

$$h(\mathbf{x}) = \inf\{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots\} \quad \text{for every } \mathbf{x},$$

then h is measurable.

Let $y_1, y_2, \ldots$ be any sequence in $\bar{E}^1$. Let

$$z_\nu = \inf\{y_\nu, y_{\nu+1}, \ldots\}, \qquad \nu = 1, 2, \ldots$$

Then $z_1 \leq z_2 \leq \cdots$. The limit of the monotone sequence $z_1, z_2, \ldots$ is called the *lower limit* of the sequence $y_1, y_2, \ldots$, and is denoted by $\liminf_{\nu\to\infty} y_\nu$. Similarly, if

$$w_\nu = \sup\{y_\nu, y_{\nu+1}, \ldots\}, \qquad \nu = 1, 2, \ldots,$$

then $w_1 \geq w_2 \geq \cdots$. The limit of the monotone sequence $w_1, w_2, \ldots$ is the *upper limit* $\limsup_{\nu\to\infty} y_\nu$. Since $z_\nu \leq w_\nu$ for each $\nu = 1, 2, \ldots$, we must have

$$\liminf_{\nu\to\infty} y_\nu \leq \limsup_{\nu\to\infty} y_\nu.$$

Equality holds if and only if the sequence $y_1, y_2, \ldots$ has a limit.

(4) *If $f_1, f_2, \ldots$ are measurable and*

$$f(\mathbf{x}) = \liminf_{\nu\to\infty} f_\nu(\mathbf{x}) \quad \textit{for every } \mathbf{x},$$

then f is measurable.

PROOF. For each $\nu = 1, 2, \ldots$ let $h_\nu(\mathbf{x}) = \inf\{f_\nu(\mathbf{x}), f_{\nu+1}(\mathbf{x}), \ldots\}$. By (3b) each h_ν is measurable; $h_1 \leq h_2 \leq \cdots$, and by the definition of "lim inf," $f(\mathbf{x}) = \sup\{h_1(\mathbf{x}), h_2(\mathbf{x}), \ldots\}$. Therefore f is measurable. $\square$

(5) *If $f_1, f_2, \ldots$ are measurable and*

$$f(\mathbf{x}) = \lim_{\nu \to \infty} f_\nu(\mathbf{x}) \quad \textit{for every } \mathbf{x},$$

then f is measurable.

Note: This is a particular case of (4).

In the next statement it is assumed that $f(\mathbf{x}) + g(\mathbf{x})$ is everywhere defined. In other words, for no $\mathbf{x}$ it is true that $f(\mathbf{x}) = +\infty$, $g(\mathbf{x}) = -\infty$ or vice versa.

(6) *If f and g are measurable, then $f + g$ is measurable.*

PROOF. For each $\nu = 1, 2, \ldots$ let

$$f_\nu(\mathbf{x}) = \begin{cases} \nu & \text{if } f(\mathbf{x}) > \nu, \\ -\nu & \text{if } f(\mathbf{x}) \leq -\nu, \\ \dfrac{j}{\nu} & \text{if } \dfrac{j-1}{\nu} < f(\mathbf{x}) \leq \dfrac{j}{\nu}, \end{cases}$$

where j is an integer, and $1 - \nu^2 \leq j \leq \nu^2$. This construction is like that in Section 5.4, Lemma 1. Then

$$f(\mathbf{x}) = \lim_{\nu \to \infty} f_\nu(\mathbf{x})$$

for every $\mathbf{x}$. Defining g_ν in the same way,

$$g(\mathbf{x}) = \lim_{\nu \to \infty} g_\nu(\mathbf{x}).$$

For each ν the functions f_ν and g_ν are measurable and take a finite number of values. It is easy to show that their sum $f_\nu + g_\nu$ is measurable. Since for every $\mathbf{x}$

$$f(\mathbf{x}) + g(\mathbf{x}) = \lim_{\nu \to \infty} [f_\nu(\mathbf{x}) + g_\nu(\mathbf{x})],$$

$f + g$ is measurable by (5). □

PROBLEMS

1. Let $a_{m\nu} = m(m + \nu)^{-1}$. Show that (5.47) does not hold. Which assumption in Lemma 1 fails to be satisfied?
2. Let f and g be measurable real-valued functions. Show that their product is measurable. [*Hint*: Show that the square of a measurable function is measurable. Then $2fg = (f + g)^2 - f^2 - g^2$.]
3. Let A be a measurable set with $V(A) < \infty$. Show that there exists a measurable set $B \subset A$ with $V(B) = \frac{1}{2}V(A)$.
4. Show that:
   (a) If A is measurable, N is a null set, and $A - N \subset B \subset A \cup N$, then B is measurable and $V(A) = V(B)$. [*Hint*: First consider the case of bounded sets.]
   (b) If f is measurable and $f(\mathbf{x}) = g(\mathbf{x})$ except for $\mathbf{x}$ in a null set N, then g is measurable.

5. A set A is called *σ-compact* if there is a sequence of compact sets $K_1 \subset K_2 \subset \cdots$, such that $A = K_1 \cup K_2 \cup \cdots$. Show that:
(a) Any closed set is σ-compact.
(b) Any open set is σ-compact.
(c) If $A_1, A_2, \ldots$ are σ-compact, then $A_1 \cup A_2 \cup \cdots$ is σ-compact.
(d) If A is any measurable set, then $A = B \cup N$ where B is σ-compact and N is a null set. [*Hint*: Show that the result is true for bounded measurable sets.]

5.11 Integrals: General definition, convergence theorems

Let us now define the integral, in the Lebesgue sense, without the special assumptions (of boundedness, or continuity) imposed in earlier sections. A second objective of this section is to find general conditions under which the symbols "$\lim_{\nu \to \alpha}$" and "$\int$" can be interchanged. Two main theorems giving such conditions are the monotone sequences theorem and Lebesgue's dominated convergence theorem.

Definition. The integral of a bounded measurable function with compact support was defined in Section 5.3. Let us extend the definition in three steps:

(a) Let f be bounded, measurable, and $f \geq 0$. For each $r > 0$, f is measurable on the r-neighborhood U_r of $\mathbf{0}$. By Theorem 5.5, f is integrable over U_r. Let $\phi(r)$ equal the integral of f over U_r. Since $f \geq 0$, ϕ is nondecreasing. Hence $\phi(r)$ tends to a limit, finite or $+\infty$, as $r \to +\infty$. Let

$$\int f\, dV = \lim_{r \to +\infty} \int_{U_r} f\, dV. \tag{5.49}$$

If the limit is finite, then f is *integrable* (over E^n). Otherwise the left-hand side of (5.49) equals $+\infty$. If $0 \leq f \leq g$, then $\int f\, dV \leq \int g\, dV$.

(b) Let f be measurable and $f \geq 0$. For any $t > 0$ consider the function ${}_t f$ such that

$${}_t f(\mathbf{x}) = \min\{f(\mathbf{x}), t\} \quad \text{for every } \mathbf{x}.$$

For each t, ${}_t f$ is a bounded, measurable function. It is called the *truncation* of f at height t. If $s \leq t$, then ${}_s f \leq {}_t f$ and hence $\int {}_s f\, dV \leq \int {}_t f\, dV$. Let

$$\int f\, dV = \lim_{t \to +\infty} \int {}_t f\, dV. \tag{5.50}$$

If the limit is finite, then f is *integrable*. Otherwise $\int f\, dV = +\infty$. If $0 \leq f \leq g$, then ${}_t f \leq {}_t g$ for every t. Consequently, $\int f\, dV \leq \int g\, dV$.

If f is bounded, then ${}_tf = f$ for all sufficiently large t. The definitions in (a) and (b) agree. Moreover, if f has compact support, then $\phi(r)$ is constant for all sufficiently large r and the definition of integral in (a) agrees with the one in Section 5.3. For Step (c), f may take both positive and negative values. We write $f = f^+ - f^-$, where f^+, f^- are nonnegative as in Section 5.6

(c) A measurable function f is *integrable* if f^+ and f^- are integrable. If f is integrable, then

$$\int f\,dV = \int f^+\,dV - \int f^-\,dV. \tag{5.51}$$

If f has compact support and is bounded, then the definition in (5.51) agrees with that in Section 5.3. This follows from Proposition 5.4 and the fact that the integrals of f^+, f^- in the new sense and in the sense of Section 5.3 agree.

Let us now give some general theorems about the validity of interchanging the symbols "$\lim_{\nu\to\infty}$" and "$\int$."

Lemma 1. *Let K be a compact set and $\phi_1, \phi_2, \ldots$ be bounded measurable functions with supports contained in K such that $\phi_1 \geq \phi_2 \geq \cdots$ and $\lim_{\nu\to\infty} \phi_\nu(\mathbf{x}) = 0$ for every $\mathbf{x}$. Then*

$$\lim_{\nu\to\infty} \int \phi_\nu\,dV = 0.$$

PROOF. There exists C such that $\phi_\nu(\mathbf{x}) \leq C$ for every $\mathbf{x}$ and $\nu = 1$, hence also for $\nu = 2, 3, \ldots$. Given $\varepsilon > 0$ let $c = \varepsilon/2V(K)$. Let

$$A_\nu = \{\mathbf{x} : \phi_\nu(\mathbf{x}) > c\}.$$

By hypothesis $K \supset A_1 \supset A_2 \supset \cdots$ and $A_1 \cap A_2 \cap \cdots$ is the empty set. Since $V(A_1)$ is finite, we may apply Formula (5.12), obtaining

$$\lim_{\nu\to\infty} V(A_\nu) = V(A_1 \cap A_2 \cap \cdots) = 0.$$

Let ν_0 be such that $V(A_\nu) < \varepsilon/2C$ for every $\nu \geq \nu_0$. Then

$$\int_{A_\nu} \phi_\nu\,dV \leq CV(A_\nu) < \frac{\varepsilon}{2}, \quad \text{if } \nu \geq \nu_0,$$

while for every ν,

$$\int_{K-A_\nu} \phi_\nu\,dV \leq cV(K - A_\nu) \leq cV(K) = \frac{\varepsilon}{2}.$$

Hence for every $\nu \geq \nu_0$,

$$\int \phi_\nu\,dV = \int_K \phi_\nu\,dV < \varepsilon. \qquad \square$$

Monotone sequences theorem. *Let $f_1, f_2, \ldots$ be measurable functions with $0 \le f_1 \le f_2 \le \cdots$, and let $f(\mathbf{x}) = \lim_{\nu\to\infty} f_\nu(\mathbf{x})$ for every $\mathbf{x}$. Then*

$$\int f\,dV = \lim_{\nu\to\infty} \int f_\nu\,dV. \tag{5.52}$$

PROOF. We have shown in Section 5.10 that the limit f is measurable. First let us assume that f is bounded and has compact support. Let $\phi_\nu = f - f_\nu$. Since $0 \le f_\nu \le f$, each function f_ν is also bounded and has compact support. Moreover,

$$\int \phi_\nu\,dV = \int f\,dV - \int f_\nu\,dV, \quad \nu = 1, 2, \ldots$$

From Lemma 1, we get (5.52).

Next let us suppose merely that f is bounded. Let

$$a_{m\nu} = \int_{U_m} f_\nu\,dV, \quad m, \nu = 1, 2, \ldots$$

and apply Lemma 1 in Section 5.10. Finally, if f is unbounded observe that for each $m = 1, 2, \ldots$,

$$0 \le {}_mf_1 \le {}_mf_2 \le \cdots, \quad {}_mf(\mathbf{x}) = \lim_{\nu\to\infty} {}_mf_\nu(\mathbf{x}),$$

where ${}_mf_\nu$ is the truncation of f_ν at height m. Apply Lemma 1 again with

$$a_{m\nu} = \int {}_mf_\nu\,dV, \quad m, \nu = 1, 2, \ldots \qquad \square$$

Note. The theorem states in particular that f is integrable if and only if the nondecreasing sequence of numbers $\int f_1\,dV, \int f_2\,dV, \ldots$ has a finite limit.

The double limiting process (5.49) and (5.50) can be replaced by a single one. Let f be measurable and $f \ge 0$. Let

$$(*) \qquad f_\nu(\mathbf{x}) = \begin{cases} \nu & \text{if } f(\mathbf{x}) \ge \nu, \quad |\mathbf{x}| \le \nu, \\ 0 & \text{if } |\mathbf{x}| > \nu, \\ f(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Then $0 \le f_1 \le f_2 \le \cdots$ and $f_\nu(\mathbf{x})$ tends to $f(\mathbf{x})$ for every $\mathbf{x}$. Hence by the theorem,

$$\int f\,dV = \lim_{\nu\to\infty} \int f_\nu\,dV.$$

Corollary 1. *If f and g are real valued integrable functions, then $f + g$ is integrable and*

$$\int (f + g)dV = \int f\,dV + \int g\,dV. \tag{5.53}$$

PROOF. By (6) (Section 5.10) $f + g$ is measurable. If $f \geq 0$, $g \geq 0$, define f_ν and g_ν by (*). For each ν, f_ν and g_ν are bounded and have compact supports. By Proposition 5.4,

$$\int (f_\nu + g_\nu) dV = \int f_\nu \, dV + \int g_\nu \, dV.$$

The sequences $f_1, f_2, \ldots, g_1, g_2, \ldots, f_1 + g_1, f_2 + g_2, \ldots$ are nondecreasing and tend respectively to $f, g, f + g$. By the monotone sequences theorem, the corollary is true when $f \geq 0, g \geq 0$.

In the general case,

$$0 \leq (f + g)^+ \leq f^+ + g^+.$$

Since f^+ and g^+ are integrable, so is $(f + g)^+$. Similarly $(f + g)^-$ is integrable. Then

$$f^+ + g^+ = (f + g)^+ + \phi,$$
$$f^- + g^- = (f + g)^- + \phi,$$

where $\phi \geq 0$. Since the corollary is true for nonnegative functions,

$$\int f^+ \, dV + \int g^+ \, dV = \int (f + g)^+ dV + \int \phi \, dV,$$
$$\int f^- \, dV + \int g^- dV = \int (f + g)^- \, dV + \int \phi \, dV.$$

Subtracting, we get (5.53.) □

Corollary 2. *A measurable function f is integrable if and only if $|f|$ is integrable.*

This is proved in the same way as Corollary 1, Section 5.6, using $f = f^+ - f^-$, $|f| = f^+ + f^-$, and Corollary 1 immediately above. It is easy to show that cf is integrable if f is integrable, and

$$\int (cf) dV = c \int f \, dV.$$

Corollary 3. *Let $f_1, f_2, \ldots$ be measurable, f_1 be integrable, $f_1 \geq f_2 \geq \cdots \geq 0$, and let $f(\mathbf{x}) = \lim_{\nu \to \infty} f_\nu(\mathbf{x})$ for every $\mathbf{x}$. Then (5.52) holds.*

PROOF. Let $g_\nu = f_1 - f_\nu$ and apply the theorem to the nondecreasing sequence $g_1, g_2 \ldots$ which has $f_1 - f$ as limit. □

For sequences that are not necessarily monotone, the result is as follows.

Fatou's lemma. *Let f_ν be measurable and $f_\nu \geq 0$ for each $\nu = 1, 2, \ldots$. Then*

$$\int (\liminf_{\nu \to \infty} f_\nu) dV \leq \liminf_{\nu \to \infty} \int f_\nu \, dV. \tag{5.54a}$$

PROOF. Let $h_\nu = \inf\{f_\nu, f_{\nu+1}, \ldots\}$. Since $h_\nu \le f_m$ whenever $\nu \le m$,

$$\int h_\nu \, dV \le \int f_m \, dV, \qquad m = \nu, \nu + 1, \ldots,$$

$$\text{(**)} \qquad \int h_\nu \, dV \le \liminf_{m \to \infty} \int f_m \, dV, \qquad \nu = 1, 2, \ldots$$

Let $f = \liminf_{\nu \to \infty} f_\nu$. The sequence $h_1, h_2, \ldots$ is nondecreasing and tends to f. By the monotone sequences theorem,

$$\int f \, dV = \lim_{\nu \to \infty} \int h_\nu \, dV.$$

Since (**) holds for each ν, the limit is no more than the right-hand side of (**). □

A statement about measurable functions is said to hold *almost everywhere* (or for *almost all* x) if it is true except for $\mathbf{x}$ in some null set.

Proposition 5.5. *If f is integrable, then $f(\mathbf{x})$ is finite almost everywhere.*

PROOF. If $f \ge 0$, let

$$A_m = \{\mathbf{x} : f(\mathbf{x}) \ge m\}, \qquad A_\infty = \{\mathbf{x} : f(\mathbf{x}) = +\infty\}.$$

Then $A_\infty = A_1 \cap A_2 \cap \cdots$ and $A_1 \supset A_2 \supset \cdots$. For each m, let $\phi_m(\mathbf{x}) = f(\mathbf{x})$ if $\mathbf{x} \in A_m$, and otherwise $\phi_m(\mathbf{x}) = 0$. Then $\phi_m \ge m 1_{A_m}$, where 1_A is the characteristic function of A (Section 5.4). Hence

$$mV(A_m) = \int m 1_{A_m} \, dV \le \int \phi_m \, dV.$$

Since $\phi_m \le f$ we have, dividing by m,

$$V(A_m) \le \frac{1}{m} \int f \, dV.$$

Since the right-hand side tends to 0 as $m \to \infty$, $V(A_\infty) = 0$.

In the general case, f^+ and f^- are integrable and

$$\{\mathbf{x} : f(\mathbf{x}) = \pm\infty\} = \{\mathbf{x} : f^+(\mathbf{x}) = +\infty\} \cup \{\mathbf{x} : f^-(\mathbf{x}) = +\infty\}. \qquad \square$$

If f is measurable and $g(\mathbf{x}) = f(\mathbf{x})$ almost everywhere, then g is measurable. Moreover, if either is integrable then so is the other and $\int f \, dV = \int g \, dV$ (Problem 1).

In Fatou's lemma the hypothesis "$f_\nu \ge 0$ for every ν" can be replaced by "$f_\nu \ge \phi$ for every ν, where ϕ is integrable." For this purpose we make the following convention. If $f \ge \phi$ and ϕ is integrable, then $\phi(\mathbf{x})$ is finite almost everywhere. By $f - \phi$ let us mean the function with value $f(\mathbf{x}) - \phi(\mathbf{x})$ if

$\phi(\mathbf{x}) \neq +\infty$, and value 0 if $\phi(\mathbf{x}) = +\infty$. Then $\int (f - \phi)dV = \int f\,dV - \int \phi\,dV$ if $f - \phi$ is integrable. If the integral of $f - \phi$ diverges to $+\infty$, then we agree that the integral of f also diverges to $+\infty$.

Let $g_\nu = f_\nu - \phi$. By assumption, $g_\nu \geq 0$ for each $\nu = 1, 2, \ldots$. Moreover,

$$\liminf_{\nu\to\infty} g_\nu(\mathbf{x}) = \liminf_{\nu\to\infty} f_\nu(\mathbf{x}) - \phi(\mathbf{x}),$$

provided $\phi(\mathbf{x}) \neq +\infty$, hence almost everywhere. Since for each ν

$$\int g_\nu\,dV = \int f_\nu\,dV - \int \phi\,dV,$$

we have

$$\liminf_{\nu\to\infty} \int g_\nu\,dV = \liminf_{\nu\to\infty} \int f_\nu\,dV - \int \phi\,dV.$$

Applying Fatou's lemma to the sequence $g_1, g_2, \ldots$ and adding $\int \phi\,dV$ to each side, we again get (5.54a).

Similarly, if for each ν, $f_\nu \leq \phi$ where ϕ is integrable, then

$$\int\left(\limsup_{\nu\to\infty} f_\nu\right)dV \geq \limsup_{\nu\to\infty} \int f_\nu\,dV. \tag{5.54b}$$

Note. In the monotone sequences theorem and its corollary the hypothesis $f_\nu \geq 0$ can also be replaced by $f_\nu \geq \phi$, where ϕ is integrable.

Lebesgue's dominated convergence theorem. *Let $f_1, f_2, \ldots$ be measurable functions such that:*

(a) $\lim_{\nu\to\infty} f_\nu(\mathbf{x}) = f(\mathbf{x})$ *almost everywhere.*
(b) *There is an integrable function g such that $|f_\nu| \leq g$ for $\nu = 1, 2, \ldots$ Then*

$$\int f\,dV = \lim_{\nu\to\infty} \int f_\nu\,dV.$$

PROOF. Since $-g \leq f_\nu \leq g$ and both $-g$ and g are integrable, we can apply (5.54a) and (5.54b). But

$$\liminf_{\nu\to\infty} f_\nu(\mathbf{x}) = \limsup_{\nu\to\infty} f_\nu(\mathbf{x}) = f(\mathbf{x})$$

almost everywhere. Hence

$$\limsup_{\nu\to\infty} \int f_\nu\,dV \leq \int f\,dV \leq \liminf_{\nu\to\infty} \int f_\nu\,dV.$$

But lim inf $\leq$ lim sup, and hence $\int f_\nu\,dV$ tends to $\int f\,dV$. □

Note: In (b) it suffices that $|f_\nu(\mathbf{x})| \leq g(\mathbf{x})$ almost everywhere. One can redefine $f_\nu(\mathbf{x})$ to be 0 on the null set where this inequality does not hold without changing $\int f_\nu\,dV$.

Integrals over measurable sets

Let A be a measurable set. Just as in Section 5.4, a function f is called *measurable on* A if, for every real c, $\{\mathbf{x} \in A : f(\mathbf{x}) > c\}$ is measurable. If the function f_A (Section 5.4) is integrable, then f is *integrable over* A and we set

$$\int_A f\,dV = \int f_A\,dV.$$

If f and g are integrable over A, then $f + g$ is integrable over A. This follows from Corollary 1, since $(f + g)_A = f_A + g_A$. Similarly cf is integrable over A. The basic properties of integrals listed in Theorem 5.4, Section 5.4, remain valid.

The monotone sequences theorem, Fatou's lemma, and Lebesgue's dominated convergence theorem remain true for integrals over A. In these theorems one has simply to write $\int_A$ in place of $\int$, and to replace the phrases

"measurable"	by "measurable on A,"
"integrable"	by "integrable over A,"
"for every $\mathbf{x}$"	by "for every $\mathbf{x} \in A$,"
"almost everywhere"	by "almost everywhere in A."
"$\lvert f_\nu \rvert \le g$"	by "$\lvert f_\nu(\mathbf{x}) \rvert \le g(\mathbf{x})$ for every $\mathbf{x} \in A$."

In each case let $F_\nu = (f_\nu)_A$, $F = f_A$. Then

$$\int F_\nu\,dV = \int_A f_\nu\,dV, \qquad \int F\,dV = \int_A f_A\,dV.$$

If, for every $\mathbf{x} \in A$, $0 \le f_1(\mathbf{x}) \le f_2(\mathbf{x}) \le \cdots$ and $f_\nu(\mathbf{x})$ tends to $f(\mathbf{x})$ as $\nu \to \infty$, then $0 \le F_1(\mathbf{x}) \le F_2(\mathbf{x}) \le \cdots$ and $F_\nu(\mathbf{x})$ tends to $F(\mathbf{x})$ for every $\mathbf{x} \in E^n$. Applying the monotone sequence theorem to the sequence $F_1, F_2, \ldots$, we get

$$\int_A f\,dV = \lim_{\nu\to\infty} \int_A f_\nu\,dV.$$

The proofs of Fatou's lemma and the dominated convergence theorem for integrals over A are similar.

Corollary 4. *Let A be a measurable set of finite measure and $f_1, f_2, \ldots$ measurable on A. Assume that:*

(a) $\lim_{\nu\to\infty} f_\nu(\mathbf{x}) = f(\mathbf{x})$ *almost everywhere in* A.

(b) *There is a number C such that $\lvert f_\nu(\mathbf{x}) \rvert \le C$ for every $\mathbf{x} \in A$ and $\nu = 1, 2, \ldots$. Then*

$$\int_A f\,dV = \lim_{\nu\to\infty} \int_A f_\nu\,dV.$$

PROOF. Let $g(\mathbf{x}) = C$ for every $\mathbf{x}$. Since $V(A)$ is finite, g is integrable over A. The corollary is then a special case of the dominated convergence theorem for integrals over A. □

Corollary 5. (a) *Let $A_1, A_2, \ldots$ be a nondecreasing sequence of measurable sets. Let f be integrable over $A = A_1 \cup A_2 \cup \cdots$. Then*

$$\int_A f\,dV = \lim_{\nu\to\infty} \int_{A_\nu} f\,dV. \tag{5.55}$$

(b) *Let $A_1, A_2, \ldots$ be a nonincreasing sequence of measurable sets, and let $A = A_1 \cap A_2 \cap \cdots$. Then* (5.55) *holds provided f is integrable over A_1.*

PROOF. To prove (a), let $f_\nu = f_{A_\nu}$, $\nu = 1, 2, \ldots$. Then $\lim_{\nu\to\infty} f_\nu(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in A$ and $|f_\nu(\mathbf{x})| \le |f(\mathbf{x})|$. The conclusion follows from the dominated convergence theorem, with $g = |f|$.

The proof of (b) is similar. □

Note. If $f \ge 0$ and the sequence $A_1, A_2, \ldots$ is nondecreasing, then we could have appealed instead to the monotone sequences theorem. In that case it is unnecessary to assume that f is integrable over A. This observation is useful in proving Corollary 6.

Corollary 6. *Let A be measurable, f be continuous on A, and $f \ge 0$. Then*

$$\int_A f\,dV = \sup\left\{\int_K f\,dV : K \subset A, K \text{ is compact}\right\}. \tag{5.56}$$

PROOF. Let s denote the right-hand side of (5.56). Since $\int_K f\,dV \le \int_A f\,dV$ whenever $K \subset A$, we must have $s \le \int_A f\,dV$. There is a sequence of compact sets $K_1 \subset K_2 \subset \cdots$ and a null set N such that $A = N \cup K_1 \cup K_2 \cup \cdots$ (see Problem 5, Section 5.10). Then $\int_{K_\nu} f\,dV \le s$ for each $\nu = 1, 2, \ldots$. Moreover, by (5.55) with $A_\nu = K_\nu$,

$$\int_A f\,dV = \int_{A-N} f\,dV = \lim_{\nu\to\infty} \int_{K_\nu} f\,dV.$$

Thus $\int_A f\,dV \le s$. Hence $\int_A f\,dV = s$. □

The right-hand side of (5.56) was taken in Section 5.6 as the definition. Corollary 6 shows that the definition there agrees with the one in the present section in case $f \ge 0$. Since the procedure for defining the integral when f also has negative values was the same in both sections [see (5.30) and (5.51)], the two definitions agree in general.

Iterated integrals

Formula (5.23) expresses $\int_A f\,dV$ as an iterated integral. In Theorem 5.6 we showed that (5.23) is correct if A is compact and f is continuous on A. Actually, (5.23) is correct if A is any measurable set and f integrable over A. This is called *Fubini's theorem.* Let us show that Fubini's theorem is correct if A is measurable and f is continuous on A. For the general case, when f is integrable but not necessarily continuous on A, we refer to the work by McShane and Botts [18, p. 143].

Consider first f continuous on A and $f \geq 0$. Let $K_1, K_2, \ldots$ and N be as in the proof of Corollary 6. Let $\tilde{A} = K_1 \cup K_2 \cup \cdots$; then $A = \tilde{A} \cup N$. Let

$$g_\nu(\mathbf{x}') = \int_{K_\nu(\mathbf{x}')} f(\mathbf{x}', \mathbf{x}'')dV_{n-s}(\mathbf{x}''), \qquad g(\mathbf{x}') = \int_{\tilde{A}(\mathbf{x}')} f(\mathbf{x}', \mathbf{x}'')dV_{n-s}(\mathbf{x}'')$$

where the sets $K_\nu(\mathbf{x}')$, $\tilde{A}(\mathbf{x}')$ are defined by (5.22) with A replaced by K_ν, $\tilde{A}$. By Theorem 5.6,

$$\int_{K_\nu} f(\mathbf{x})dV_n(\mathbf{x}) = \int_R g_\nu(\mathbf{x}')dV_s(\mathbf{x}').$$

Since $K_1(\mathbf{x}'), K_2(\mathbf{x}'), \ldots$ is a nondecreasing sequence of sets with union $\tilde{A}(\mathbf{x}')$, Corollary 5 implies that $g_\nu(\mathbf{x}') \to g(\mathbf{x}')$ as $\nu \to \infty$ for each $\mathbf{x}'$. Since $f \geq 0$, $g_1 \leq g_2 \leq \cdots$. The monotone sequences theorem and Corollary 5 then imply

$$(*) \qquad \int_A f(\mathbf{x})dV_n(\mathbf{x}) = \int_R g(\mathbf{x}')dV_s(\mathbf{x}').$$

We leave it to the reader to show that $V_{n-s}[N(\mathbf{x}')] = 0$, for almost all $\mathbf{x}' \in R$ (Problem 11). Since $A(\mathbf{x}') = \tilde{A}(\mathbf{x}') \cup N(\mathbf{x}')$, we have for almost all $\mathbf{x}'$

$$g(\mathbf{x}') = \int_{A(\mathbf{x}')} f(\mathbf{x}', \mathbf{x}'')dV_{n-s}(\mathbf{x}'').$$

Then (*) becomes the desired formula (5.23). The proof shows that if either side of (*) diverges to $+\infty$, then so does the other side, provided $f \geq 0$.

If f is continuous and integrable on A, with both positive and negative values, we write $f = f^+ - f^-$ with f^+, f^- as above. Since both f^+ and f^- satisfy (5.23), f does also.

Note. If A is σ-compact (Problem 5, Section 5.10), then we can take $\tilde{A} = A$ and avoid mentioning the sets $N(\mathbf{x}')$ in the proof.

PROBLEMS

1. Let f be integrable, and $f(\mathbf{x}) = g(\mathbf{x})$ almost everywhere. Show that g is integrable, and that $\int f\,dV = \int g\,dV$. [*Hint*: This is known from Section 5.4 if f and g are bounded and have compact supports.]

2. Let $f(x, y) = (x^2 + y^2)^{-p/2}$ if $0 < x^2 + y^2 < 1$; and $f(x, y) = 0$ otherwise ($p > 0$). Find ${}_tf$ and $\int_t f\, dV_2$. What happens as $t \to \infty$?

3. Let f be measurable and $f \geq 0$. Show that if $\int f\, dV = 0$, then $f(\mathbf{x}) = 0$ almost everywhere.

4. Let $f_\nu(x) = \sin \nu\pi x$. Show that:
(a) $\liminf_{\nu\to\infty} f_\nu(x) = -1$ and $\limsup_{\nu\to\infty} f_\nu(x) = 1$ whenever x is irrational. [*Hint*: If x is irrational, then every arc of the circle $s^2 + t^2 = 1$ contains $(\cos \nu\pi x, \sin \nu\pi x)$ for infinitely many ν.]
(b) $\int_0^1 (\liminf_{\nu\to\infty} f_\nu) dx = -1$, $\liminf_{\nu\to\infty} \int_0^1 f_\nu\, dx = 0$.

5. Let $f_\nu(x) = \nu$ if $x \in (0, \nu^{-1})$ and $f_\nu(x) = 0$ otherwise, $\nu = 1, 2, \ldots$. Show that $\lim_{\nu\to\infty} f_\nu(x) = 0$ for every x, but $\int f_\nu\, dx = 1$. Why does this not contradict Lebesgue's dominated convergence theorem?

6. Let $f_\nu(x) = \nu/(x^2 + \nu^2)$, $\nu = 1, 2, \ldots$. Show that $0 \leq f_\nu(x) \leq 1$, $\lim_{\nu\to\infty} f_\nu(x) = 0$ for every x, and $\int f_\nu\, dx = \pi$. Why does this not contradict the dominated convergence theorem?

7. Let $f_1, f_2, \ldots$ be functions on E^1, which satisfy the hypotheses of Lebesgue's dominated convergence theorem. Let $F_\nu(x) = \int_0^x f_\nu(t)dt$, $F(x) = \int_0^x f(t)dt$. Show that F_ν tends to F uniformly on E^1 as $\nu \to \infty$. [*Note*: As is customary notation in elementary calculus, $\int_0^x = \int_{[0,x]}$ if $x \geq 0$, and $\int_0^x = -\int_x^0$ if $x < 0$.]

8. (a) Let $f_1, f_2, \ldots$ be integrable over a measurable set A, and assume that $\sum_{k=1}^\infty \int_A |f_k|\, dV$ is finite. Let $G(\mathbf{x}) = \sum_{k=1}^\infty |f_k(\mathbf{x})|$. Since the terms of this series are nonnegative, it either converges or diverges to $+\infty$ for every $\mathbf{x} \in A$. Show that G is integrable over A and hence $G(\mathbf{x})$ is finite for almost every $\mathbf{x} \in A$. [*Hint*: Apply the monotone sequences theorem to the sequence $G_1, G_2, \ldots$, where $G_\nu = |(f_1)_A| + \cdots + |(f_\nu)_A|$.]
(b) By (a) the series $\sum_{k=1}^\infty f_k(\mathbf{x})$ converges absolutely for almost every $\mathbf{x} \in A$. Let $F(\mathbf{x})$ be the sum of the series. Show that $\int_A F\, dV = \sum_{k=1}^\infty \int_A f_k\, dV$. [*Hint*: Apply the dominated convergence theorem to the sequence $F_1, F_2, \ldots$, where $F_\nu = (f_1)_A + \cdots + (f_\nu)_A$.]

9. Let $f_1, f_2, \ldots$ be integrable over A, where A has finite measure. Assume that $|f_k(\mathbf{x})| \leq C_k$ for every $\mathbf{x} \in A$ and $k = 1, 2, \ldots$, and that the series $\sum_{k=1}^\infty C_k$ converges. Show that the series $\sum_{k=1}^\infty f_k(\mathbf{x})$ converges absolutely for almost every $\mathbf{x} \in A$; and if $F(\mathbf{x})$ is the sum of the series, then $\int_A F\, dV = \sum_{k=1}^\infty \int_A f_k\, dV$. [*Hint*: Use Problem 8.]

10. Suppose that $f_\nu \geq 0$ for $\nu = 1, 2, \ldots$ and that $\int f_\nu\, dV \to 0$ as $\nu \to \infty$. Show that $\liminf_{\nu\to\infty} f_\nu(\mathbf{x}) = 0$ almost everywhere. [*Hint*: Problem 3, Fatou's lemma.]

11. Write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ with $\mathbf{x}' \in E^s$, $\mathbf{x}'' \in E^{n-s}$. Suppose that $V_n(N) = 0$. Let $N(\mathbf{x}') = \{\mathbf{x}'' : (\mathbf{x}', \mathbf{x}'') \in N\}$. Show that $V_{n-s}[N(\mathbf{x}')] = 0$ except for $\mathbf{x}' \in N'$ where $V_s(N') = 0$. [*Hint*: First take N bounded. Let $G_1 \supset G_2 \supset \cdots$ be open sets such that $N \subset G_\nu$ for $\nu = 1, 2, \ldots$ and $V_n(G_\nu) \to 0$ as $\nu \to \infty$. Let $g_\nu(\mathbf{x}') = V_{n-s}[G_\nu(\mathbf{x}')]$. Then $V_n(G_\nu) = \int g_\nu\, dV_s$. Apply the monotone sequences theorem to $g_1, g_2, \ldots$.

5.12 Differentiation under the integral sign

Let A be a measurable subset of E^n, B be an open subset of E^l, and $A \times B = \{(\mathbf{x}, \mathbf{t}) : \mathbf{x} \in A, \mathbf{t} \in B\}$ be their cartesian product. In this section we are concerned with the validity of the formula

$$\frac{\partial}{\partial t^j} \int_A f(\mathbf{x}, \mathbf{t}) dV_n(\mathbf{x}) = \int_A \frac{\partial f}{\partial t^j}(\mathbf{x}, \mathbf{t}) dV(\mathbf{x}), \qquad j = 1, \ldots, l. \tag{5.57}$$

Lemma 1. *Let f be continuous on $A \times B$. Assume that there is a function g integrable over A such that $|f(\mathbf{x}, \mathbf{t})| \leq g(\mathbf{x})$ for every $\mathbf{x} \in A$, $\mathbf{t} \in B$. Let*

$$\phi(\mathbf{t}) = \int_A f(\mathbf{x}, \mathbf{t}) dV_n(\mathbf{x}), \qquad \mathbf{t} \in B. \tag{5.58}$$

Then ϕ is continuous on B.

PROOF. Let $\mathbf{t}_0$ be any point of B. If $\mathbf{t}_1, \mathbf{t}_2, \ldots$ is any sequence in B tending to $\mathbf{t}_0$, then since $f(\mathbf{x}, \ )$ is continuous at $\mathbf{t}_0$

$$f(\mathbf{x}, \mathbf{t}_0) = \lim_{m \to \infty} f(\mathbf{x}, \mathbf{t}_m) \quad \text{for every } \mathbf{x} \in A.$$

Since $|f(\mathbf{x}, \mathbf{t}_m)| \leq g(\mathbf{x})$, by Lebesgue's dominated convergence theorem

$$\int_A f(\mathbf{x}, \mathbf{t}_0) dV_n(\mathbf{x}) = \lim_{m \to \infty} \int_A f(\mathbf{x}, \mathbf{t}_m) dV_n(\mathbf{x}).$$

Thus $\phi(\mathbf{t}_m) \to \phi(\mathbf{t}_0)$ as $m \to \infty$. Since this is true for every sequence in B tending to $\mathbf{t}_0$, ϕ is continuous at $\mathbf{t}_0$. □

Lemma 2. *Let $l = 1$. Assume that f and $\partial f/\partial t$ are continuous on $A \times B$ and satisfy*

$$|f(\mathbf{x}, t)| \leq g(\mathbf{x}), \left|\frac{\partial f}{\partial t}(\mathbf{x}, t)\right| \leq h(\mathbf{x}) \quad \text{for every } \mathbf{x} \in A, t \in B,$$

where g and h are integrable over A. Then

$$\phi'(t) = \int_A \frac{\partial f}{\partial t}(\mathbf{x}, t) dV_n(\mathbf{x}), \quad t \in B. \tag{5.59}$$

PROOF. Let $t_0 \in B$, and let $\delta > 0$ be such that B contains the interval $(t_0 - \delta, t_0 + \delta)$. If $0 < u < \delta$, then $\partial f/\partial t$ is integrable over $A \times [t_0, t_0 + u]$. This set is measurable, and by the iterated integral theorem,

$$\int_{t_0}^{t_0+u} dt \int_A \frac{\partial f}{\partial t}(\mathbf{x}, t) dV_n(\mathbf{x}) = \int_A \left\{ \int_{t_0}^{t_0+u} \frac{\partial f}{\partial t}(\mathbf{x}, t) dt \right\} dV_n(\mathbf{x}). \tag{*}$$

By the fundamental theorem of calculus the inner integral on the right-hand side is $f(\mathbf{x}, t_0 + u) - f(\mathbf{x}, t_0)$. If we let $\psi(t)$ denote the right-hand side of

(5.59), then (*) becomes

$$(**) \qquad \int_{t_0}^{t_0+u} \psi(t)dt = \phi(t_0 + u) - \phi(t_0).$$

Similarly (**) is true when $-\delta < u < 0$. Lemma 1 implies that ψ is continuous; hence by the fundamental theorem of calculus, $\psi(t_0) = \phi'(t_0)$. □

If, instead of an open set, B is a closed interval $[a, b]$, then the proof of Lemma 2 shows that (5.59) is still true provided $\phi'(t)$ means the one-sided derivative at the endpoints.

Theorem 5.9. *Let A be measurable and B open. Let f and $\partial f/\partial t^j$, $j = 1, \ldots, l$, be continuous on $A \times B$ and satisfy*

$$|f(\mathbf{x}, \mathbf{t})| \leq g(\mathbf{x}), \left|\frac{\partial f}{\partial t^j}(\mathbf{x}, \mathbf{t})\right| \leq h_j(\mathbf{x}) \quad \textit{for every } \mathbf{x} \in A, \mathbf{t} \in B,$$

where g and $h_1, \ldots, h_l$ are integrable over A. Then the function ϕ in (5.58) *is of class $C^{(1)}$ on B and its partial derivatives are given by* (5.57).

PROOF. Applying Lemma 2 with $t^1, \ldots, t^{j-1}, t^{j-1}, \ldots, t^l$ fixed, we get

$$\frac{\partial \phi}{\partial t^j}(\mathbf{t}) = \int_A \frac{\partial f}{\partial t^j}(\mathbf{x}, \mathbf{t})dV_n(\mathbf{x}), \quad j = 1, \ldots, l.$$

Applying Lemma 1 to the function $\partial f/\partial t^j$, we find that $\partial \phi/\partial t^j$ is continuous on B for each j. Hence ϕ is of class $C^{(1)}$. □

Corollary. *The conclusion of the theorem holds if A is compact, B is open, and f together with $\partial f/\partial t^j$, $j = 1, \ldots, l$, are continuous on $A \times B$.*

PROOF. Let U be any neighborhood whose closure is contained in B. Since $A \times \text{cl } U$ is compact, $|f(\mathbf{x}, \mathbf{t})|$ and

$$\left|\frac{\partial f}{\partial t^j}(\mathbf{x}, \mathbf{t})\right|, \quad j = 1, \ldots, l,$$

are bounded on $A \times U$ by some number C. Let $g = h_j = C$. By the theorem with B replaced by U, ϕ is of class $C^{(1)}$ on U and (5.57) holds there. Since this is true for every such U, ϕ is of class $C^{(1)}$ on B and (5.57) holds for every $\mathbf{t} \in B$. □

EXAMPLE 1. Let $\phi(t) = \int_1^\infty x^{-1} \exp(-xt)dx$, $t > 0$. Find $\phi'(t)$. Using (5.59),

$$\phi'(t) = -\int_1^\infty \exp(-xt)dx = -t^{-1} \exp(-t).$$

If B is an interval (a, ∞), $a > 0$, the hypotheses of Lemma 2 are satisfied with $g(x) = h(x) = \exp(-ax)$. The formula for $\phi'(t)$ is correct for all t in any such interval, and hence for every $t > 0$.

Problems

1. Find $\phi'(t)$ if $\phi(t)$ is:
   (a) $\int_0^1 \log(x^2 + t^2)dx$, $t \neq 0$. (b) $\int_0^\pi x^{-1} \exp(xt)\sin x\, dx$.

2. Let $\phi(t) = \int_0^1 \log(2 - x^2t^2)dx$. Show that $\phi'(0) = 0$ and that ϕ is concave on the interval $(-1, 1)$.

3. For $x \in E^1$ let $\phi(x) = \int_0^\infty \exp(-t^2)\cos xt\, dt$. Show that $\phi'(x) = -\frac{1}{2}x\phi(x)$ and find $\phi(x)$.

4. (*Leibniz's rule*). Show that

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} f(x, t)dx\right] = f[b(t), t]b'(t) - f[a(t), t]a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x, t)dx,$$

   provided f and $\partial f/\partial t$ are continuous on $[a_0, b_0] \times B$ where B is open, that $a_0 \leq a(t) \leq b_0$ and $a_0 \leq b(t) \leq b_0$ for every $t \in B$, and that the functions a, b are of class $C^{(1)}$. [*Hint*: Let $G(x, t) = \int_{a_0}^x f(s, t)ds$, so that $\partial G/\partial x = f$. Calculate the derivative of $G[b(t), t] - G[a(t), t]$.]

5. Let $\phi(t) = \int_{\log t}^t x^{-1} \exp(x^2t^2)dx$, $t > 1$. Find $\phi'(t)$.

6. Let $\phi(x, t) = \int_{x-ct}^{x+ct} f(s)ds$, where $c > 0$ and f is of class $C^{(1)}$ on E^1. Show that $(\partial^2\phi/\partial t^2) = c^2(\partial^2\phi/\partial x^2)$.

7. Let $\phi(x, y) = \int_0^y dt \int_0^x f(s, t)ds$, $x > 0$, $y > 0$. Show that if f is continuous on $\{(x, y): x \geq 0, y \geq 0\}$, then $(\partial^2\phi/\partial x\, \partial y) = f$.

8. Let

$$\phi(x) = \int_0^\infty \exp\left(-t^2 - \frac{x^2}{t^2}\right)dt, \text{ for } x \in E^1.$$

   (a) Show that $\phi(x) = \frac{1}{2}\sqrt{\pi}\exp(-2|x|)$ for all x. [*Hint*: For $x > 0$, show that $\phi'(x) = -2\phi(x)$ using the substitution $s = 1/t$.]
   (b) Note that application of (5.59) gives a false result at $x = 0$. Why is this not surprising?

9. Let $\phi(t) = \int_0^\infty f(x, t)dx$, for $t > 0$, where $f(x, t) = \exp(-xt)x^{-1}\sin x$. Show that:
   (a) $\phi'(t) = -1/(1 + t^2)$.
   (b) $\phi(t) \to 0$ as $t \to +\infty$. [*Hint*: Apply the dominated convergence theorem with $f_\nu(x) = f(x, t_\nu)$ where $1 \leq t_1 \leq t_2 \leq \cdots$ and $t_\nu \to +\infty$ as $\nu \to \infty$.]
   (c) $\lim_{r\to\infty} \int_0^r \frac{\sin x}{x}dx = \frac{\pi}{2}$. (d) $\int_0^\infty \left|\frac{\sin x}{x}\right|dx = +\infty$.

   [*Hints*: From (a) and (b), $\phi(t) = \pi/2 - \tan^{-1} t$.] Integrate by parts to show that

$$\left|\int_r^\infty f(x, t)dx\right| \leq \frac{c}{r} \text{ for suitable } c.$$

   By the dominated convergence theorem,

$$\lim_{t\to 0^+} \int_0^r f(x, t)dx = \int_0^r \frac{\sin x}{x}dx.$$

5.13 L^p-Spaces

The concept of normed vector space is defined in Section 2.9. The spaces $\mathscr{C}(S)$ of bounded, continuous functions in Section 2.10 furnish one interesting class of infinite dimensional normed vector spaces. We now consider another class, the spaces $L^p(A)$, which are defined as follows.

Definition. Let A be a measurable set of positive measure, and let p be a number such that $p \geq 1$. The collection of all measurable real valued functions f with domain A such that $|f|^p$ is integrable over A is denoted by $L^p(A)$. For instance, $L^1(A)$ is just the collection of all real valued functions integrable over A.

We regard two functions f and g as defining the same element of $L^p(A)$ if $f(\mathbf{x}) = g(\mathbf{x})$ for almost all $\mathbf{x} \in A$.

Since $p \geq 1$, by Problem 3, Section 2.11,

$$|f(\mathbf{x}) + g(\mathbf{x})|^p \leq 2^{p-1}(|f(\mathbf{x})|^p + |g(\mathbf{x})|^p).$$

Hence if $|f|^p$ and $|g|^p$ are integrable over A, then $|f + g|^p$ is also integrable over A. This shows that the sum of two elements in $L^p(A)$ is also in $L^p(A)$. Clearly, $cf \in L^p(A)$ if $f \in L^p(A)$ and c is a real number. Thus $L^p(A)$ is a vector space over the real number field. The *p-norm* of a function $f \in L^p(A)$ is the number

$$\|f\|_p = \left(\int_A |f|^p \, dV_n\right)^{1/p}. \tag{5.60}$$

The p-norm has the following three properties:

(a) $\|f\|_p = 0$ if and only if $f(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in A$.
(b) $\|cf\|_p = |c| \|f\|_p$ for every real c.
(c) $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

The proofs of (a) and (b) are left to the reader (Problem 4). Property (c) is called *Minkowski's inequality* (let us defer the proof to later in this section). A finite dimensional version of Minkowski's inequality is given in Problem 4, Section 2.11. Properties (a), (b), and (c) state that $\| \ \|_p$ is a norm on the space $L^p(A)$. (Recall that any function f such that $f(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in A$ is considered as equivalent to the zero function.) In any normed vector space $\mathscr{V}$, the norm defines a distance (Section 2.9). The distance between two functions $f, g \in L^p(A)$, taken in this sense, is $\|f - g\|_p$. It is called the distance *in mean of order p*. The concepts of convergent and Cauchy sequences are defined in any metric space (Section 2.9). In $L^p(A)$ these concepts become the following. A sequence $f_1, f_2, \ldots$ converges to f in mean of order p if $\|f_\nu - f\|_p \to 0$ as $\nu \to \infty$. If for every $\varepsilon > 0$ there exists N such that $\|f_\nu - f_l\|_p < \varepsilon$ for every $l, \nu \geq N$, then the sequence $f_1, f_2, \ldots$ is *Cauchy* in mean of order p.

Theorem 5.10. *Every Cauchy sequence $f_1, f_2, \ldots$ in $L^p(A)$ converges in mean of order p to a limit $f \in L^p(A)$.*

This theorem is one of the remarkable features of the Lebesgue theory. It states that $L^p(A)$ is a complete normed vector space (i.e., a Banach space.)

PROOF. Let $f_1, f_2, \ldots$ be a Cauchy sequence in $L^p(A)$. There is an increasing sequence of positive integers $N_1, N_2, \ldots$ such that $\|f_\nu - f_l\|_p < 2^{-k-k/p}$ for every $\nu, l \geq N_k$. Let $g_k = f_{N_k}$, and let

$$(*) \qquad F(\mathbf{x}) = \sum_{k=1}^{\infty} 2^{kp} |g_k(\mathbf{x}) - g_{k+1}(\mathbf{x})|^p.$$

Since its terms are nonnegative, this series either converges or diverges to $+\infty$ for every $\mathbf{x} \in A$. But

$$\sum_{k=1}^{\infty} \int_A 2^{kp} |g_k - g_{k+1}|^p \, dV = \sum_{k=1}^{\infty} 2^{kp} (\|g_k - g_{k+1}\|_p)^p < \sum_{k=1}^{\infty} 2^{-k}.$$

Since the series on the right converges, so does the one on the left. Therefore, F is integrable over A, and $B = \{\mathbf{x} \in A : F(\mathbf{x}) = +\infty\}$ is a null set [Problem 8(a), Section 5.11]. Now each term of a nonnegative series is no more than the sum of the series. Applying this observation to (*), we find that for $\mathbf{x} \in A - B$

$$|g_k(\mathbf{x}) - g_{k+1}(\mathbf{x})| \leq 2^{-k} [F(\mathbf{x})]^{1/p}.$$

Therefore, for $s = 1, 2, \ldots$ and $\mathbf{x} \in A - B$,

$$|g_k(\mathbf{x}) - g_{k+s}(\mathbf{x})| \leq \sum_{r=0}^{s-1} |g_{k+r}(\mathbf{x}) - g_{k+r+1}(\mathbf{x})|$$

$$(**) \qquad \leq [F(\mathbf{x})]^{1/p} \sum_{r=0}^{s-1} 2^{-k+r} < 2^{1-k} [F(\mathbf{x})]^{1/p}.$$

Since the right-hand side approaches 0 as $k \to \infty$, the sequence of real numbers $g_1(\mathbf{x}), g_2(\mathbf{x})_1 \ldots$ is Cauchy. Let

$$f(\mathbf{x}) = \lim_{k \to \infty} g_k(\mathbf{x}), \quad \text{if } \mathbf{x} \in A - B,$$

and let $f(\mathbf{x}) = 0$ if $\mathbf{x} \in B$. Letting $s \to \infty$ in (**),

$$|g_k(\mathbf{x}) - f(\mathbf{x})| \leq 2^{1-k} [F(\mathbf{x})]^{1/p}$$

for $\mathbf{x} \in A - B$. Since B is a null set,

$$\int_A |g_k - f|^p \, dV \leq 2^{(1-k)p} \int_A F \, dV.$$

Since F is integrable over A, the right-hand side tends to 0. Therefore

$$\lim_{k \to \infty} \|g_k - f\|_p = 0.$$

If $v \geq N_k$, then $\|g_k - f_v\|_p < 2^{-k-k/p}$ (recall the definition of g_k). Using Minkowski's inequality

$$\|f_v - f\|_p \leq \|f_v - g_k\|_p + \|g_k - f\|_p.$$

Since the right-hand side tends to 0 as $k \to \infty$, this shows that

$$\lim_{v \to \infty} \|f_v - f\|_p = 0. \qquad \square$$

If $p > 1$, then the number p' such that

$$\frac{1}{p} + \frac{1}{p'} = 1$$

is called *conjugate* to p.

Theorem 5.11. *If $f \in L^p(A)$ and $g \in L^{p'}(A)$, then fg is integrable over A and*

(5.61) $$\|fg\|_1 \leq \|f\|_p \|g\|_{p'} \qquad \textit{(Hölder's inequality).}$$

PROOF. Let $\phi(t) = t^{1/p}$ for $t \geq 0$. Since $0 < 1/p < 1$, $\phi''(s) < 0$ for all $s > 0$ (this implies that ϕ is concave). Hence $\phi(t) \leq \phi(1) + \phi'(1)(t - 1)$, or

$$t^{1/p} \leq 1 + \frac{1}{p}(t - 1) = \frac{t}{p} + \frac{1}{p'}.$$

Setting $t = u^p v^{-p'}$, where $u \geq 0$, $v > 0$, we find since $1 - p' = -p'/p$ that

(5.62) $$uv \leq \frac{u^p}{p} + \frac{v^{p'}}{p'}.$$

Obviously, this inequality also holds when $v = 0$.

If $\|f\|_p = 0$, then $f(\mathbf{x}) = 0$ almost everywhere in A and both sides of Hölder's inequality are 0. Similarly both sides are 0 if $\|g\|_{p'} = 0$. Suppose that $\|f\|_p > 0$, $\|g\|_{p'} > 0$, and let

$$\tilde{f} = (1/\|f\|_p)f, \qquad \tilde{g} = (1/\|g\|_{p'})g.$$

Then

$$\int |\tilde{f}|^p \, dV = \int |\tilde{g}|^{p'} \, dV = 1,$$

and setting $u = |\tilde{f}(\mathbf{x})|$, $v = |\tilde{g}(\mathbf{x})|$ in (5.62)

$$\int_A |\tilde{f}(\mathbf{x})||\tilde{g}(\mathbf{x})| \, dV_n(\mathbf{x}) \leq \frac{1}{p} + \frac{1}{p'} = 1.$$

But the left-hand side is $\|fg\|_1 / \|f\|_p \|g\|_{p'}$. This proves Hölder's inequality. $\square$

Note. The finite dimensional version of Hölder's inequality (Problem 8, Section 4.8) expresses the fact that the p-norm on E^n and the p'-norm on the dual space $(E^n)^*$ are dually related. There is an infinite dimensional analog which we state without proof. Let $[L^p(A)]'$ denote the set of all real valued linear functions on $L^p(A)$ that are continuous. On $[L^p(A)]'$ there is defined a dual norm, just as in the finite dimensional case. Then $[L^p(A)]'$ is isomorphic with $L^{p'}(A)$, and the dual norm is just the p'-norm [18, p. 211] for $p > 1$.

PROOF OF MINKOWSKI'S INEQUALITY

$$(*) \qquad \int_A |f+g|^p\,dV \le \int_A |f||f+g|^{p-1}\,dV + \int_A |g||f+g|^{p-1}\,dV.$$

By Hölder's inequality,

$$\int_A |f||f+g|^{p-1}\,dV \le \left(\int_A |f|^p\,dV\right)^{1/p}\left(\int_A |f+g|^{(p-1)p'}\,dV\right)^{1/p'}.$$

But $(p-1)p' = p$. Estimating similarly the last term in (*), we get

$$\int_A |f+g|^p\,dV \le (\|f\|_p + \|g\|_p)\left(\int_A |f+g|^p\,dV\right)^{1/p'}.$$

If $\|f+g\|_p = 0$, both sides are 0. Otherwise we divide both sides by $(\int_A |f+g|^p\,dV)^{1/p'}$. Since $1 - 1/p' = 1/p$, we get Minkowski's inequality. □

If $p = 1$, then (formally) $p' = \infty$. Let us call f *essentially bounded* if f is equivalent to a bounded function g [equivalent means that $f(\mathbf{x}) = g(\mathbf{x})$ almost everywhere in A]. Let $L^\infty(A)$ be the collection of all essentially bounded measurable functions. For $f \in L^\infty(A)$ let

$$\|f\|_\infty = \text{ess sup}\{|f(\mathbf{x})| : \mathbf{x} \in A\},$$

where the right-hand side means $\inf\{(\sup\{|g(\mathbf{x})| : \mathbf{x} \in A\}) : g$ is equivalent to $f\}$. Then Hölder's inequality is still true when $p = 1$.

If $p = 2$, then $p' = 2$. In $L^2(A)$ let us introduce an inner product $\cdot$ as follows:

$$f \cdot g = \int_A fg\,dV_n.$$

Then $f \cdot f = (\|f\|_2)^2$. Moreover, $|f \cdot g| \le \|fg\|_1$ and hence

$$|f \cdot g| \le \|f\|_2 \|g\|_2. \tag{5.63}$$

This formula corresponds to Cauchy's inequality in E^n. In fact, the space $L^2(A)$ is an infinite dimensional analog of euclidean E^n.

An *inner product space* H is a vector space with an inner product $\cdot$ satisfying a list of axioms corresponding to those in Section 1.2, Problem 2. If H is infinite dimensional and complete, then H is a *Hilbert space*. The preceding theorem shows that $L^2(A)$ is a Hilbert space.

Problems

1. Let $f(\mathbf{x}) = |\mathbf{x}|^{-\alpha}$. Show that if A is the unit n-ball, then $f \in L^p(A)$ for $p < n/\alpha$ but not for $p \geq n/\alpha$. [*Hint*: See Section 5.6, Example 4.]

2. Let $f(x) = x^{-1}(\log x)^{-2}$, $A = (0, \frac{1}{2})$. Show that $f \in L^p(A)$ for $p = 1$, but not for $p > 1$.

3. Let $1 \leq q < p$. Using Hölder's inequality show that if A has finite measure, then $f \in L^p(A)$ implies that $f \in L^q(A)$. Give an example to show that it is necessary to assume A has finite measure. [*Hint*: Apply Hölder's inequality to the functions $|f|^q$ and 1 with p replaced by p/q.]

4. Prove properties (a) and (b) for the p-norm.

5. Let $A = [0, 1]$.
   (a) Describe a sequence $I_1, I_2, \ldots$ of closed intervals such that $V_1(I_\nu) \to 0$ as $\nu \to \infty$, but every $x \in [0, 1]$ belongs to infinitely many I_ν.
   (b) Let $f_\nu(x) = 1$ for $x \in I_\nu$, $f_\nu(x) = 0$ for $x \notin I_\nu$. Show that $\| f_\nu \|_p \to 0$ as $\nu \to \infty$, for $1 \leq p < \infty$, but $f_\nu(x)$ tends to a limit as $\nu \to \infty$ for no $x \in [0, 1]$.

Curves and line integrals

Consider a vector-valued function $\mathbf{g}$, from a 1-dimensional interval J into E^n with derivative $\mathbf{g}'(t) \neq \mathbf{0}$. One can think of the point $\mathbf{x} = \mathbf{g}(t)$ as traversing a curve while t traverses J from left to right. We do not call $\mathbf{g}$ itself a curve. Instead, any vector-valued function $\mathbf{f}$ obtained from $\mathbf{g}$ by a suitable parameter change is regarded as representing the same curve γ as $\mathbf{g}$. In Section 6.3 we define the concept of differential form ω of degree 1. Then the line integral of a differential form ω along a curve γ is defined in Section 6.4. The differential df of a function f is called an *exact differential form of degree 1*. It is shown that the line integral of ω depends just on the endpoints of γ if and only if ω is exact (Theorem 6.1).

This chapter may be read independently of Chapters 4 and 5, except for the chain rule (Section 4.4). Section 6.1 repeats some material included in Chapter 4.

6.1 Derivatives

Let $\mathbf{g}$ be a function from a set $J \subset E^1$ into E^n. Let t be an interior point of J. Then the *derivative* of $\mathbf{g}$ at t is the vector

$$\mathbf{g}'(t) = \lim_{u \to 0} \frac{1}{u} [\mathbf{g}(t + u) - \mathbf{g}(t)], \tag{6.1}$$

provided the limit exists.

The derivative of a vector-valued function has many of the same properties as in the case of real-valued functions. If $\mathbf{f}$ and $\mathbf{g}$ both have a derivative at t, then

$$\begin{aligned} (\mathbf{f} + \mathbf{g})'(t) &= \mathbf{f}'(t) + \mathbf{g}'(t), \\ (\mathbf{f} \cdot \mathbf{g})'(t) &= \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t). \end{aligned} \tag{6.2}$$

Here $\mathbf{f} \cdot \mathbf{g}$ is the real-valued function whose value at each t is the inner product $\mathbf{f}(t) \cdot \mathbf{g}(t)$.

The derivative has a geometric interpretation as a tangent vector. Let us suppose that J is an interval. As t traverses J from left to right, the point $\mathbf{g}(t)$ traverses some curve in E^n. A precise definition of the term "curve" is given in Section 6.2.

Let us assume that t_0 is a point of J at which $\mathbf{g}'(t_0) \neq \mathbf{0}$. Let

$$\mathbf{x}_0 = \mathbf{g}(t_0), \qquad \mathbf{v}_0 = \mathbf{g}'(t_0),$$
$$\mathbf{x} = \mathbf{g}(t), \qquad \mathbf{y} = \mathbf{x}_0 + u\mathbf{v}_0,$$

where $t = t_0 + u$ and $|u|$ is small enough that $t \in J$. The ratio of the distances $|\mathbf{x} - \mathbf{y}|$ and $|\mathbf{x} - \mathbf{x}_0|$ may be written, upon multiplying numerator and denominator by $1/|u|$, in the form

$$\frac{|\mathbf{x} - \mathbf{y}|}{|\mathbf{x} - \mathbf{x}_0|} = \left| \frac{1}{u}[\mathbf{g}(t) - \mathbf{g}(t_0)] - \mathbf{g}'(t_0) \right| \frac{1}{|(1/u)[\mathbf{g}(t) - \mathbf{g}(t_0)]|}.$$

Hence

$$\lim_{u \to 0} \frac{|\mathbf{x} - \mathbf{y}|}{|\mathbf{x} - \mathbf{x}_0|} = 0 \frac{1}{|\mathbf{g}'(t_0)|} = 0.$$

This justifies calling $\mathbf{v}_0$ a *tangent vector* at $\mathbf{x}_0$, and the line through $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{v}_0$ a *tangent line* at $\mathbf{x}_0$ (Figure 6.1). Note that we have used the assumption that $\mathbf{g}'(t_0) \neq \mathbf{0}$.

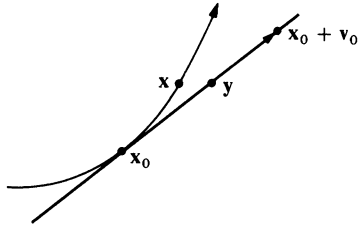


Figure 6.1

The number t is often called a *parameter*. It need not have any geometric or physical significance. However, if $n = 3$ and t happens to denote time in a physical problem, then $\mathbf{g}'(t)$ is the velocity vector.

A vector-valued function $\mathbf{g}$ has components $g^1, \ldots, g^n$, which are the real valued functions such that

$$\mathbf{g}(t) = \sum_{i=1}^{n} g^i(t)\mathbf{e}_i$$

for every $t \in J$. If $\mathbf{g}'(t)$ exists, then the ith component $u^{-1}[g^i(t + u) - g^i(t)]$ of the expression on the right side of (6.1) tends to $g^{i\prime}(t)$ as $u \to 0$, by Proposition 2.2, and

$$\mathbf{g}'(t) = \sum_{i=1}^{n} g^{i\prime}(t)\mathbf{e}_i. \tag{6.3}$$

Conversely, if $g^{i\prime}(t)$ exists for each $i = 1, \ldots, n$, then $\mathbf{g}'(t)$ exists and is given by (6.3).

EXAMPLE 1. Let $n = 2$, $\mathbf{g}(t) = t^2\mathbf{e}_1 + (\log t)\mathbf{e}_2$. Find the tangent line at $\mathbf{e}_1$. In this example $g^1(t) = t^2$, $g^2(t) = \log t$, and $t_0 = 1$, $\mathbf{x}_0 = \mathbf{g}(1) = \mathbf{e}_1$. Then $\mathbf{g}'(t) = 2t\mathbf{e}_1 + t^{-1}\mathbf{e}_2$, and $\mathbf{v}_0 = \mathbf{g}'(1) = 2\mathbf{e}_1 + \mathbf{e}_2$. The tangent line goes through $\mathbf{e}_1$ and $3\mathbf{e}_1 + \mathbf{e}_2$. Its equation is $2y = x - 1$.

PROBLEMS

1. Find the tangent line at $2^{-1/2}\mathbf{e}_1 - 2^{1/2}\mathbf{e}_2$ to the ellipse represented by $\mathbf{g}(t) = (\cos t)\mathbf{e}_1 + (2 \sin t)\mathbf{e}_2$, $J = [0, 2\pi]$. Illustrate with a sketch.

2. Find the tangent line at $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ to the curve represented by

$$\mathbf{g}(t) = t\mathbf{e}_1 + t^{1/2}\mathbf{e}_2 + t^{1/3}\mathbf{e}_3, \tfrac{1}{2} \leq t \leq 2.$$

3. A particle moves along the parabola $y^2 = 4x$ with constant speed 2 and so that $dy/dt = g^{2\prime}(t) > 0$. Find the velocity vector $\mathbf{g}'(t)$ at $\mathbf{e}_1 - 2\mathbf{e}_2$. [*Note*: The speed is $|\mathbf{g}'(t)|$.]

4. Give a proof of Formulas (6.2):
   (a) Using the corresponding formulas for derivatives of real valued functions and (6.3).
   (b) Directly from the definition (6.1).

5. Let $\mathbf{g}(t) = [3t/(1 + t^3)]\mathbf{e}_1 + [3t^2/(1 + t^3)]\mathbf{e}_2$, $t \neq -1$.
   (a) Sketch the curve traversed by $\mathbf{g}(t)$ on the interval $(-\infty, -1)$. On the interval $(-1, \infty)$.
   (b) Show that $\{\mathbf{g}(t) : t \neq -1\} = \{(x, y) : x^3 + y^3 = 3xy\}$. This set is called the *folium of Descartes*.

6.2 Curves in E^n

Let $\mathbf{g}$ be a function from an interval $J \subset E^1$ into E^n. Then $\mathbf{g}(t)$ traverses a curve in E^n as the "parameter" t traverses J. It is better not to call $\mathbf{g}$ itself a curve. Instead one should regard any vector-valued function $\mathbf{f}$ obtained from $\mathbf{g}$ by a suitable change of parameter as representing the same curve as $\mathbf{g}$. We define a curve as an equivalence class of equivalent parametric representations. To simplify matters we first consider only curves with continuously changing tangents.

Let us now be more precise. Let us for simplicity assume that $J = [a, b]$, a closed bounded interval, and that the components $g^1, \ldots, g^n$ are of class $C^{(1)}$ on $[a, b]$. By $g^{i\prime}(a)$ and $g^{i\prime}(b)$ we mean, respectively, right-hand and left-hand derivatives. They are equal to the derivatives at a and b of any class $C^{(1)}$ extension of g^i to an open set containing $[a, b]$.

Definition. If $\mathbf{g}'(t) \neq \mathbf{0}$ for every $t \in [a, b]$, then $\mathbf{g}$ is a *parametric representation of class* $C^{(1)}$ on $[a, b]$.

To motivate the definition of equivalence that we are going to make, let us first consider an example.

EXAMPLE 1. Let $\mathbf{g}(t) = t\mathbf{e}_1 + t^2\mathbf{e}_2$, $1 \leq t \leq 2$. Then $\mathbf{g}^1(t) = t$, $g^2(t) = t^2$, $\mathbf{g}'(t) = \mathbf{e}_1 + 2t\mathbf{e}_2 \neq \mathbf{0}$. Hence $\mathbf{g}$ is a parametric representation of class $C^{(1)}$ on the interval $[1, 2]$. In fact, it represents the arc of the parabola $y = x^2$ between $(1, 1)$ and $(2, 4)$, traversed from left to right (Figure 6.2). If we let $\mathbf{f}(\tau) = (\exp \tau)\mathbf{e}_1 + (\exp 2\tau)\mathbf{e}_2$, $0 \leq \tau \leq \log 2$, then $\mathbf{f}$ also represents this same parabolic arc. In effect, $\mathbf{f}$ is obtained from $\mathbf{g}$ by the parameter change $t = \exp \tau$. It is reasonable to regard $\mathbf{f}$ and $\mathbf{g}$ as equivalent, and we do so.

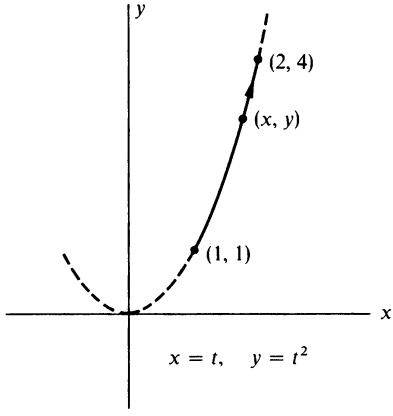


Figure 6.2

Now let $\mathbf{g}$ be any parametric representation of class $C^{(1)}$ on $[a, b]$. Let ϕ be any real-valued function of class $C^{(1)}$ on some closed interval $[\alpha, \beta]$ such that

$$\phi'(\tau) > 0 \quad \text{for every } \tau \in [\alpha, \beta], \qquad \phi(\alpha) = a, \qquad \phi(\beta) = b. \tag{6.4}$$

Let $\mathbf{f}$ be the composite of $\mathbf{g}$ and ϕ, denoted by $\mathbf{f} = \mathbf{g} \circ \phi$. Then

$$\mathbf{f}(\tau) = \mathbf{g}[\phi(\tau)] \quad \text{for every } \tau \in [\alpha, \beta].$$

From the composite function theorem

$$f^{i\prime}(\tau) = g^{i\prime}[\phi(\tau)]\phi'(\tau), \quad \text{for } i = 1, \ldots, n,$$

which is the same as

$$\mathbf{f}'(\tau) = \mathbf{g}'[\phi(\tau)]\phi'(\tau) \tag{6.5}$$

In particular, $\mathbf{f}'(\tau) \neq \mathbf{0}$ and $\mathbf{f}$ is also a parametric representation of class $C^{(1)}$. The tangent vector $\mathbf{f}'(\tau)$ differs from the tangent vector $\mathbf{g}'[\phi(\tau)]$ by the positive scalar multiple $\phi'(\tau)$. (Scalar multiplication on the right means the same thing as on the left, $\mathbf{v}c = c\mathbf{v}$.)

Definition. We say that $\mathbf{f}$ is *equivalent* to $\mathbf{g}$ if there exists ϕ satisfying the above conditions such that $\mathbf{f} = \mathbf{g} \circ \phi$.

The properties of reflexivity, symmetry, and transitivity required of an equivalence relation hold (Problem 6). By an *equivalence class* is meant the collection of all parametric representations of class $C^{(1)}$ equivalent to a given one. The reader may have encountered the notion of equivalence class elsewhere in mathematics. An example is the definition of the rational numbers starting from the integers.

Definition. A *curve* γ *of class* $C^{(1)}$ is an equivalence class of parametric representations of class $C^{(1)}$.

By requiring that the components $g^1, \ldots, g^n$ be of class $C^{(q)}$, $q \geq 2$, and allowing only parameter changes ϕ of class $C^{(q)}$, the notion of curve of class $C^{(q)}$ can be defined in the same way. To study curvature of curves one needs to assume class $C^{(2)}$ at least [25]. However, for present purposes we need only class $C^{(1)}$. From now on we say "curve" instead of "curve of class $C^{(1)}$," and "representation" instead of "parametric representation of class $C^{(1)}$."

Each curve has an infinite number of representations. If $\mathbf{g}$ is one such representation, then each parameter change ϕ leads to another. It is often highly advantageous to make a judicious choice of parameter. In a physical problem, time (measured according to some preassigned scale) may be the preferred parameter. For certain curves one of the components $x^1, \ldots, x^n$ can be taken as the parameter. For example, if the first component $g^{1\prime}(t)$ of the tangent vector $\mathbf{g}'(t)$ is everywhere positive, then g^1 has an inverse (Section A.4). Let us take for ϕ the inverse of g^1. Formally this amounts simply to solving the equation $x^1 = g^1(t)$ for t, obtaining $t = \phi(x^1)$. Set $\tau = x^1$. Then x^1 is the new parameter and $f^1(x^1) = x^1$. Figure 6.3 illustrates this situation for $n = 2$.

A curve γ is to be regarded as the path traversed by a moving point, and we have not excluded the possibility that γ passes through the same point several times. The *multiplicity* of a point $\mathbf{x}$ is the number of points $t \in [a, b]$ such that $\mathbf{g}(t) = \mathbf{x}$. The multiplicity does not depend on the particular representation $\mathbf{g}$ chosen for γ, since any ϕ satisfying (6.4) is a univalent

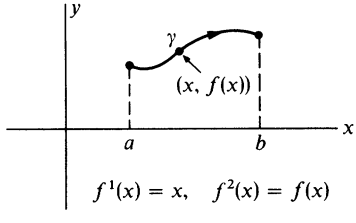
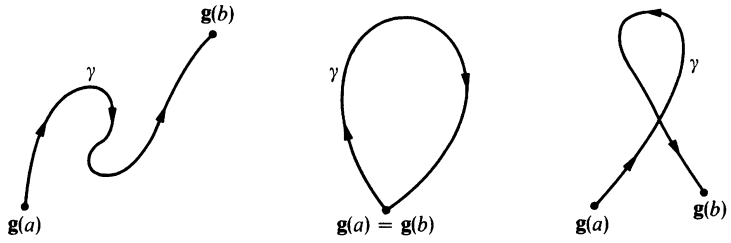


Figure 6.3



Simple arc | Simple closed curve | Not simple

Figure 6.4

function, namely, $\phi(\tau_1) \neq \phi(\tau_2)$ if $\tau_1 \neq \tau_2$. The *trace* of γ is the set of points of positive multiplicity, that is, the set of points through which γ passes at least once. If $\mathbf{x}$ has multiplicity 1, then $\mathbf{x}$ is called a *simple point*. If every point of the trace is simple, then γ is called a *simple arc*.

The point $\mathbf{g}(a)$ is called the *initial endpoint* of γ and $\mathbf{g}(b)$ the *final endpoint*. If $\mathbf{g}(a) = \mathbf{g}(b)$, then γ is called a *closed curve*. A closed curve is called *simple* if every point of the trace is simple except $\mathbf{g}(a)$, which has multiplicity 2 (Figure 6.4).

EXAMPLE 2. Let $\mathbf{g}(t) = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0), 0 \leq t \leq 1$. Then γ is the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_0$, traversed from $\mathbf{x}_0$ to $\mathbf{x}_1$. It is a simple arc.

EXAMPLE 3. Let $\mathbf{g}(t) = (\cos mt)\mathbf{e}_1 + (\sin mt)\mathbf{e}_2$, $0 \leq t \leq 2\pi$, where m is an integer not 0. The trace is the unit circle $x^2 + y^2 = 1$. The closed curve γ that $\mathbf{g}$ represents goes around the circle $|m|$ times, counterclockwise if $m > 0$ and clockwise if $m < 0$. If $m = \pm 1$, then γ is a simple closed curve.

At this point we need some properties of integrals, which are reviewed in Section A.3. In the present chapter we employ the Riemann definition of integral, as is customary in calculus. The more sophisticated Lebesgue theory of integrals was developed in Chapter 5.

Definition. The *length* l of a curve γ is

$$l = \int_a^b |\mathbf{g}'(t)|\,dt. \tag{6.6}$$

If $\mathbf{f}$ is equivalent to $\mathbf{g}$, then

$$\int_\alpha^\beta |\mathbf{f}'(\tau)|\,d\tau = \int_\alpha^\beta |\mathbf{g}'[\phi(\tau)]|\,|\phi'(\tau)d\tau = \int_a^b |\mathbf{g}'(t)|\,dt,$$

by (6.5) and the theorem about change of variables in integrals (Section A.3). Thus l does not depend on the particular representation chosen for γ.

Formula (6.6) is suggested by considering inscribed polygons. Let $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$, and let $\mu = \max\{t_1 - t_0, t_2 - t_1, \ldots, t_m - t_{m-1}\}$. The polygon which joins successively $\mathbf{g}(t_{j-1})$ with $\mathbf{g}(t_j)$ has elementary length

$$(*) \qquad \sum_{j=1}^{m} |\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})|.$$

The length l is the limit of the elementary lengths of polygons inscribed in γ. More precisely, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|(*) - l| < \varepsilon$ whenever $\mu < \delta$. Since we mention this fact just to motivate the definition (6.6), the proof will only be indicated. Since the derivative $\mathbf{g}'$ is continuous, $\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})$ can be replaced by $\mathbf{g}'(s_j)(t_j - t_{j-1})$ and the sum (*) by

$$(**) \qquad \sum_{j=1}^{m} |\mathbf{g}'(s_j)|(t_j - t_{j-1}),$$

with error tending to 0 as $\mu \to 0$, where s_j can be chosen arbitrarily in $[t_{j-1}, t_j]$. But (**) is a Riemann sum for the integral (6.6), and tends to l as $\mu \to 0$. The proof that (*) can be replaced by (**) with small error makes use of the fact that the continuous function $\mathbf{g}'$ is uniformly continuous on the compact set $[a, b]$ (Section 2.5, Problem 8).

Every smooth curve γ has a representation of particular geometric interest. It is called the standard representation, or representation with arc length s as parameter, and is defined in the following way. Let $\mathbf{g}$ represent γ on $[a, b]$, and let

$$S(t) = \int_a^t |\mathbf{g}'(u)|\,du \quad \text{for every } t \in [a, b].$$

The length of the part of γ represented on $[a, t]$ is $S(t)$. Clearly, $S(a) = 0$ and $S(b) = l$. By the fundamental theorem of calculus,

$$S'(t) = |\mathbf{g}'(t)| > 0$$

for every $t \in [a, b]$. In particular, if t signifies time then $S'(t)$ is the length of the velocity vector, that is, the speed of motion.

Since $S'(t) > 0$ the equation $s = S(t)$ can be solved for t. More precisely, the function S has an inverse ϕ of class $C^{(1)}$ on $[0, l]$. Let $\mathbf{G} = \mathbf{g} \circ \phi$. Then $\mathbf{G}$ is the *standard representation* of γ. From (6.5)

$$\mathbf{G}'(s) = \mathbf{g}'[\phi(s)]\phi'(s), \quad \text{for every } s \in [0, l].$$

Since

$$\phi'(s) = \frac{1}{S'[\phi(s)]} = \frac{1}{|\mathbf{g}'[\phi(s)]|},$$

we find that

$$|\mathbf{G}'(s)| = 1 \tag{6.7a}$$

for every $s \in [0, l]$. Hence $\mathbf{G}'(s)$ is a *unit tangent vector* at the point $\mathbf{G}(s)$. If we write dx^i/ds for $G^{i\prime}(s)$, then (6.7a) can be rewritten

$$\left(\frac{dx^1}{ds}\right)^2 + \cdots + \left(\frac{dx^n}{ds}\right)^2 = 1. \tag{6.7b}$$

EXAMPLE 3 (*continued*). Let $m > 0$. Then $|\mathbf{g}'(t)| = m$, $S(t) = mt$. Solving the equation $s = S(t)$ for t, we obtain the standard representation

$$\mathbf{G}(s) = (\cos s)\mathbf{e}_1 + (\sin s)\mathbf{e}_2, \qquad 0 \leq s \leq 2m\pi.$$

Piecewise smooth curves

It is not difficult to adapt the preceding discussion to curves of class $C^{(1)}$ except for a finite number of corners and cusps. By a parametric representation of a piecewise smooth curve is meant a continuous function $\mathbf{g}$ on an interval $[a, b]$ with the following property: There exist $t_0, t_1, \ldots, t_p$ with

$$a = t_0 < t_1 < \cdots < t_{p-1} < t_p = b$$

such that the restriction of $\mathbf{g}$ to each of the closed subintervals $[t_{j-1}, t_j]$, $j = 1, \ldots, p$, is a parametric representation of class $C^{(1)}$. In particular, $\mathbf{g}$ has at each t_j interior to $[a, b]$ right- and left-hand derivatives which need not be equal. Parameter changes which are piecewise of class $C^{(1)}$ are admitted. A piecewise smooth curve is an equivalence class of parametric representations which are piecewise of class $C^{(1)}$.

EXAMPLE 4. Let $n = 2$ and $\mathbf{g}(t) = t\mathbf{e}_1 + |t - 1|\mathbf{e}_2, 0 \leq t \leq 2$. This is a piecewise $C^{(1)}$ representation of the polygon from $\mathbf{e}_2$ to $\mathbf{e}_1$ to $2\mathbf{e}_1 + \mathbf{e}_2$ with a corner at $\mathbf{e}_1$. Let $\phi(\tau) = \tau^3 + 1$ and $\mathbf{f}(\tau) = \mathbf{g}(\tau^3 + 1) = (\tau^3 + 1)\mathbf{e}_1 + |\tau^3|\mathbf{e}_2$, $-1 \leq \tau \leq 1$. Then $f^1(\tau) = \tau^3 + 1$ and $f^2(\tau) = |\tau^3|$. The components f^1, f^2 are of class $C^{(1)}$, which might lead one to think that there is no corner. However, ϕ does not define an admissible parameter change, since $\phi'(0) = 0$ contrary to (6.4). Since $\mathbf{f}'(0) = \mathbf{0}$, $\mathbf{f}$ is not a parametric representation of class $C^{(1)}$. This example emphasizes the importance of the restriction $\phi'(\tau) > 0$ in (6.4).

PROBLEMS

1. Which of the following represent simple arcs? Simple closed curves? Illustrate with a sketch.
   (a) $\mathbf{g}(t) = (a \cos t)\mathbf{e}_1 + (b \sin t)\mathbf{e}_2, a > 0, b > 0, J = [0, 2\pi]$.
   (b) Same as (a) except $J = [-\pi, 2\pi]$.
   (c) $\mathbf{g}(t) = (-\cosh t)\mathbf{e}_1 + (\sinh t)\mathbf{e}_2$, $J = [-1, 1]$ (see Section 3.5 for the definition of cosh and sinh).

2. (a) Let γ be represented by $\mathbf{f}(x) = x\mathbf{e}_1 + f(x)\mathbf{e}_2$, $a \leq x \leq b$, where f is of class $C^{(1)}$ on $[a, b]$. Show that

$$l = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

   (b) Find l in case $f(x) = |x|^{3/2}$, $a = -b$.

3. Find the standard representation of the helical curve represented on $[0, 2\pi]$ by $\mathbf{g}(t) = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + t\mathbf{e}_3$. Sketch the trace.

4. Sketch the trace of the curve γ represented on $[0, 2\pi]$ by $\mathbf{g}(t) = (\cos t)\mathbf{e}_1 + (\sin 2t)\mathbf{e}_2$. Find the tangent vectors to γ at the double point $(0, 0)$.

5. Let $g^1(t) = \cos(1/t)\exp(-1/t)$, $g^2(t) = \sin(1/t)\exp(-1/t)$ if $0 < t \leq 1$, and $g^1(0) = g^2(0) = 0$.
   (a) Show that g^1 and g^2 are of class $C^{(1)}$ on $[0, 1]$. [*Hint*: $u^k \exp(-u) \to 0$ as $u \to +\infty$.]
   (b) Does $\mathbf{g} = g^1\mathbf{e}_1 + g^2\mathbf{e}_2$ represent a curve of class $C^{(1)}$? Illustrate with a sketch.

6. Let us write $\mathbf{f} \sim \mathbf{g}$ to mean $\mathbf{f}$ is equivalent to $\mathbf{g}$. Prove that:
   (a) $\mathbf{g} \sim \mathbf{g}$ (reflexivity).
   (b) If $\mathbf{f} \sim \mathbf{g}$, then $\mathbf{g} \sim \mathbf{f}$ (symmetry).
   (c) If $\mathbf{g}_1 \sim \mathbf{g}_2$ and $\mathbf{g}_2 \sim \mathbf{g}_3$, then $\mathbf{g}_1 \sim \mathbf{g}_3$ (transitivity).

7. Let γ be a curve of class $C^{(1)}$. Prove that the multiplicity of any point $\mathbf{x}$ is finite.

8. Let γ_0 and γ_1 be curves represented on $[a, b]$ by $\mathbf{g}_0$ and $\mathbf{g}_1$, respectively. For every $u \in [0, 1]$ let γ_u be the curve represented by $\mathbf{g}_u(t) = u\mathbf{g}_1(t) + (1 - u)\mathbf{g}_0(t)$, $a \leq t \leq b$. Let $l(u)$ be the length of γ_u. Prove that l is a convex function on $[0, 1]$. When is the convexity strict?

9. Let $\mathbf{G}$ be the standard representation of a curve γ of class $C^{(2)}$.
   (a) Show that $\mathbf{G}'(s) \cdot \mathbf{G}''(s) = 0$. [*Hint*: Use the fact that $|\mathbf{G}'(s)|^2 = 1$.]
   (b) Let $\mathbf{g}$ be any parametric representation of class $C^{(2)}$ of γ and define $S(t)$ as in this section. Show that $S''(t) = \mathbf{g}'(t) \cdot \mathbf{g}''(t)/S'(t)$.
   (c) If $\mathbf{G}''(s) \neq \mathbf{0}$, then $\mathbf{G}''(s)$ is called the *principal normal vector* and $|\mathbf{G}''(s)|$ the *absolute curvature* at $\mathbf{G}(s)$. Show that

$$|\mathbf{G}''[S(t)]| = \frac{[|\mathbf{g}'(t)|^2 |\mathbf{g}''(t)|^2 - (\mathbf{g}'(t) \cdot \mathbf{g}''(t))^2]^{1/2}}{|\mathbf{g}'(t)|^3}.$$

6.3 Differential 1-forms

Let us first give a rough description of this notion and afterward be more precise. A differential form $\boldsymbol{\omega}$ of degree 1 is supposed to be an "expression linear in the differentials $dx^1, \ldots, dx^n$":

$$\boldsymbol{\omega} = \omega_1\, dx^1 + \cdots + \omega_n\, dx^n, \tag{6.8}$$

where the coefficients $\omega_1, \ldots, \omega_n$ are real valued functions. In case there is a real valued differentiable function f such that ω_i is the ith partial derivative f_i for each $i = 1, \ldots, n$, then $\boldsymbol{\omega}$ is called the differential of f and is written df.

Thus

$$df = f_1\,dx^1 + \cdots + f_n\,dx^n. \tag{6.9}$$

It is important to know whether a given differential form $\boldsymbol{\omega}$ is the differential of a function. A considerable part of the discussion in this section and Section 6.4 is directed to just this question. One gets a necessary condition (6.11) from the fact that the mixed partial derivatives f_{ij} and f_{ji} of a function f of class $C^{(2)}$ are equal. This necessary condition turns out to be sufficient if the domain is simply connected (Section 8.10).

We recall from Section 3.2 that the elements of the space $(E^n)^*$ dual to E^n are called *covectors*, and that the components a_i of a covector $\mathbf{a}$ are written with subscripts. No matter what precise meaning we give to the symbols $dx^1, \ldots, dx^n$, the functions $\omega_1, \ldots, \omega_n$ must determine the differential form $\boldsymbol{\omega}$. For each $\mathbf{x}$, the numbers $\omega_1(\mathbf{x}), \ldots, \omega_n(\mathbf{x})$ are the components of a covector. This suggests that we may define a differential form as a function whose values are covectors.

To state this precisely:

Definition. A *differential form of degree* 1 is a function $\boldsymbol{\omega}$ with domain $D \subset E^n$ and values in $(E^n)^*$.

For short we usually say "1-form" instead of "differential form of degree 1." In Chapter 7 differential forms of any degree $r = 0, 1, 2, \ldots, n$ are defined.

The value of $\boldsymbol{\omega}$ at $\mathbf{x}$ is denoted by $\boldsymbol{\omega}(\mathbf{x})$. It is the covector

$$\boldsymbol{\omega}(\mathbf{x}) = \omega_1(\mathbf{x})\mathbf{e}^1 + \cdots + \omega_n(\mathbf{x})\mathbf{e}^n, \tag{6.10}$$

where, as in Section 3.2, $\mathbf{e}^1, \ldots, \mathbf{e}^n$ are the standard basis covectors.

A 1-form $\boldsymbol{\omega}$ is a constant form if there is a covector $\mathbf{a}$ such that $\boldsymbol{\omega}(\mathbf{x}) = \mathbf{a}$ for every $\mathbf{x} \in D$. In particular, for each $i = 1, \ldots, n$ let us consider the constant 1-form with value $\mathbf{e}^i$. This 1-form is denoted by dx^i. Since $(E^n)^*$ is a vector space, the sum $\boldsymbol{\omega} + \boldsymbol{\zeta}$ of two functions $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$ with the same domain D and values in $(E^n)^*$ is defined (Section 2.1). Similarly, the product $f\boldsymbol{\omega}$ is defined if f is a real-valued function and $\boldsymbol{\omega}$ is a 1-form, with the same domain D. In particular, $\omega_i\,dx^i$ is the 1-form whose value at each $\mathbf{x}$ is $\omega_i(\mathbf{x})\mathbf{e}^i$. From (6.10), $\omega_1\,dx^1 + \cdots + \omega_n\,dx^n$ is the 1-form whose value at each $\mathbf{x}$ is $\boldsymbol{\omega}(\mathbf{x})$. Therefore Formula (6.8) is correct.

The differential of a function

Let us now suppose that D is an open set. Let f be a real-valued differentiable function with domain D. The differential of f at $\mathbf{x}$ is the covector $df(\mathbf{x})$ whose components are the partial derivatives $f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})$.

Definition. The *differential* of f is the differential form df of degree 1 whose value at each $\mathbf{x} \in D$ is the covector $df(\mathbf{x})$.

Some authors define df as the real valued function whose domain is the cartesian product $D \times E^n$ and whose value at each pair $(\mathbf{x}, \mathbf{h})$ is the number $df(\mathbf{x}) \cdot \mathbf{h}$. Knowing $df(\mathbf{x})$, one can find $df(\mathbf{x}) \cdot \mathbf{h}$ for every $\mathbf{h} \in E^n$, and vice versa. Hence this definition is equivalent to the one we have given.

If f and g are differentiable functions with the same domain D, then

$$d(f + g) = df + dg, \qquad d(fg) = f\,dg + g\,df.$$

These formulas follow from Problem 7, Section 3.3. Similarly, writing c for the constant function with value c,

$$dc = \mathbf{0}$$

where $\mathbf{0}$ denotes the "zero form" whose value is $\mathbf{0}$ everywhere. If D is connected, then, conversely, $df = \mathbf{0}$ implies that f is a constant function. This is just a restatement of Corollary 2, Section 3.3.

If L is a linear function, then dL is a constant 1-form. For let $L(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$, as in Proposition 3.1. Then the ith partial derivative of L is a_i and $dL(\mathbf{x}) = \mathbf{a}$ for every $\mathbf{x}$.

In particular, the standard cartesian coordinate functions $X^1, \ldots, X^n$ (Section 3.2) are linear. In fact $X^i(\mathbf{x}) = \mathbf{e}^i \cdot \mathbf{x} = x^i$, and $dX^i(\mathbf{x}) = \mathbf{e}^i$. Hence dX^i is just the constant 1-form which we have denoted by dx^i. The common practice of writing dx^i instead of dX^i arises from the habit of confusing notationally a function with its value at some particular point $\mathbf{x}$, in this case of confusing X^i with $x^i = X^i(\mathbf{x})$. Nevertheless, following custom, we adhere to the notation dx^i.

Definition. A 1-form $\boldsymbol{\omega}$ is *exact* if there is a function f such that $\boldsymbol{\omega} = df$.

If $df = dg$, then $d(f - g) = \mathbf{0}$. If D is connected, $f - g$ is a constant function. Hence the function f whose differential is a given exact 1-form $\boldsymbol{\omega}$ is determined up to the addition of a constant function, if the domain is connected.

A 1-form $\boldsymbol{\omega}$ is of *class* $C^{(q)}$ if its components ω_i are functions of class $C^{(q)}$. If $\boldsymbol{\omega} = df$, then $\omega_i = \partial f / \partial x^i$. In this case $\boldsymbol{\omega}$ is of class $C^{(q)}$ if and only if f is of class $C^{(q+1)}$.

Let us look for some criteria to determine whether a 1-form $\boldsymbol{\omega}$ is exact. If $\boldsymbol{\omega}$ is of class $C^{(1)}$ and $\boldsymbol{\omega} = df$, then f is of class $C^{(2)}$. Using Theorem 3.3 and the $\partial/\partial x^i$ notation for partial derivatives,

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial^2 f}{\partial x^j \, \partial x^i} = \frac{\partial^2 f}{\partial x^i \, \partial x^j} = \frac{\partial \omega_j}{\partial x^i}.$$

Thus the conditions

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}, \qquad i, j = 1, \ldots, n, \tag{6.11}$$

are necessary for exactness of $\boldsymbol{\omega}$.

Definition. A 1-form ω of class $C^{(1)}$ that satisfies (6.11) is called a *closed* 1-form.

In (6.11) we may as well suppose that $i < j$. Thus the definition says in effect that ω is a closed differential form if its components $\omega_1, \ldots, \omega_n$ satisfy these $n(n-1)/2$ conditions. For instance, if $n = 2$ let us write dx and dy instead of dx^1 and dx^2, and

$$M(x, y) = \omega_1(x, y), \qquad N(x, y) = \omega_2(x, y).$$

The expression for a 1-form is then

$$\omega = M\,dx + N\,dy.$$

The condition that ω be closed is that the components M and N satisfy

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

We have shown that every exact 1-form is closed. The converse is false, as Example 2 (below) shows. It is comparatively easy to check whether conditions (6.11) are satisfied. Therefore, it is very desirable to find some additional condition that will guarantee that the converse holds. Such a condition is that the domain D be simply connected, a term to be defined in Section 8.10. We prove in Chapter 8 that if D is simply connected, then every closed 1-form with domain D is exact. It turns out that any convex set, and in particular E^n, is simply connected. For $n = 2$, an open, connected set D is simply connected if and only if, roughly speaking, D has no holes.

We have been careful to distinguish notationally between functions and their values. One can scarcely attain a sound knowledge of calculus until this distinction is recognized. Nevertheless, in examples we sometimes abuse the notation for brevity. For instance, $d(x^2y) = 2xy\,dx + x^2\,dy$ is short for the statement "$df = f_1\,dx + f_2\,dy$, where $f(x, y) = x^2y$, $f_1(x, y) = 2xy$, $f_2(x, y) = x^2$ for every $(x, y) \in E^2$."

EXAMPLE 1. Let $\omega = 2xy\,dx + (x^2 + 2y)\,dy$, $D = E^2$. This is, of course, an abbreviation for $\omega = M\,dx + N\,dy$, where $M(x, y) = 2xy$, $N(x, y) = x^2 + 2y$, for every $(x, y) \in E^2$. In this example,

$$\frac{\partial M}{\partial y}(x, y) = 2x = \frac{\partial N}{\partial x}(x, y)$$

for every (x, y). Hence ω is a closed 1-form. Since E^2 is connected and simply connected, $\omega = df$ where f is determined up to the addition of a constant function. The function f can be found by partial integration with respect to the first variable, as follows:

$$\frac{\partial f}{\partial x}(x, y) = M(x, y) = 2xy, \qquad f(x, y) = x^2y + \phi(y),$$

where the function ϕ is determined from

$$\frac{\partial f}{\partial y}(x, y) = N(x, y) = x^2 + 2y, \qquad x^2 + 2y = x^2 + \phi'(y).$$

Of course these equations hold for every $(x, y) \in E^2$. Then $\phi'(y) = 2y$, and $\phi(y) = y^2 + c$ for every y, where the "constant of integration" c is a number that may be chosen arbitrarily. Hence for every $(x, y) \in E^2$,

$$f(x, y) = x^2y + y^2 + c.$$

EXAMPLE 2. Let $D = E^2 - \{(0, 0)\}$. By removing $(0, 0)$ we have made a hole, and D is not simply connected. Let $\omega = M\,dx + N\,dy$, where for every $(x, y) \in D$

$$M(x, y) = -\frac{y}{x^2 + y^2},$$

$$N(x, y) = \frac{x}{x^2 + y^2}.$$

A computation shows that $\partial M/\partial y = \partial N/\partial x$ in D, hence ω is closed. Let us show that ω is not exact. Let D_1 be the open subset of D obtained by deleting the positive x-axis. For every $(x, y) \in D_1$ let $\Theta(x, y)$ be the angle from the positive x-axis to (x, y), $0 < \Theta(x, y) < 2\pi$ (Figure 6.5). Using elementary calculus, we find that in D_1, $d\Theta = \omega$ [Problem 6(a)]. If there were a function f of class $C^{(2)}$ on D such that $\omega = df$, then upon restricting f to D_1 we would have $d(f - \Theta) = \mathbf{0}$. Since D_1 is connected, $f - \Theta$ would be constant on D_1. This would imply that Θ can be continuously extended across the positive x-axis, which is false. Hence ω is not exact.

EXAMPLE 3. In some cases it can be seen by inspection that ω is exact. For instance, if $\omega = 2x^1dx^1 + \cdots + 2x^n\,dx^n$ and $D = E^n$, then

$$\omega = d[(x^1)^2 + \cdots + (x^n)^2 + c] = d(\mathbf{x} \cdot \mathbf{x} + c).$$

The reader may have also discovered by inspection that the form ω in Example 1 is exact.

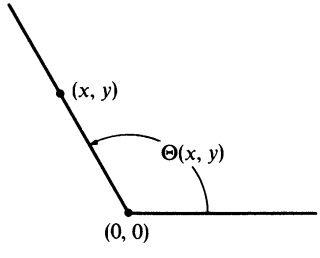


Figure 6.5

Problems

1. Let $n = 1$. Give a precise interpretation of the formula $df/dx = f'$ from elementary calculus. [*Hint*: The quotient of two real valued functions is defined wherever the denominator does not have the value 0.]

2. Let $n = 3$ and $\omega = M\,dx + N\,dy + O\,dz$. What do conditions (6.11) become in this case?

3. In each case determine whether ω is exact. If exact, find all functions f such that $\omega = df$.
   (a) $\omega = xy\,dx + (x^2/2)dy$, $D = E^2$.
   (b) $\omega = x\,dx + xz\,dy + xy\,dz$, $D = E^3$.
   (c) $\omega = y\,dx$, $D = E^2$.
   (d) $\omega = (1/x^2 + 1/y^2)(y\,dx - x\,dy)$, $D = \{(x, y) : x \neq 0 \text{ and } y \neq 0\}$.

4. Let $\omega = dy + p(x)y\,dx$ and D be the vertical strip $\{(x, y) : a < x < b\}$. Let p be continuous on (a, b) and P be an antiderivative of p, that is, a function such that $P'(x) = p(x)$ for every $x \in (a, b)$. Let $f(x) = \exp[P(x)]$. Show that $f\omega$ is exact.

5. Show that
   (a) $d[(\sum_{i=1}^n x^i)^2] = 2\sum_{i,j=1}^n x^i\,dx^j$.
   (b) $d[\sum_{i\neq j} x^i x^j] = 2\sum_{i=1}^n \sum_{k\neq i} x^k\,dx^i$.
   [*Hint for* (b): What is $(\sum x^i)^2 - \sum (x^i)^2$?]

6. In Example 2: (a) Show that $\Theta_1(x, y) = M(x, y), \Theta_2(x, y) = N(x, y)$ for every $(x, y) \in D_1$. You may use the formulas for the derivatives of the inverse trigonometric functions. (b) Verify that $\partial M/\partial y = \partial N/\partial x$ by calculating these partial derivatives.

7. Let g be continuous on E^1. Show that

$$g(|\mathbf{x}|) \sum_{i=1}^n x^i\,dx^i$$

   is an exact 1-form. [*Hint*: Let $h(u) = ug(u)$ for every $u \in E^1$. The function h has an antiderivative.]

6.4 Line integrals

Let D be an open subset of E^n. Let ω be a 1-form with domain D, and γ a curve whose trace is contained in D. We assume that ω is continuous. Let us consider an inscribed polygon joining successively the points $\mathbf{g}(t_{j-1})$ and $\mathbf{g}(t_j)$ as in Section 6.2. If $s_j \in [t_{j-1}, t_j]$, then $\omega[\mathbf{g}(s_j)]$ is a covector, and its scalar product with the vector $\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})$ is a number. Let us consider the sum

$$(*) \qquad \sum_{j=1}^m \omega[\mathbf{g}(s_j)] \cdot [\mathbf{g}(t_j) - \mathbf{g}(t_{j-1})].$$

By reasoning such as that indicated in Section 6.2, the sum (*) tends to the integral (6.12a) as $\mu \to 0$. This integral is called a *line integral.*

Definition. Let $\boldsymbol{\omega}$ be continuous and γ piecewise smooth. The line integral of $\boldsymbol{\omega}$ along γ is

$$\int_a^b \boldsymbol{\omega}[\mathbf{g}(t)] \cdot \mathbf{g}'(t)\, dt. \tag{6.12a}$$

The line integral exists, since if $\mathbf{g}$ is piecewise of class $C^{(1)}$ the integrand in (6.12a) is bounded and has a finite number of discontinuities. If $\mathbf{f}$ is equivalent to $\mathbf{g}$, then using (6.5)

$$\int_a^b \boldsymbol{\omega}[\mathbf{g}(t)] \cdot \mathbf{g}'(t)dt = \int_\alpha^\beta \boldsymbol{\omega}[\mathbf{g}(\phi(\tau))] \cdot \mathbf{g}'[\phi(\tau)]\phi'(\tau)d\tau = \int_\alpha^\beta \boldsymbol{\omega}[\mathbf{f}(\tau)] \cdot \mathbf{f}'(\tau)d\tau.$$

Hence the line integral does not depend on the particular representation $\mathbf{g}$ chosen for γ.

Line integrals have an important role in many parts of mathematical analysis, such as in the theory of complex analytic functions. Several fundamental physical concepts are also expressed in terms of line integrals. The concept of work is mentioned at the end of the present section. In Section 6.6 we discuss thermal systems, and in Section 8.5, circulation in a flowing fluid.

The notation for line integral is $\int_\gamma \boldsymbol{\omega}$. Writing out the scalar product in (6.12a),

$$\int_\gamma \boldsymbol{\omega} = \int_a^b \left[\sum_{i=1}^n \omega_i[\mathbf{g}(t)]g^{i\prime}(t) \right] dt. \tag{6.12b}$$

The notation for differential form is supposed to suggest (6.12b). Let us write $\boldsymbol{\omega} = \omega_1\, dx^1 + \cdots + \omega_n\, dx^n$ as in (6.8) and formally multiply and divide the right-hand side by dt. If we set $\mathbf{x} = \mathbf{g}(t)$ and write dx^i/dt for $g^{i\prime}(t)$, then we get the integrand on the right-hand side of (6.12b).

EXAMPLE 1. Let γ be the semicircle with center $(0, 0)$ and endpoints $\pm a\mathbf{e}_2$, directed from $-a\mathbf{e}_2$ to $a\mathbf{e}_2$. Let us evaluate $\int_\gamma x\, dy - y\, dx$. Points (x, y) of the semicircle satisfy the equations $x = a\cos\theta, y = a\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$. The most convenient representation for γ is on $[-\pi/2, \pi/2]$ with $\mathbf{g}(\theta) = (a\cos\theta)\mathbf{e}_1 + (a\sin\theta)\mathbf{e}_2$. Then

$$\int_\gamma (x\, dy - y\, dx) = \int_{-\pi/2}^{\pi/2} \left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta} \right) d\theta = \int_{-\pi/2}^{\pi/2} a^2\, d\theta = a^2\pi.$$

Elementary properties of line integrals

From the corresponding linearity property of ordinary integrals,

$$\int_\gamma (\boldsymbol{\omega} + \boldsymbol{\zeta}) = \int_\gamma \boldsymbol{\omega} + \int_\gamma \boldsymbol{\zeta}, \qquad \int_\gamma (c\boldsymbol{\omega}) = c\int_\gamma \boldsymbol{\omega}, \tag{6.13}$$

for any pair of 1-forms ω, ζ, and scalar c. Let $\mathbf{g}$ represent γ on $[a, b]$, and let ϕ be of class $C^{(1)}$ on $[\alpha, \beta]$ with

$$\phi'(\tau) < 0 \quad \text{for every } \tau \in [\alpha, \beta], \qquad \phi(\alpha) = b, \qquad \phi(\beta) = a.$$

The formula for change of variables in integrals still holds if we agree, as usual in calculus, that $\int_b^a = -\int_a^b$. The composite $\mathbf{f} = \mathbf{g} \circ \phi$ represents a curve, which is denoted by $-\gamma$ and is called the *curve obtained by reversing the sense of direction of* γ. From the change of variables formula

$$\int_{-\gamma} \omega = -\int_{\gamma} \omega. \tag{6.14}$$

Let $\gamma_1, \ldots, \gamma_p$ be piecewise smooth curves such that the final endpoint of γ_j is the initial endpoint of γ_{j+1} for $j = 1, \ldots, p-1$. Let γ be obtained by "joining together" the curves $\gamma_1, \ldots, \gamma_p$. More precisely, let us divide $[0, 1]$ into p subintervals $[(j-1)/p, j/p]$ of the same length. Each curve γ_j has a representation on an interval $[a_j, b_j]$. By a linear change of parameter we may assume that $a_j = (j-1)/p$, $b_j = j/p$. Let $\mathbf{g}_j$ be such a representation of γ_j for each $j = 1, \ldots, p$. Then $\mathbf{g}_j(j/p) = \mathbf{g}_{j+1}(j/p)$. Let $\mathbf{g}$ be the function such that $\mathbf{g}(t) = \mathbf{g}_j(t)$ for $t \in [(j-1)/p, j/p]$. Then $\mathbf{g}$ is a parametric representation that is piecewise of class $C^{(1)}$, and γ is the curve that $\mathbf{g}$ represents. Let us call γ the *sum* of these curves and write $\gamma = \gamma_1 + \cdots + \gamma_p$. Since an ordinary integral over $[a, b]$ is the sum of the integrals over the subintervals $[(j-1)/p, j/p]$, we have

$$\int_{\gamma_1 + \cdots + \gamma_p} \omega = \int_{\gamma_1} \omega + \cdots + \int_{\gamma_p} \omega. \tag{6.15}$$

EXAMPLE 2. Let γ be the boundary of a rectangle in E^2, directed counterclockwise as in Figure 6.6. Let $\omega = M\,dx + N\,dy$. Then

$$\int_{\gamma} \omega = \sum_{j=1}^{4} \int_{\gamma_j} \omega.$$

The most convenient representation for γ_1 is obtained by setting $g^1(t) = t$, $g^2(t) = c$, $a \le t \le b$. Taking similar representations for γ_2, $-\gamma_3$, $-\gamma_4$ and

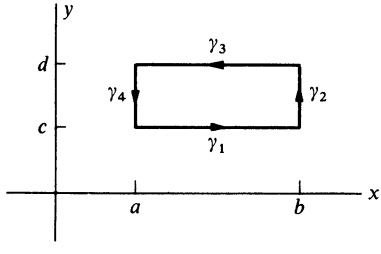


Figure 6.6

using (6.14) we find that

$$\int_\gamma \omega = \int_a^b M(t, c)dt + \int_c^d N(b, t)dt - \int_a^b M(t, d)dt - \int_c^d N(a, t)dt.$$

Let us now consider the case when ω is an exact differential form, $\omega = df$, where f is of class $C^{(1)}$ on an open set D containing the trace of γ. By the definition (6.12a),

$$\int_\gamma df = \int_a^b df[\mathbf{g}(t)] \cdot \mathbf{g}'(t)dt.$$

Let us use the formula

$$(f \circ \mathbf{g})'(t) = df[\mathbf{g}(t)] \cdot \mathbf{g}'(t).$$

This is a special case of the chain rule, (Section 4.4). When arc length is the parameter it says that the derivative at s of $f \circ \mathbf{G}$ equals the derivative of f in the direction of the unit tangent vector at $\mathbf{G}(s)$. By the fundamental theorem of calculus,

$$\int_a^b (f \circ \mathbf{g})'(t)dt = f[\mathbf{g}(b)] - f[\mathbf{g}(a)].$$

Let $\mathbf{x}_0 = \mathbf{g}(a)$ and $\mathbf{x}_1 = \mathbf{g}(b)$ be the endpoints of γ. Then

$$\int_\gamma df = f(\mathbf{x}_1) - f(\mathbf{x}_0). \tag{6.16}$$

This is a generalization of the fundamental theorem of calculus. It shows that the line integral of an exact 1-form depends only on the endpoints of γ. In particular, if γ is closed then the line integral is 0.

The following theorem shows that each of these properties characterizes exact 1-forms. We say that γ *lies in* D if its trace is a subset of D. By curve we mean here piecewise smooth curve.

Theorem 6.1. *Let $D \subset E^n$ be open, and ω a continuous 1-form with domain D. The following three statements are equivalent:*

(1) *ω is exact.*
(2) *For every closed curve γ lying in D, $\int_\gamma \omega = 0$.*
(3) *If γ_1 and γ_2 are any two curves lying in D with the same initial endpoint and the same final endpoint, then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ (see Figure 6.7).*

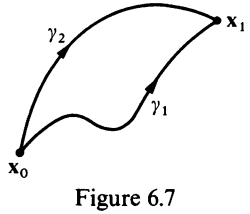


Figure 6.7

PROOF. We have seen that (1) implies (2) in Theorem 6.1. If γ_1 and γ_2 have the same endpoints, then $\gamma_1 - \gamma_2$ is closed. If (2) holds, then

$$0 = \int_{\gamma_1 - \gamma_2} \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega.$$

Hence (2) implies (3).

It remains to show that (3) implies (1). For simplicity let us assume that D is connected. If D is not connected, the construction to follow must be applied separately to each component (that is, maximal connected subset) of D.

Let $\mathbf{x}_0$ be some point of D, and define f as follows. Since D is open and connected, any point of D can be joined to $\mathbf{x}_0$ by a curve whose trace is contained in D (see Problem 10, Section 2.7). For every $\mathbf{x} \in D$ let

$$f(\mathbf{x}) = \int_{\gamma} \omega,$$

where γ is any curve lying in D with initial endpoint $\mathbf{x}_0$ and final endpoint $\mathbf{x}$. Since we are assuming (3) in Theorem 6.1, it does not matter which curve with these properties is chosen. Let us show that $df = \omega$.

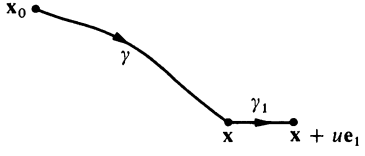


Figure 6.8

Given $\mathbf{x} \in D$, let U be a neighborhood of $\mathbf{x}$ contained in D and δ the radius of U. Let $0 < u < \delta$ and for each $i = 1, \ldots, n$ let γ_i be the line segment from $\mathbf{x}$ to $\mathbf{x} + u\mathbf{e}_i$ (see Figure 6.8). Then

$$f(\mathbf{x} + u\mathbf{e}_i) - f(\mathbf{x}) = \int_{\gamma + \gamma_i} \omega - \int_{\gamma} \omega = \int_{\gamma_i} \omega.$$

Let $\mathbf{g}_i(t) = \mathbf{x} + t\mathbf{e}_i$, $\psi_i(t) = \omega_i(\mathbf{x} + t\mathbf{e}_i)$. Then $\mathbf{g}_i$ represents γ_i on $[0, u]$, and hence

$$\frac{1}{u}[f(\mathbf{x} + u\mathbf{e}_i) - f(\mathbf{x})] = \frac{1}{u} \int_{\gamma_i} \omega = \frac{1}{u} \int_0^u \psi_i(t)dt.$$

Since ω_i is a continuous function, ψ_i is continuous. Consequently, the right-hand side tends to $\psi_i(0) = \omega_i(\mathbf{x})$ as $u \to 0^+$ by the fundamental theorem of calculus. Similarly, $u^{-1}[f(\mathbf{x} + u\mathbf{e}_i) - f(\mathbf{x})]$ tends to $\omega_i(\mathbf{x})$ as $u \to 0^-$.

We have shown that each partial derivative of f of order 1 exists at $\mathbf{x}$ and that

$$\frac{\partial f}{\partial x^i}(\mathbf{x}) = \omega_i(\mathbf{x}), \qquad i = 1, \ldots, n.$$

Therefore $df(\mathbf{x}) = \omega(\mathbf{x})$. Since this is true for every $\mathbf{x} \in D$, $\omega = df$. □

Work

Let D be an open connected subset of E^3. In mechanics the idea of force field is considered. A force field assigns at each $\mathbf{x} \in D$ a linear function, which we call the *force covector acting at* $\mathbf{x}$ and denote by $\boldsymbol{\omega}(\mathbf{x})$. If $\mathbf{h}$ is a "small displacement" from $\mathbf{x}$, then the work done moving a particle along the line segment from $\mathbf{x}$ to $\mathbf{x} + \mathbf{h}$ is approximately $\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{h}$. The force field is the differential form $\boldsymbol{\omega}$ of degree 1 whose value at each $\mathbf{x} \in D$ is the force covector $\boldsymbol{\omega}(\mathbf{x})$.

For present purposes it is more natural to regard force as a covector rather than a vector. However, one can also consider the *force vector* $\mathbf{F}(\mathbf{x}) = \omega_1(\mathbf{x})\mathbf{e}_1 + \omega_2(\mathbf{x})\mathbf{e}_2 + \omega_3(\mathbf{x})\mathbf{e}_3$ with the same components as $\boldsymbol{\omega}(\mathbf{x})$. This simple device for changing covectors into vectors is justified since we use the standard euclidean inner product. If E^n is given another inner product $B(\mathbf{x}, \mathbf{y})$, then the components of $\boldsymbol{\omega}(\mathbf{x})$ and $\mathbf{F}(\mathbf{x})$ would be related by

$$\omega_i = \sum_{j=1}^{n} c_{ij} F^j, \qquad F^i = \sum_{j=1}^{n} c^{ij} \omega_j,$$

where (c_{ij}) is the matrix of B and $(c^{ij}) = (c_{ij})^{-1}$. Compare with (2.2) and (3.12).

Let γ be a piecewise smooth curve lying in D. Using the notation in Section 6.2, with for sake of simplicity $s_j = t_{j-1}$, the vector $\mathbf{h}_j = \mathbf{g}(t_j) - \mathbf{g}(t_{j-1})$ is a displacement from $\mathbf{g}(t_{j-1})$, which is small if μ is small. The work done going along γ from $\mathbf{g}(t_{j-1})$ to $\mathbf{g}(t_j)$ should be approximately $\boldsymbol{\omega}[\mathbf{g}(t_{j-1})] \cdot \mathbf{h}_j$. This suggests the following.

Definition. The *work* w done in moving a particle along γ is

$$w = \int_{\gamma} \boldsymbol{\omega}.$$

If arc length is used as parameter, then from the definition (6.12a),

$$w = \int_0^l \boldsymbol{\omega}[\mathbf{G}(s)] \cdot \mathbf{G}'(s) ds.$$

The expression $\boldsymbol{\omega}[\mathbf{G}(s)] \cdot \mathbf{G}'(s)$ is called the *component of the field* at $\mathbf{G}(s)$ in the direction of the unit tangent vector $\mathbf{G}'(s)$ to γ.

A force field $\boldsymbol{\omega}$ of class $C^{(1)}$ is called *conservative* if $\boldsymbol{\omega}$ is closed. In Section 8.10 we define the concept of simply connected open set. It is shown there that any closed 1-form $\boldsymbol{\omega}$ is exact if D is simply connected. If $\boldsymbol{\omega}$ is exact and $\boldsymbol{\omega} = df$, then f is a *potential* of the field $\boldsymbol{\omega}$. If D is connected, f is determined up to the addition of a constant function.

EXAMPLE 3. Let $\omega = -\rho^{-3}(x\,dx + y\,dy + z\,dz)$, where $\rho^2 = x^2 + y^2 + z^2$ and $D = E^3 - \{\mathbf{0}\}$. If we agree that $f(x, y, z) \to 0$ as $\rho \to \infty$, then the potential f is given by $f(\mathbf{x}) = \rho^{-1}$. Except for a multiplicative constant it is the Newtonian potential due to a mass concentrated at $\mathbf{0}$.

PROBLEMS

1. Evaluate $\frac{1}{2}\int_\gamma x\,dy - y\,dx$ in case:
   (a) γ bounds the triangle shown in Figure 6.9.
   (b) γ is represented by $\mathbf{g}(t) = (a\cos t)\mathbf{e}_1 + (b\sin t)\mathbf{e}_2$, $0 \le t \le 2\pi$, where $a, b > 0$. Your answer should be the area of the set enclosed by γ. This is a very special case of Green's theorem (Section 8.7).

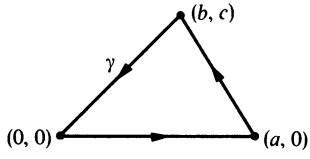


Figure 6.9

2. Let $\mathbf{g}_1(t) = t\mathbf{e}_1 + (2t - 1)\mathbf{e}_2$, $1 \le t \le 2$, and $\mathbf{g}_2(t) = (t + 1)\mathbf{e}_1 + (t^2 + t + 1)\mathbf{e}_2$, $0 \le t \le 1$. Evaluate $\int_{\gamma_1} \omega$ and $\int_{\gamma_2} \omega$ for each of the 1-forms (a), (c), and (d) in Problem 3, Section 6.3, where γ_1 and γ_2 are the curves represented by $\mathbf{g}_1$ and $\mathbf{g}_2$, respectively.

3. Let $n = 1$, $\omega = f\,dx$, and γ be the interval $[a, b]$ directed from a to b. Show that

$$\int_\gamma \omega = \int_a^b f(x)dx.$$

4. Let $D \subset E^2$ be open and simply connected, and u, v functions of class $C^{(1)}$ that satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

   [*Note*: These two first-order partial differential equations are called the *Cauchy–Riemann* equations. They are fundamental to the theory of complex analytic functions.]
   Show that for any closed curve γ lying in D,

$$\int_\gamma u\,dx - v\,dy = 0, \qquad \int_\gamma v\,dx + u\,dy = 0.$$

   [*Hint*: Any closed 1-form is exact since D is simply connected.]

5. Show that the force field $\omega = \psi(\rho^2)(x\,dx + y\,dy + z\,dz)$ is conservative, where ψ is any function of class $C^{(1)}$ on E^1 and $\rho^2 = x^2 + y^2 + z^2$. Find its potential f, if $f(\mathbf{0}) = 0$.

(*Other line integrals.*) The following problems deal not with integrals of 1-forms, but with some other types of line integrals that often occur.

6. If f is continuous on D, then (by definition)

$$\int_\gamma f\,ds = \int_a^b f[\mathbf{g}(t)]\,|\mathbf{g}'(t)|\,dt.$$

Show that this integral does not depend on the particular representation $\mathbf{g}$ chosen for γ. In particular, if arc length is the parameter, then

$$\int_\gamma f\,ds = \int_0^1 f[\mathbf{G}(s)]ds.$$

7. The *moment of inertia* of a curve γ about a point $\mathbf{x}_0$ is $\int_\gamma |\mathbf{x} - \mathbf{x}_0|^2\,ds$. Find the moment of inertia about $\mathbf{0}$ of the line segment in E^3 joining $\mathbf{e}_1$ and $\mathbf{e}_2 + 2\mathbf{e}_3$.

8. The *centroid* $\bar{\mathbf{x}}$ of a curve γ is the point such that

$$\bar{x}^i = \frac{\int_\gamma x^i\,ds}{l}, \qquad i = 1, \ldots, n,$$

where l is the length of γ.

(a) Find the centroid of the helical curve in Problem 3, Section 6.2.

(b) Find its moment of inertia about $\pi\mathbf{e}_3$.

9. Let W be continuous on $D \times E^n$ and satisfy the homogeneity condition $W(\mathbf{x}, c\mathbf{h}) = cW(\mathbf{x}, \mathbf{h})$ whenever $c \geq 0$. Then (by definition)

$$\int_\gamma W = \int_a^b W[\mathbf{g}(t), \mathbf{g}'(t)]dt.$$

Show that:

(a) This integral does not depend on the particular representation $\mathbf{g}$ of γ.

(b) If $W(\mathbf{x}, \mathbf{h}) = \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{h}$, then $\int_\gamma W = \int_\gamma \boldsymbol{\omega}$; and if $W(\mathbf{x}, \mathbf{h}) = f(\mathbf{x})|\mathbf{h}|$, $\int_\gamma W = \int_\gamma f\,ds$.

(c) Let $W(\mathbf{x}, \mathbf{h}) = \|\mathbf{h}\|$, where $\|\ \|$ is any norm on E^n (Section 2.11). Then $\int_\gamma W$ is called the *length* of γ with respect to this norm. Show that if γ is the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$, then the length is $\|\mathbf{x}_1 - \mathbf{x}_2\|$.

*6.5 Gradient method

In Section 3.5 we found relative extrema of a function f by calculating the critical points and testing them by Theorem 3.4. However, in practice the equation $df(\mathbf{x}) = \mathbf{0}$ for the critical points can be explicitly solved only when f has some special form. In Section 3.5 a method for finding critical points approximately was indicated. It is called the *gradient* method, or method of steepest ascent, and will now be described more precisely.

Let D be an open set and $\mathbf{F} = (F^1, \ldots, F^n)$ a vector-valued function whose components F^i are of class $C^{(1)}$ on D. A function $\mathbf{g}$ from an interval J into E^n is called a *solution* of the system of first-order ordinary differential equations

$$\frac{dx^i}{dt} = F^i(\mathbf{x}), \qquad i = 1, \ldots, n,$$

if $\mathbf{g}'(t) = \mathbf{F}[\mathbf{g}(t)]$ for every $t \in J$. An existence theorem for such systems [5, 11] states that given $\mathbf{x}_1 \in D$ there is a solution $\mathbf{g}$ on some open interval J containing 0 such that $\mathbf{g}(0) = \mathbf{x}_1$.

Now let f be of class $C^{(2)}$ on D. Let $\mathbf{F}(\mathbf{x}) = \operatorname{grad} f(\mathbf{x})$ be the gradient vector of f at $\mathbf{x}$. Assume that $\mathbf{x}_1$ is not a critical point of f. A solution $\mathbf{g}$ of

$$\mathbf{g}'(t) = \operatorname{grad} f[\mathbf{g}(t)], \qquad \mathbf{g}(0) = \mathbf{x}_1, \tag{6.17}$$

is called a *gradient trajectory of f through $\mathbf{x}_1$*. It is shown in Section 4.7 that $\operatorname{grad} f(\mathbf{x})$ is a normal vector to the level set of f containing $\mathbf{x}$. Therefore, the gradient trajectories are normal to the level sets, as indicated in Figure 6.10.

Let $\phi(t) = f[\mathbf{g}(t)]$. By the chain rule,

$$\phi'(t) = \operatorname{grad} f[\mathbf{g}(t)] \cdot \mathbf{g}'(t) = |\operatorname{grad} f[\mathbf{g}(t)]|^2 > 0.$$

Hence ϕ is increasing. In other words, *the values of f increase along each gradient trajectory as t increases.*

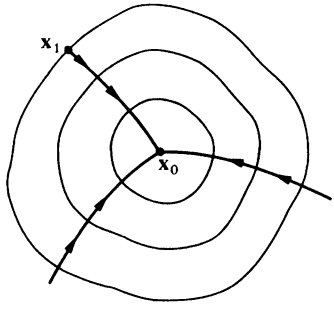


Figure 6.10

Let us assume that there is a gradient trajectory $\mathbf{g}$ through $\mathbf{x}_1$ which is defined for every $t \geq 0$.

By the uniqueness theorem for systems of differential equation [5, 11], $\mathbf{g}(t)$ is never a critical point. However, one may ask whether $\mathbf{g}(t)$ approaches a critical point $\mathbf{x}_0$ as $t \to +\infty$. While this is not always true, we prove two partial results in this direction.

Proposition 6.1. *If $\mathbf{g}(t)$ tends to a limit $\mathbf{x}_0$ as $t \to +\infty$, then $\mathbf{x}_0$ is a critical point of f.*

PROOF. Define ϕ as above. Then ϕ is increasing and

$$f(\mathbf{x}_0) = \lim_{t \to +\infty} \phi(t).$$

Suppose that $\operatorname{grad} f(\mathbf{x}_0) \neq \mathbf{0}$. Since $\operatorname{grad} f$ is continuous there exists a neighborhood U of $\mathbf{x}_0$ and $m > 0$ such that $|\operatorname{grad} f(\mathbf{x})| \geq m$ for every $\mathbf{x} \in U$. There exists t_1 such that $\mathbf{g}(t) \in U$ for every $t \geq t_1$. By the fundamental theorem of

calculus, if $t_1 < t_2$ then

$$\phi(t_2) = \phi(t_1) + \int_{t_1}^{t_2} \phi'(t)dt \geq \phi(t_1) + m^2(t_2 - t_1).$$

The right-hand side tends to $+\infty$ as $t_2 \to +\infty$, but $\phi(t_2) \leq f(\mathbf{x}_0)$. This is a contradiction. Hence grad $f(\mathbf{x}_0) = \mathbf{0}$. □

Note. It may happen that a trajectory $\mathbf{g}$ remains in a compact set $K \subset D$ for every $t \geq 0$, but $\mathbf{g}(t)$ does not approach a limit $\mathbf{x}_0$ at $t \to +\infty$. In that case it can be shown that $\mathbf{g}$ has a "limit set" B, which consists of all accumulation points of sequences $[\mathbf{g}(t_k)]$ for all possible sequences $t_1, t_2, \ldots$ tending to $+\infty$. B is a compact, connected subset of the level set $\{\mathbf{x} : f(\mathbf{x}) = C\}$, where $C = \lim_{t\to\infty} \phi(t)$, and every point of B is critical. If f has only isolated critical points, then B is a single point $\mathbf{x}_0$ and $\mathbf{g}(t) \to \mathbf{x}_0$ as $t \to \infty$. We do not prove this.

However, let us show that if $\mathbf{x}_0$ is a nondegenerate critical point at which f has a relative maximum, then any trajectory starting sufficiently near $\mathbf{x}_0$ leads to $\mathbf{x}_0$.

Proposition 6.2. *Let $\mathbf{x}_0$ be a nondegenerate critical point such that $Q(\mathbf{x}_0, \ )$ is negative definite. Then there is a neighborhood U of $\mathbf{x}_0$ such that $\mathbf{x}_0 = \lim_{t\to\infty} \mathbf{g}(t)$ provided $\mathbf{x}_1 \in U$.*

Proof. By Problem 8, Section 3.5, there exists $m > 0$ such that $Q(\mathbf{x}_0, \mathbf{h}) \leq -m|\mathbf{h}|^2$ for every $\mathbf{h}$. Since f is of class $C^{(2)}$ there is a neighborhood U of $\mathbf{x}_0$ such that $|f_{ij}(\mathbf{x}_0 + s\mathbf{h}) - f_{ij}(\mathbf{x}_0)| \leq m/2n^2$ for $i, j = 1, \ldots, n$, whenever $\mathbf{x}_0 + \mathbf{h} \in U$ and $s \in (0, 1)$. Since $f_i(\mathbf{x}_0) = 0$ there is by the mean value theorem $s_i \in (0, 1)$ such that

$$f_i(\mathbf{x}_0 + \mathbf{h}) = \sum_{j=1}^{n} f_{ij}(\mathbf{x}_0 + s_i\mathbf{h})h^j.$$

If $\mathbf{x} \in U$ and $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$, then

$$\text{grad } f(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0) = \sum_{i,j=1}^{n} f_{ij}(\mathbf{x}_0 + s_i\mathbf{h})h^i h^j \leq -\frac{m}{2}|\mathbf{h}|^2.$$

Now let $\psi(t) = |\mathbf{g}(t) - \mathbf{x}_0|^2$, the square of the distance from $\mathbf{x}_0$, Taking $\mathbf{x} = \mathbf{g}(t)$ and using (6.17), we have provided $\mathbf{g}(t) \in U$

$$\psi'(t) = 2[\mathbf{g}(t) - \mathbf{x}_0] \cdot \mathbf{g}'(t) \leq -m|\mathbf{g}(t) - \mathbf{x}_0|^2,$$

which becomes $\psi'(t) \leq -m\psi(t)$. Now $\mathbf{g}(0) = \mathbf{x}_1 \in U$, and since ψ is decreasing, $\mathbf{g}(t) \in U$ for every $t \geq 0$. Dividing by $\psi(t)$ in the inequality $\psi'(t) \leq -m\psi(t)$ and integrating over $[0, t]$,

$$\log \psi(t) - \log \psi(0) \leq -mt.$$

The right-hand side tends to $-\infty$ as $t \to \infty$. Hence so does $\log \psi(t)$; and $\psi(t)$ tends to 0. □

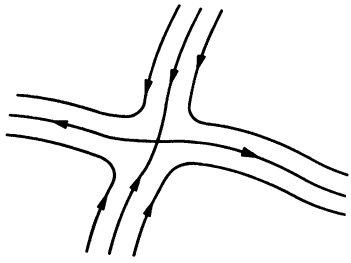


Figure 6.11

Figure 6.10 indicates the behavior of the level sets and gradient trajectories near a nondegenerate maximum. The situation is similar near a nondegenerate minimum. The gradient trajectories in that case are followed as $t \to -\infty$.

Near a saddle point ($n = 2$) the behavior of the trajectories is indicated in Figure 6.11.

EXAMPLE 1. Let $f(x, y) = x^2 - y^2$. The equations of the gradient trajectories are

$$\frac{dx}{dt} = 2x, \qquad \frac{dy}{dt} = -2y,$$

whose solutions are $g^1(t) = x_1 \exp(2t)$, $g^2(t) = y_1 \exp(-2t)$. The trajectories lie on the hyperbolas $xy = k$ orthogonal to the level sets $x^2 - y^2 = c$, and on the coordinate axes. Only trajectories starting from points $(0, y_1)$ lead to the saddle point $(0, 0)$.

PROBLEMS

1. Sketch the level sets and gradient trajectories.
   (a) $f(x, y) = 1 - x^2 - 2y^2$.
   (b) $f(x, y) = xy + y$.

2. Let f be a strictly concave function on E^n which has an absolute maximum at $\mathbf{x}_0$. By Theorem 3.7, f has no other critical points. Consider any gradient trajectory $\mathbf{g}$.
   (a) Let $\psi(t) = |\mathbf{g}(t) - \mathbf{x}_0|^2$. Show that ψ is nonincreasing.
   (b) Show that $\psi(t) \to 0$, and hence $\mathbf{g}(t) \to \mathbf{x}_0$, as $t \to +\infty$.

3. Let $\mathbf{G}$ be the representation with arc length as parameter of a gradient trajectory $\mathbf{g}$. Show that $\mathbf{G}'(s)$ is the direction of the gradient at $\mathbf{G}(s)$ (see Section 6.2).

*6.6 Integrating factors; thermal systems

Let ω be a differential form of degree 1, with domain $D \subset E^n$. It may happen that ω is not exact, but that the product of a suitable real valued function ϕ and ω is exact. Let us assume that $\omega(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in D$.

Definition. A function ϕ is an *integrating factor* if $\phi(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in D$ and $\phi\omega = df$ for some function f.

In some cases an integrating factor can be found by inspection.

EXAMPLE 1. Let $D = \{(x, y) : x > 0\}$, $\omega = 2y\,dx + x\,dy$. Then ω is not exact, but

$$x\omega = 2xy\,dx + x^2\,dy = d(x^2y).$$

In this example, $\phi(x, y) = x$ is the integrating factor and $f(x, y) = x^2y$.

Let us require ϕ and ω to be of class $C^{(1)}$ on D. Then f is of class $C^{(2)}$, and its first-order partial derivatives are $f_i = \phi\omega_i$, $i = 1, \ldots, n$. By Theorem 3.3, $f_{ij} = f_{ji}$. Thus an integrating factor ϕ must satisfy the equations

$$\frac{\partial}{\partial x^j}(\phi\omega_i) = \frac{\partial}{\partial x^i}(\phi\omega_j), \quad i < j. \tag{6.18}$$

When $n = 2$, there is only one equation in (6.18) to be satisfied by ϕ. We shall see that when $n = 2$ integrating factors exist, at least locally (Problem 2). For $n > 2$, there are $n(n-1)/2$ equations in (6.18). Further conditions must be imposed on ω in order that there be an integrating factor. Theorem 6.2 (below) gives one set of such conditions. These conditions have a natural interpretation in the discussion of thermal systems to follow. Another, more easily verifiable, set of conditions will be given later in Section 7.10.

Let γ be a curve in E^n, and $\mathbf{g}$ a parametric representation of γ on $[a, b]$.

Definition. We say that $\omega = \mathbf{0}$ *on* γ if $\omega[\mathbf{g}(t)] \cdot \mathbf{g}'(t) = 0$ for $a \leq t \leq b$.

If $\phi\omega = df$ and $\phi(\mathbf{x}) \neq 0$, then the curves on which $\omega = \mathbf{0}$ are just those on which $f(\mathbf{x})$ is constant. To see this, note that $f[\mathbf{g}(t)]$ is constant if and only if $df = \mathbf{0}$ on γ. But $df = \mathbf{0}$ on γ if and only $\omega = \mathbf{0}$ on γ.

Note. In the terminology of differential equations, a curve γ on which $\omega = \mathbf{0}$ is called a *solution curve* for the differential equation $\omega = \mathbf{0}$. Such a differential equation is called of *Pfaffian* type. Since $\omega(\mathbf{x}) \neq \mathbf{0}$, we have also $df(\mathbf{x}) \neq \mathbf{0}$ if there is an integrating factor ϕ. Thus the level sets $B_c = \{\mathbf{x} \in D : f(\mathbf{x}) = c\}$ are $(n-1)$-manifolds (see Section 4.7). Every solution curve lies in some level set B_c.

Let us now assume that $\omega_1(\mathbf{x}) = 1$, i.e., that ω has the special type

$$\omega = dx^1 + \zeta, \qquad \zeta = \sum_{i=2}^{n} \omega_i\,dx^i. \tag{6.19}$$

This assumption holds in the application to thermal systems below. Moreover, the problem of local existence of an integrating factor reduces to considering ω of this type (see the remark following Theorem 6.2).

Let us denote points of E^n by $\mathbf{x} = (x^1, \mathbf{z})$ where $\mathbf{z} = (x^2, \ldots, x^n)$. If $\boldsymbol{\omega} = \mathbf{0}$ on γ, then

$$0 = \int_\gamma (dx^1 + \zeta).$$

Hence, if $\mathbf{x}_0 = (x_0^1, \mathbf{z}_0)$ and $\mathbf{x}_1 = (x_1^1, \mathbf{z}_1)$ are the endpoints of γ,

$$x_1^1 - x_0^1 = -\int_\gamma \zeta. \tag{6.20}$$

Definition. We say that a curve γ is *of type* $\mathscr{T}$ if $\mathbf{z}_0 = \mathbf{z}_1$ and $\boldsymbol{\omega} = \mathbf{0}$ on γ.

Let $\Omega \subset D$. We say that $\boldsymbol{\omega}$ satisfies *property* (P) in Ω if $\int_\gamma \zeta = 0$ for any curve γ of type $\mathscr{T}$ lying in Ω.

By (6.20), property (P) is equivalent to the statement that any curve γ of type $\mathscr{T}$ lying in Ω is a closed curve ($\mathbf{x}_0 = \mathbf{x}_1$).

Theorem 6.2. *Let Ω_1 be a neighborhood of $\mathbf{x}^*$, with $\Omega_1 \subset D$.*

(a) *If $\boldsymbol{\omega}$ has an integrating factor in Ω_1, then $\boldsymbol{\omega}$ satisfies property (P) in Ω_1.*
(b) *If $\boldsymbol{\omega}$ satisfies property (P) in Ω_1, then there exists an open set Δ containing $\mathbf{x}^*$ such that $\Delta \subset \Omega_1$ and $\boldsymbol{\omega}$ has an integrating factor in Δ.*

Remark. In Theorem 6.2, $\boldsymbol{\omega}$ is assumed of type (6.19). However, this is actually no restriction. Since $\boldsymbol{\omega}(\mathbf{x}^*) \neq \mathbf{0}$, some component $\omega_i(\mathbf{x})$ is not 0 for all $\mathbf{x}$ in a neighborhood Ω_1 of $\mathbf{x}^*$. We may rearrange indices so that $i = 1$. Let $\tilde{\boldsymbol{\omega}} = (\omega_1)^{-1}\boldsymbol{\omega}$. Then $\tilde{\omega}_1(\mathbf{x}) = 1$, and for $\Delta \subset \Omega_1$, $\boldsymbol{\omega}$ has an integrating factor in Δ if and only if $\tilde{\boldsymbol{\omega}}$ has an integrating factor in Δ.

PROOF. To prove Theorem 6.2(a), suppose that $\phi\boldsymbol{\omega} = df$ in Ω_1. Since $\omega_1 = 1$, $\partial f/\partial x^1 = \phi \neq 0$ in Ω_1. Consider any curve γ of type $\mathscr{T}$ lying in Ω_1. Then $\gamma \subset B_c \cap \Omega_1$, for some level set $B_c = \{\mathbf{x} : f(\mathbf{x}) = c\}$.

By the implicit function theorem,

$$B_c \cap \Omega_1 = \{(\psi(\mathbf{z}), \mathbf{z}) : \mathbf{z} \in R\},$$

where $R \subset E^{n-1}$ is open. At the endpoints $\mathbf{x}_0$, $\mathbf{x}_1$ of γ we have $x_0^1 = \psi(\mathbf{z}_0)$ and $x_1^1 = \psi(\mathbf{z}_1)$. Since $\mathbf{z}_0 = \mathbf{z}_1$, we have $x_0^1 = x_1^1$ which by (6.20) is equivalent to property (P).

For proof of Theorem 6.2(b), to simplify the notation let us suppose that $\mathbf{x}^* = \mathbf{0}$. Suppose that $\boldsymbol{\omega}$ has property (P) in Ω_1. Let us define $F(u, \mathbf{z})$ for $(u, \mathbf{z})$ in a suitable neighborhood Ω_2 of $\mathbf{0}$ as follows ($\Omega_2 \subset \Omega_1$). Let γ be any curve lying in Ω_1, such that $\boldsymbol{\omega} = \mathbf{0}$ on γ and γ has endpoints $(u, \mathbf{0})$, $(x^1, \mathbf{z})$ (see Figure 6.12). We set $F(u, \mathbf{z}) = x^1$. Let us show that $F(u, \mathbf{z})$ does not depend on the particular choice of γ. Suppose that $\tilde{\gamma}$ is another such curve, with endpoints $(u, \mathbf{0})$ and $(\tilde{x}^1, \mathbf{z})$. The curve γ_1 obtained by traversing $\tilde{\gamma}$ in the opposite sense followed by γ is of type $\mathscr{T}$. By Property (P), $\tilde{x}^1 = x^1$.

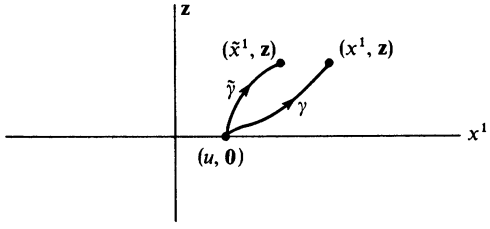


Figure 6.12

In order to show that $F(u, \mathbf{z})$ is of class $C^{(1)}$, we make a particular choice for γ and appeal to a theorem about differential equations. This curve γ is represented on $[0, 1]$ by $\mathbf{g}(t) = (g^1(t), t\mathbf{z})$, where g^1 is the solution of the differential equation

$$\frac{dg^1}{dt} = -\sum_{i=2}^{n} \omega_i[g^1(t), t\mathbf{z}]x^i, \qquad 0 \le t \le 1,$$

with $g^1(0) = u$, $\mathbf{z} = (x^2, \ldots, x^n)$. Then $\boldsymbol{\omega} = \mathbf{0}$ on γ and γ has endpoints $(u, \mathbf{0})$, $(g^1(1), \mathbf{z})$. Thus $F(u, \mathbf{z}) = g^1(1)$. By [11, Section I.3], F is of class $C^{(1)}$ on a suitable neighborhood Ω_2 of $\mathbf{0}$.

Let us next show that

$$\frac{\partial F}{\partial x^i}(u, \mathbf{z}) = -\omega_i(x^1, \mathbf{z}), \qquad i = 2, \ldots, n, \tag{6.21}$$

where $x^1 = F(u, \mathbf{z})$. For this purpose, given $v > 0$ sufficiently small, consider the curve represented on $[0, 1]$ by $\mathbf{G}(t) = (G^1(t), \mathbf{z} + tv\mathbf{e}_i)$, where

$$\frac{dG^1}{dt} = -\omega_i[G^1(t), \mathbf{z} + tv\mathbf{e}_i]v,$$

with $G^1(0) = x^1$. Then $\boldsymbol{\omega} = \mathbf{0}$ on this curve; its endpoints are $(x^1, \mathbf{z})$, $(G^1(1), \mathbf{z} + v\mathbf{e}_i)$. Thus $F(u, \mathbf{z} + v\mathbf{e}_i) = G^1(1)$. By the mean value theorem,

$$\frac{F(u, \mathbf{z} + v\mathbf{e}_i) - F(u, \mathbf{z})}{v} = \frac{G^1(1) - G^1(0)}{v}$$

$$= -\omega_i[G^1(s), \mathbf{z} + sv\mathbf{e}_i]$$

for some $s \in (0, 1)$. When $v \to 0$ we get (6.21).

Let $\boldsymbol{\omega}^*$ be the 1-form obtained by substituting, in (6.19), $x^1 = F(u, \mathbf{z})$ and replacing dx^1 by dF. Then by (6.21),

$$\boldsymbol{\omega}^* = dF + \sum_{i=2}^{n} \omega_i\, dx^i = \frac{\partial F}{\partial u}\, du.$$

Since $F(u, \mathbf{0}) = u$, we have

$$\frac{\partial F}{\partial u}(u, \mathbf{0}) = 1. \tag{6.22}$$

Consider the transformation $\mathbf{F}(u, \mathbf{z}) = (F(u, \mathbf{z}), \mathbf{z})$ from Ω_2 into E^n. By (6.22), $J\mathbf{F}(u, \mathbf{0}) = 1$. The inverse function theorem implies that $\mathbf{0}$ has a neighborhood $\Omega \subset \Omega_2$ such that $\mathbf{F}|\Omega$ has an inverse of class $C^{(1)}$, defined on an open set $\Delta = \mathbf{F}(\Omega)$ containing $\mathbf{F}(\mathbf{0}) = \mathbf{0}$. The inverse has the form $(\mathbf{F}|\Omega)^{-1}(\mathbf{x}) = (f(\mathbf{x}), \mathbf{z})$, where $f(\mathbf{x})$ is that u for which $F(u, \mathbf{z}) = x^1$. We set

$$\tau(\mathbf{x}) = \frac{\partial F}{\partial u}(f(\mathbf{x}), \mathbf{z}), \qquad \phi(\mathbf{x}) = \frac{1}{\tau(\mathbf{x})}.$$

Then $\omega = \tau\, df$ in Δ and ϕ is an integrating factor. □

When $n = 2$, the property (P) is not needed to construct the function F in the proof of Theorem 6.2. See Problem 2. Thus an integrating factor always exists in a neighborhood of any (x, y) when $n = 2$.

Thermal systems

The subject of thermodynamics is concerned with relationships between heat and other forms of energy such as mechanical, chemical, electrical, or magnetic. It deals with gross effects such as temperature and pressure, without reference to the underlying atomic structure of the matter being studied.

The following mathematical model is used to describe various thermal systems [16]. A possible state of the system is a vector $\mathbf{x} = (x^1, \ldots, x^n)$. The number x^1 represents the total energy of the system, and $x^2, \ldots, x^n$ other quantities such as volume and mass. Let D denote the set of possible states. It is assumed that D is open, convex, and that $\mathbf{x} \in D$ implies $c\mathbf{x} \in D$ for any $c > 0$. A 1-form $\omega = dx^1 + \zeta$ of type (6.19) is given. The coefficients $\omega_2, \ldots, \omega_n$ are functions on D homogeneous of degree 0, namely, $\omega_i(c\mathbf{x}) = \omega_i(\mathbf{x})$ for $c > 0$. [*Note*: In thermodynamics literature the components x^i are called *extensive* variables and the functions ω_i, *intensive* variables.] The physical significance of ζ is as follows. If a thermal system in a state $\mathbf{x}$ is displaced to a new state $\mathbf{x} + \mathbf{h}$, then for $|\mathbf{h}|$ small, $\zeta(\mathbf{x}) \cdot \mathbf{h}$ represents approximately the work done (cf. Section 6.4). Let us now suppose that the states of a thermal system change with time t, and let $\mathbf{x} = \mathbf{g}(t)$ be the state at time $t \in [a, b]$. Let γ be the curve represented by $\mathbf{g}$. Then $\int_\gamma \zeta$ is the work done in moving the thermal system along γ. On the other hand,

$$\int_\gamma dx^1 = g^1(b) - g^1(a)$$

is the change in total energy. The sum $\int_\gamma \omega$ is interpreted as the heat energy transferred to the system from outside it.

The curve γ is called *adiabatic* if $\omega = \mathbf{0}$ on γ. In other words, no heat is transferred as the thermal system states move along an adiabatic curve γ.

As above, let $\mathbf{z} = (x^2, \ldots, x^n)$ denote the vector of system states, omitting energy x^1, and denote by $\mathbf{x}_0 = (x_0^1, \mathbf{z}_0)$, $\mathbf{x}_1 = (x_1^1, \mathbf{z}_1)$, the initial, final states as the system moves along γ. It is assumed that:

Principle 1 (Conservation of energy). *If γ is an adiabatic curve such that $\mathbf{z}_0 = \mathbf{z}_1$, then $x_0^1 = x_1^1$.*

Principle 1 is equivalent to property (P) above. Theorem 6.2(b) guarantees that integrating factors exist, at least locally. Principle 2 is an additional assumption that there is an integrating factor of a special form.

Let $\theta(\mathbf{x})$ denote the temperature of a thermal system in state $\mathbf{x}$, measured on some scale. It is assumed that θ is of class $C^{(1)}$, homogeneous of degree 0, and that $\partial\theta/\partial x^1 > 0$.

Principle 2. *There exist functions T and S such that $\omega = T\,dS$ and $T = \tau \circ \theta$, where τ is of class $C^{(1)}$, $\tau(\theta) > 0$ and $\tau'(\theta) > 0$ for all θ.*

$T(\mathbf{x})$ is called the *thermodynamic temperature* and $S(\mathbf{x})$, the *entropy* of a thermal system in state $\mathbf{x}$. Thermodynamic temperature T is measured on a scale natural to the problem. Adiabatic curves are just those on which $dS = 0$, in other words, curves on which entropy S is constant. The function T is homogeneous of degree 0, and S is homogeneous of degree 1 [$S(c\mathbf{x}) = cS(\mathbf{x})$ for $c > 0$].

A curve γ is called *isothermal* if $T(\mathbf{x})$ is constant on γ; that is, if $T[\mathbf{g}(t)] = k$ for some k, where $\mathbf{g}$ represents γ. If γ is isothermal, then

$$\int_\gamma \omega = k \int_\gamma dS = k[S(\mathbf{x}_1) - S(\mathbf{x}_0)]. \tag{6.23}$$

We have

$$\frac{\partial T}{\partial x^1} = \tau'[\theta(\mathbf{x})]\frac{\partial \theta}{\partial x^1} > 0.$$

Hence, T is an increasing function of x^1 for fixed $\mathbf{z}$. Suppose that γ is isothermal, with endpoints $\mathbf{x}_0 = (x_0^1, \mathbf{z}_0)$, $\mathbf{x}_1 = (x_1^1, \mathbf{z}_1)$. If $\mathbf{z}_0 = \mathbf{z}_1$, then we must have $x_0^1 = x_1^1$ since $T(\mathbf{x}_0) = T(\mathbf{x}_1)$. By (6.23) we have $\int_\gamma \omega = 0$ and hence $\int_\gamma \zeta = 0$. Thus there is neither a change in the total energy nor work done as the system moves along an isothermal curve γ for which $\mathbf{z}_0 = \mathbf{z}_1$.

Principles 1 and 2 are intimately related to the laws of thermodynamics for thermal systems that can be described by the above model. This model is an idealization of actual physical systems. For one thing, a curve γ leading from thermal state $\mathbf{x}_0$ to state $\mathbf{x}_1$ can be traversed in the opposite sense, from $\mathbf{x}_1$ to $\mathbf{x}_0$. Thus the model applies to reversible processes. Moreover, in the model $\mathbf{x}$ is to be regarded as an equilibrium state of the system. For example, if the model describes a gas in a container, then such variables as pressure, volume, and temperature should be the same at all points of the container. For additional references concerning classical thermodynamics and its relation to Pfaffian differential forms see [26] [27].

EXAMPLE 2 (*Ideal gas*). Let $n = 3$ and $x^1 = U$, $x^2 = V$, $x^3 = N$, where U is the total kinetic energy of the molecules in the gas, V the volume, and N the mass. Let

$$\omega = dU + P\,dV - \mu\,dN,$$

where P is pressure and μ is called the *chemical potential.* If $\omega = T\,dS$, then $\phi = 1/T$ is an integrating factor and

$$\frac{\partial S}{\partial U} = \frac{1}{T}, \qquad \frac{\partial S}{\partial V} = \frac{P}{T}, \qquad \frac{\partial S}{\partial N} = -\frac{\mu}{T}.$$

Since S is homogeneous of degree 1, Euler's formula (Problem 8, Section 3.3) gives

$$S = U\frac{\partial S}{\partial U} + V\frac{\partial S}{\partial V} + N\frac{\partial S}{\partial N}.$$

Therefore

$$S = \frac{U + PV - \mu N}{T}. \tag{6.24}$$

For an ideal gas, it is required that

$$P = a\frac{U}{V}, \qquad T = b\frac{U}{N}, \tag{6.25}$$

for suitable constants a, b.

Formula (6.18) gives three equations that $\phi = 1/T$ must satisfy. One of these holds by (6.25). The other two equations determine μ, and then S, from (6.24):

$$S = \frac{a}{b}N\log\frac{V}{N} + \frac{1}{b}N\log\frac{U}{N} + CN \tag{6.26}$$

where C is any constant (see Problem 4).

Problems

1. Let $n = 3$, $\mathbf{x} = (x, y, z)$. Show that the 1-form $\omega = dx + y\,dz$ has no integrating factor. [*Hint*: Show that property (P) does not hold. Try curves γ on which $y = a\cos t$, $z = a\sin t$, $0 \le t \le 2\pi$, and $\omega = \mathbf{0}$ on γ].

2. Let $n = 2$ and $\omega = dx + N(x, y)dy$. Given (x^*, y^*), define $F(u, y)$ by

$$\frac{\partial F}{\partial y} = -N(F(u, y), y), \text{ with } F(u, y^*) = u.$$

   [In other words, for fixed u, $F(u, y)$ satisfies the differential equation $dx/dy = -N(x, y)$.] Following the last part of the proof of Theorem 6.2 (b), show that $\omega = \tau\,df$ in some open set Δ containing (x^*, y^*).

3. Let $n = 2$ and $\omega = dU + P\,dV = T\,dS$.
   (a) Show that $S = T^{-1}(U + PV)$.
   (b) Find T if $P = aUV^{-1}$.

4. In Example 2, derive (6.26).

Exterior algebra and differential calculus

7

In this chapter we introduce the algebra and calculus of differential forms, which also goes by the name "exterior differential calculus." In Section 6.3 we defined the concept of differential form of degree 1, and in Section 6.4, the integral $\int_\gamma \omega$ of such a differential form ω along a curve γ. This integral changes sign if the direction in which γ is traversed is reversed. In Chapter 8 we define the integral of a differential form ω of higher degree r over a portion A of an r-dimensional manifold. For example, for $r = 2$ a differential form of degree 2 can be integrated over a piece of surface A. The integral of a differential form depends on the orientation assigned to A and changes sign if the orientation is reversed.

We begin in Section 7.1 with the special case of degree $r = 2$. The purpose is to motivate the treatment of differential forms of arbitrary degree r, which follows in Section 7.4. For similar reasons we begin the discussion of multivectors, frames, and orientations in Section 7.5 with the case $r = 2$.

We recall from Section 6.3 that a differential form of degree 1 is a covector-valued function. In order to define differential forms of higher degree r we first introduce in Sections 7.2 and 7.3 multicovectors of degree r. For brevity, they are called r-covectors. An r-covector is an alternating, multilinear function with domain the r-fold cartesian product $E^n \times \cdots \times E^n$. It turns out that the r-covectors form a vector space of dimension $\binom{n}{r}$, which is denoted by $(E^n_r)^*$.

There is a natural product for multicovectors called the *exterior product* and denoted by the symbol $\wedge$. If ω is an r-covector and ζ an s-covector, then $\omega \wedge \zeta$ is a certain $(r + s)$-covector. The exterior product is associative, and it is commutative except for a possible sign change (Proposition 7.3).

A differential form of degree r is defined as an r-covector-valued function. For brevity we say r-form for differential form of degree r. The exterior product $\omega \wedge \zeta$ of an r-form ω and an s-form ζ is defined using the exterior

product for multicovectors. Its properties are summarized in Theorem 7.1. Every r-form can be represented as a homogeneous polynomial in $dx^1, \ldots, dx^n$ of degree r, whose coefficients are real valued functions [see Formula (7.23)]. As in Section 6.3, the 1-forms $dx^1, \ldots, dx^n$ are the differentials of the standard cartesian coordinate functions. Their exterior products are anticommutative, namely, $dx^j \wedge dx^i = -dx^i \wedge dx^j$.

Every r-form ω of class $C^{(1)}$ has an exterior differential $d\omega$, which is a form of degree $r + 1$. The usual formulas for differentials of sums and products remain true, except for a possible change of sign in the product rule. Another important fact is that $d(d\omega) = \mathbf{0}$ for any form of class $C^{(2)}$ (see Theorem 7.2).

One of the main results of Chapter 8 is Stokes's formula (8.20). This equates the integral of $d\omega$ over a portion A of an $(r + 1)$-dimensional manifold with the integral of ω over the boundary of A, provided A and its boundary are oriented consistently. It is Stokes's formula that justifies the way in which the exterior differential is defined.

In Section 7.5 we define the concept of r-vector. The r-vectors form a vector space E^n_r, whose dual space turns out to be the space $(E^n_r)^*$ of r-covectors.

Certain multivectors, called *decomposable*, have an interesting geometric interpretation. An r-vector α is decomposable if there are vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ such that $\alpha = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$. It turns out (Theorem 7.3) that if $\alpha \neq \mathbf{0}$, then $\mathbf{h}_1, \ldots, \mathbf{h}_r$ span an r-dimensional vector subspace P of E^n. With α is associated an orientation of P. If two of the vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ are interchanged, then α changes sign and the orientation of P changes. The norm $|\alpha|$ of a decomposable r-vector α equals the r-dimensional measure of a certain r-parallelepiped.

Besides its differential $d\omega$, an r-form ω of class $C^{(1)}$ has a codifferential $\tilde{d}\omega$, which is a form of degree $r - 1$ (Section 7.8). In Chapter 8 the codifferential is used only for $r = 1$, in which case it becomes the divergence. In Section 7.9 the basic formulas of vector analysis in E^3 are derived.

7.1 Covectors and differential forms of degree 2

Let us call a real valued function L with domain E^n 1-linear if L is linear. We recall from Section 3.2 that a 1-linear function L may be identified with the corresponding covector $\mathbf{a} = a_1\mathbf{e}^1 + \cdots + a_n\mathbf{e}^n$, where $a_i = L(\mathbf{e}_i)$. Here $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{e}^1, \ldots, \mathbf{e}^n\}$ are, respectively, the standard bases for E^n and its dual $(E^n)^*$. The linear function corresponding to $\mathbf{e}^i$ is the standard cartesian coordinate function X^i, such that $X^i(\mathbf{x}) = x^i$ for all $\mathbf{x}$. In this chapter a covector $\mathbf{a}$ is also called a *covector of degree* 1, or for brevity, a 1-*covector*.

Alternating bilinear functions

In order to define covectors of degree 2, let us first recall the concept of bilinear function. Let B be a real valued function with domain the cartesian product $E^n \times E^n$. The elements of $E^n \times E^n$ are pairs of vectors, denoted by

$(\mathbf{h}, \mathbf{k})$. For fixed $\mathbf{h}$, let $B(\mathbf{h}, \ )$ denote the function whose value at $\mathbf{k}$ is $B(\mathbf{h}, \mathbf{k})$. Similarly, for fixed $\mathbf{k}$, $B(\ , \mathbf{k})$ is the function whose value at $\mathbf{h}$ is $B(\mathbf{h}, \mathbf{k})$. The function B is *bilinear* if $B(\mathbf{h}, \ )$ and $B(\ , \mathbf{k})$ are linear functions for every $\mathbf{h}, \mathbf{k}$. We have

$$\mathbf{h} = \sum_{i=1}^{n} h^i \mathbf{e}_i, \qquad \mathbf{k} = \sum_{j=1}^{n} k^i \mathbf{e}_i.$$

If B is bilinear, then

$$B(\mathbf{h}, \mathbf{k}) = \sum_{i=1}^{n} h^i B(\mathbf{e}_i, \mathbf{k}) = \sum_{i,j=1}^{n} h^i k^j B(\mathbf{e}_i, \mathbf{e}_j).$$

Let

$$\omega_{ij} = B(\mathbf{e}_i, \mathbf{e}_j), \qquad i, j = 1, \ldots, n; \tag{7.1}$$

then for every $(\mathbf{h}, \mathbf{k})$

$$B(\mathbf{h}, \mathbf{k}) = \sum_{i,j=1}^{n} \omega_{ij} h^i k^j. \tag{7.2}$$

In this chapter we are interested in a special class of r-linear functions, called *alternating*. For $r = 2$, B is alternating if $B(\mathbf{h}, \mathbf{k}) = -B(\mathbf{k}, \mathbf{h})$ for every $(\mathbf{h}, \mathbf{k})$. If B is bilinear and alternating, then $\omega_{ij} = -\omega_{ji}$, and in particular $\omega_{ii} = 0$. Formula (7.2) can be rewritten

$$B(\mathbf{h}, \mathbf{k}) = \sum_{i<j} (\omega_{ij} h^i k^j + \omega_{ji} h^j k^i),$$

or

$$B(\mathbf{h}, \mathbf{k}) = \sum_{i<j} \omega_{ij} (h^i k^j - h^j k^i). \tag{7.3}$$

Conversely, given $n(n-1)/2$ numbers ω_{ij}, $i < j$, Formula (7.3) defines an alternating bilinear function.

It is convenient to introduce a different name and notation for alternating bilinear functions.

Definition. A *covector of degree* 2 (or 2-*covector*) is an alternating bilinear function on $E^n \times E^n$. The space of 2-covectors is denoted by $(E_2^n)^*$.

If B_1 and B_2 are alternating bilinear functions, then the sum $B_1 + B_2$ is also alternating and bilinear. Similarly, cB is alternating and bilinear if c is real, B alternating and bilinear. Thus $(E_2^n)^*$ satisfies the axioms for a vector space (Appendix A.1).

From now on, we generally use lowercase Greek letters, such as $\boldsymbol{\omega}, \boldsymbol{\zeta}, \boldsymbol{\eta}, \ldots$, to denote 2-covectors rather than B. Let us introduce some particular 2-covectors, denoted by $\mathbf{e}^{ij}$. For every pair of vectors $(\mathbf{h}, \mathbf{k}) \in E^n \times E^n$, let

$$\mathbf{e}^{ij}(\mathbf{h}, \mathbf{k}) = h^i k^j - h^j k^i.$$

The function $\mathbf{e}^{ij}$ is alternating and bilinear, for any $i, j = 1, \ldots, n$. Thus $\mathbf{e}^{ij}$ is a 2-covector. We also have

$$\mathbf{e}^{ji} = -\mathbf{e}^{ij}, \qquad \mathbf{e}^{ii} = \mathbf{0}. \tag{7.4}$$

Since (7.3) holds for all $(\mathbf{h}, \mathbf{k})$, we have upon writing ω instead of B to denote a 2-covector,

$$\omega = \sum_{i<j} \omega_{ij} \mathbf{e}^{ij}. \tag{7.5}$$

Conversely, given $n(n-1)/2$ numbers $\omega_{ij}, i < j$, Formula (7.5) defines an alternating bilinear function. The 2-covectors $\mathbf{e}^{ij}, i < j$, are linearly independent (Problem 2). These $n(n-1)/2$ 2-covectors form a basis for $(E_2^n)^*$, called the *standard basis*. In particular, $(E_2^n)^*$ is of dimension $n(n-1)/2$.

EXAMPLE 1. Let $n = 3$. The 2-covectors $\{\mathbf{e}^{12}, \mathbf{e}^{13}, \mathbf{e}^{23}\}$ form the standard basis for $(E_2^3)^*$, which is 3-dimensional. Every $\omega \in (E_2^3)^*$ can be uniquely written as $\omega = \omega_{12}\mathbf{e}^{12} + \omega_{13}\mathbf{e}^{13} + \omega_{23}\mathbf{e}^{23}$, where the coefficients $\omega_{12}, \omega_{13}, \omega_{23}$ are real numbers.

Exterior products of 1-covectors

One way to get a 2-covector is by multiplying two 1-covectors, using a new kind of product (called the *exterior* product). The symbol for this kind of multiplication is $\wedge$. To define the exterior product $\mathbf{a} \wedge \mathbf{b}$ of 1-covectors $\mathbf{a}$ and $\mathbf{b}$, we first specify how to multiply standard basis covectors $\mathbf{e}^i$ and $\mathbf{e}^j$, and then we agree that $\wedge$ distributes with addition and scalar multiplication. Let us set $\mathbf{e}^i \wedge \mathbf{e}^j = \mathbf{e}^{ij}$. If

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}^i, \qquad \mathbf{b} = \sum_{j=1}^{n} b_j \mathbf{e}^j,$$

we then set

$$\mathbf{a} \wedge \mathbf{b} = \sum_{i,j=1}^{n} a_i b_j \mathbf{e}^{ij} = \sum_{i<j} (a_i b_j - a_j b_i)\mathbf{e}^{ij}.$$

In the last equality, we have used (7.4). Thus, the components of $\mathbf{a} \wedge \mathbf{b}$ in (7.5) are $\omega_{ij} = a_i b_j - a_j b_i$.

We see in Section 7.5 that every 2-covector is of the form $\omega = \mathbf{a} \wedge \mathbf{b}$ for dimensions $n = 2, 3$, but this does not apply for $n > 3$.

Differential forms of degree 2

The definition is quite analogous to the case of degree 1 in Section 6.3.

Definition. A *differential form of degree* 2 (or 2-*form*) is a function ω with domain $D \subset E^n$ and values in $(E_2^n)^*$. The value of ω at $\mathbf{x}$ is denoted by $\omega(\mathbf{x})$.

The values of ω are 2-covectors. The same Greek letters $\omega, \zeta, \eta, \ldots$ used above to denote 2-covectors are now used to denote differential forms.

One way to get a 2-form is by multiplying two 1-forms, using the product $\wedge$. If ζ and η are 1-forms, then $\zeta \wedge \eta$ is the 2-form obtained by multiplying their values at each $\mathbf{x}$:

$$(\zeta \wedge \eta)(\mathbf{x}) = \zeta(\mathbf{x}) \wedge \eta(\mathbf{x}).$$

In particular, dx^i denotes the 1-form, whose (constant) value at any $\mathbf{x}$ is the 1-covector $\mathbf{e}^i$. Then $dx^i \wedge dx^j$ has the constant value $\mathbf{e}^i \wedge \mathbf{e}^j = \mathbf{e}^{ij}$. By (7.5),

$$\omega(\mathbf{x}) = \sum_{i<j} \omega_{ij}(\mathbf{x})\mathbf{e}^{ij}$$

for each $\mathbf{x} \in D$. Thus

$$\omega = \sum_{i<j} \omega_{ij}\, dx^i \wedge dx^j. \tag{7.6}$$

By (7.4), we have

$$dx^i \wedge dx^j = -dx^j \wedge dx^i, \qquad dx^i \wedge dx^i = \mathbf{0}. \tag{7.7}$$

Formula (7.6) represents any 2-form ω as a homogeneous quadratic polynomial in $dx^1, \ldots, dx^n$, whose coefficients ω_{ij} are real valued functions on D.

EXAMPLE 2. Let $n = 3$ and $\zeta = dz - y\,dy$, $\eta = z\,dx - \cos x\,dz$. Then

$$\begin{aligned}\zeta \wedge \eta &= (dz - y\,dy) \wedge (z\,dx - \cos x\,dz) \\ &= yz\,dx \wedge dy - z\,dx \wedge dz + y\cos x\,dy \wedge dz.\end{aligned}$$

We have used the fact that $\wedge$ obeys the usual rules of algebra, except that (7.7) replaces the commutative law. For instance, $dy \wedge dx = -dx \wedge dy$ and $dz \wedge dz = \mathbf{0}$.

The exterior differential of a 1-form

We recall from Section 6.3 that the differential df of a real valued function f of class $C^{(1)}$ is the 1-form whose value at $\mathbf{x}$ is $df(\mathbf{x})$. Now consider a 1-form $\omega = \sum_{i=1}^n \omega_i\, dx^i$, where the coefficient functions ω_i are of class $C^{(1)}$ on D. The exterior differential $d\omega$ is the 2-form defined by

$$d\omega = \sum_{i=1}^n d\omega_i \wedge dx^i.$$

We have

$$d\omega = \sum_{i=1}^n \left\{ \sum_{j=1}^n \frac{\partial \omega_i}{\partial x^j}\, dx^j \right\} \wedge dx^i.$$

By (7.7), this can be rewritten

$$d\omega = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j. \tag{7.8}$$

In Section 6.3 we defined the concepts of closed and exact differential forms of degree 1. By (6.11) and (7.8), a 1-form ω of class $C^{(1)}$ is closed if and only if $d\omega = \mathbf{0}$. If f is of class $C^{(2)}$, then df is exact (hence closed) and of class $C^{(1)}$. Therefore $d(df) = \mathbf{0}$.

EXAMPLE 3. Let $n = 2$ and $\omega = M\,dx + N\,dy$. Then

$$dM = \frac{\partial M}{\partial x}\,dx + \frac{\partial M}{\partial y}\,dy, \qquad dN = \frac{\partial N}{\partial x}\,dx + \frac{\partial N}{\partial y}\,dy,$$

$$d\omega = dM \wedge dx + dN \wedge dy = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx \wedge dy.$$

As in Section 6.3, the condition that ω be closed is $\partial M/\partial y = \partial N/\partial x$.

EXAMPLE 4. If $n = 3$ and $\omega = 2\,dx + z^2\,dy + x^2y\,dz$, then

$$\begin{aligned} d\omega &= d(2) \wedge dx + d(z^2) \wedge dy + d(x^2y) \wedge dz \\ &= 2z\,dz \wedge dy + 2xy\,dx \wedge dz + x^2\,dy \wedge dz. \end{aligned}$$

The definition of $d\omega$ given above may at first seem rather arbitrary. We show at the end of Section 7.4 that the exterior differential is uniquely specified by a few reasonable properties that may be required of it. However, let us now show how to arrive at $d\omega$ in another way, starting from a special case of Stokes's formula. We require that Stokes's formula hold for any parallelogram (in Section 8.8 it is proved under much more general circumstances).

The following discussion is intended purely as motivation, and can be omitted if desired. Consider a parallelogram K with vertices $\mathbf{x}_0$, $\mathbf{x}_0 + \mathbf{h}$, $\mathbf{x}_0 + \mathbf{k}$, $\mathbf{x}_0 + \mathbf{h} + \mathbf{k}$, and let

$$\mathbf{g}(s, t) = \mathbf{x}_0 + s\mathbf{h} + t\mathbf{k}.$$

Then $K = \mathbf{g}(I)$, where $I = [0, 1] \times [0, 1]$ is a square in the (s, t) plane. The pair of vectors $(\mathbf{h}, \mathbf{k})$, written in the given order, assigns an orientation to K (we postpone a formal definition of the concept of orientation until Section 7.5). The integral of a 2-form η over an oriented parallelogram K is denoted by $\int_{K^o} \eta$. It is defined as follows. Let $\eta(\mathbf{x})$ be the value of η at $\mathbf{x}$; $\eta(\mathbf{x})$ is a 2-covector, that is, an alternating bilinear function, which assigns to the pair $(\mathbf{h}, \mathbf{k})$ the real number $\eta(\mathbf{x})(\mathbf{h}, \mathbf{k})$. Let

$$(*) \qquad \int_{K^o} \eta = \int_I \eta[\mathbf{g}(s, t)](\mathbf{h}, \mathbf{k}) dV_2(s, t).$$

Note that for a constant 2-form, namely, $\eta(\mathbf{x}) = \eta_0$ for all $\mathbf{x}$, the integral becomes simply $\eta_0(\mathbf{h}, \mathbf{k})$. The definition (*) can then be motivated by dividing K into a large number m^2 of small parallelograms congruent to K. The right

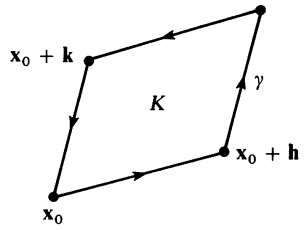


Figure 7.1

side of (*) is the limit as $m \to \infty$ of the sum

$$\sum_{l=1}^{m^2} \boldsymbol{\eta}(\mathbf{x}_l)(m^{-1}\mathbf{h}, m^{-1}\mathbf{k}) = m^{-2} \sum_{l=1}^{m^2} \boldsymbol{\eta}(\mathbf{x}_l)(\mathbf{h}, \mathbf{k}),$$

where $\mathbf{x}_l$ is some point of the lth small parallelogram.

Now let γ be the polygon bounding K, traversed in the sense indicated in Figure 7.1. Let $\boldsymbol{\omega}$ be a 1-form of class $C^{(1)}$. Then

$$(**) \qquad \int_\gamma \boldsymbol{\omega} = \int_0^1 \{\boldsymbol{\omega}[\mathbf{g}(1,t)] - \boldsymbol{\omega}[\mathbf{g}(0,t)]\} \cdot \mathbf{k}\, dt - \int_0^1 \{\boldsymbol{\omega}[\mathbf{g}(s,1)] - \boldsymbol{\omega}[\mathbf{g}(s,0)]\} \cdot \mathbf{h}\, ds.$$

The first integral on the right side is the contribution from the second and fourth segments of γ, starting from $\mathbf{x}_0$, and the second integral the contribution from the first and third segments. Stokes's formula asserts that

$$(7.9) \qquad \int_\gamma \boldsymbol{\omega} = \int_{K^\circ} \boldsymbol{\eta},$$

for a suitable 2-form $\boldsymbol{\eta}$. Let us show that $\boldsymbol{\eta} = d\boldsymbol{\omega}$.

Now

$$\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{k} = \sum_{i=1}^{n} \omega_i(\mathbf{x}) k^i$$

where $\omega_1, \ldots, \omega_n$ are the components of the 1-form $\boldsymbol{\omega}$. By the fundamental theorem of calculus

$$\omega_i[\mathbf{g}(1,t)] - \omega_i[\mathbf{g}(0,t)] = \int_0^1 \frac{\partial}{\partial s} \omega_i[\mathbf{g}(s,t)] ds = \int_0^1 d\omega_i[\mathbf{g}(s,t)] \cdot \frac{\partial \mathbf{g}}{\partial s} ds = \int_0^1 d\omega_i[\mathbf{g}(s,t)] \cdot \mathbf{h}\, ds.$$

Similarly,

$$\omega_i[\mathbf{g}(s,1)] - \omega_i[\mathbf{g}(s,0)] = \int_0^1 d\omega_i[\mathbf{g}(s,t)] \cdot \mathbf{k}\, dt.$$

By comparing these formulas with (*) and (**) we see that, if (7.9) holds for any parallelogram K, we must require

$$\boldsymbol{\eta}(\mathbf{x})(\mathbf{h}, \mathbf{k}) = \sum_{i=1}^{n} \{(d\omega_i(\mathbf{x}) \cdot \mathbf{h})k^i - (d\omega_i(\mathbf{x}) \cdot \mathbf{k})h^i\}.$$

But then

$$\boldsymbol{\eta}(\mathbf{x})(\mathbf{h}, \mathbf{k}) = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x^j} (h^j k^i - k^j h^i) \right\}.$$

Recall that $dx^j \wedge dx^i$ is the constant 2-form, whose value is the 2-covector $\mathbf{e}^{ji}$ which assigns to $(\mathbf{h}, \mathbf{k})$ the number $h^j k^i - k^j h^i$. Thus

$$\boldsymbol{\eta} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i,$$

which is just the formula preceding (7.8) for $d\boldsymbol{\omega}$. Thus $\boldsymbol{\eta} = d\boldsymbol{\omega}$.

Problems

1. Following Example 1, write out for $n = 2$ and $n = 4$ the standard basis for $(E_2^n)^*$ and express any 2-covector as a linear combination of the basis covectors $\mathbf{e}^{ij}$.

2. Show that the 2-covectors $\mathbf{e}^{ij}$, $1 \leq i < j \leq n$, are linearly independent.

3. Let $n = 3$. Simplify:
(a) $(\mathbf{e}^1 + \mathbf{e}^2) \wedge (\mathbf{e}^1 - 3\mathbf{e}^3)$.
(b) $(dy - x\,dz) \wedge (xy\,dx + 3\,dy + z\,dz)$.

4. Let $n = 2$. Find the exterior differential of:
(a) $x^2 y\,dy - xy^2\,dx$.
(b) $x\,dy + y\,dx$.
(c) $f(x)dx + g(y)dy$.
(d) $f(x, y)dy$.

5. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be 1-covectors. Show that:
(a) $(\mathbf{a} + \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c}$.
(b) $(k\mathbf{a}) \wedge \mathbf{b} = k(\mathbf{a} \wedge \mathbf{b})$, if k is real.
(c) $\mathbf{b} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{b}$.
(d) $\mathbf{a} \wedge \mathbf{a} = \mathbf{0}$.
(e) $(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} - \mathbf{b}) = -2\mathbf{a} \wedge \mathbf{b}$.

6. Show that if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are 1-covectors, then

$$\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a} = (\mathbf{a} - \mathbf{b}) \wedge (\mathbf{b} - \mathbf{c}).$$

7. Show that if $\boldsymbol{\omega} = \mathbf{a} \wedge \mathbf{b}$, then $\omega_{ij}\omega_{kl} + \omega_{ik}\omega_{lj} + \omega_{il}\omega_{jk} = 0$ for $i, j, k, l = 1, \ldots, n$. [*Hint*: $\omega_{ij} = a_i b_j - a_j b_i$. Show that

$$\det \begin{pmatrix} \omega_{ij} & \omega_{ik} & \omega_{il} \\ a_j & a_k & a_l \\ b_j & b_k & b_l \end{pmatrix} = 0.]$$

7.2 Alternating multilinear functions

In preparation for studying differential forms of any degree r, let us first discuss alternating multilinear functions of degree r.

For any $r \geq 2$ let M be a real valued function with domain the r-fold cartesian product $E^n \times \cdots \times E^n$. The elements of $E^n \times \cdots \times E^n$ are r-tuples of vectors, denoted by $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$.

Definition. The function M is *multilinear of degree r* if for each $l = 1, \ldots, r$ and $\mathbf{h}_1, \ldots, \mathbf{h}_{l-1}, \mathbf{h}_{l+1}, \ldots, \mathbf{h}_r$ the function $M(\mathbf{h}_1, \ldots, \mathbf{h}_{l-1}, \quad, \mathbf{h}_{l+1}, \ldots, \mathbf{h}_r)$ is linear.

For brevity we write r-linear instead of multilinear of degree r. When $r = 2$ we wrote $\mathbf{h}_1 = \mathbf{h}$, $\mathbf{h}_2 = \mathbf{k}$. The new definition agrees for $r = 2$ with the definition of bilinear function. The formula which generalizes (7.2) to multilinear functions is

$$M(\mathbf{h}_1, \ldots, \mathbf{h}_r) = \sum_{i_1, \ldots, i_r = 1}^{n} \omega_{i_1 \ldots i_r} h_1^{i_1} \cdots h_r^{i_r}, \tag{7.10}$$

where

$$\omega_{i_1 \ldots i_r} = M(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_r}). \tag{7.11}$$

This is proved by induction on r.

Interchanges

Let S be some set. For our purposes we shall take either $S = E^n$ or $S = \{1, 2, \ldots, n\}$. If $(p_1, \ldots, p_r)$ and $(p'_1, \ldots, p'_r)$ are r-tuples of elements of S, let us say that the second r-tuple is obtained from the first by interchanging p_s and p_t if $p'_s = p_t$, $p'_t = p_s$, and $p'_l = p_l$ for $l \neq s, t$.

EXAMPLE 1. The triple of vectors $(\mathbf{h}_3, \mathbf{h}_2, \mathbf{h}_1)$ is obtained from $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ by interchanging $\mathbf{h}_3$ and $\mathbf{h}_1$. The 4-tuple of integers (1, 5, 3, 7) is obtained from the 4-tuple (1, 7, 3, 5) by interchanging 5 and 7.

Definition. An r-linear function M is *alternating* if $M(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ changes sign whenever two vectors in an r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ are interchanged.

We recall that an r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is called *linearly dependent* if there exist scalars $c^1, \ldots, c^r$, not all 0, such that $c^1\mathbf{h}_1 + \cdots + c^r\mathbf{h}_r = \mathbf{0}$.

Proposition 7.1. *Let M be r-linear and alternating. If $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is a linearly dependent r-tuple, then $M(\mathbf{h}_1, \ldots, \mathbf{h}_r) = 0$.*

PROOF. First of all, the conclusion is true if some vector in the r-tuple is repeated. For instance, suppose that $\mathbf{h}_1 = \mathbf{h}_2$. Since M is alternating,

$$M(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \ldots, \mathbf{h}_r) = -M(\mathbf{h}_2, \mathbf{h}_1, \mathbf{h}_3, \ldots, \mathbf{h}_r).$$

Then $M(\mathbf{h}_1, \mathbf{h}_1, \mathbf{h}_3, \ldots, \mathbf{h}_r)$ is its own negative, and must be 0.

Suppose for instance that $\mathbf{h}_r$ is a linear combination of the vectors preceding it,

$$\mathbf{h}_r = c^1\mathbf{h}_1 + \cdots + c^{r-1}\mathbf{h}_{r-1}.$$

Since $M(\mathbf{h}_1, \ldots, \mathbf{h}_{r-1}, \ )$ is a linear function,

$$M(\mathbf{h}_1, \ldots, \mathbf{h}_r) = \sum_{l=1}^{r-1} c^l M(\mathbf{h}_1, \ldots, \mathbf{h}_{r-1}, \mathbf{h}_l).$$

In the lth term on the right-hand side, the vector $\mathbf{h}_l$ is repeated, and hence each term is 0. Thus $M(\mathbf{h}_1, \ldots, \mathbf{h}_r) = 0$. □

For any $r \geq 2$ there is the trivial alternating r-linear function $\mathbf{0}$, which has the value 0 for every r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$. If $r > n$, then $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ must be linearly dependent and from the proposition we get the following.

Corollary. *If $r > n$, then $\mathbf{0}$ is the only alternating r-linear function.*

When $r = n$, an alternating n-linear function is

$$D(\mathbf{h}_1, \ldots, \mathbf{h}_n) = \det(h_j^i), \tag{7.12}$$

namely, the determinant of the matrix that has $\mathbf{h}_1, \ldots, \mathbf{h}_n$ as its column vectors. We show at the end of this section that any alternating n-linear function M is a scalar multiple of D. The determinant is the alternating n-linear function with the additional property $D(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.

Let us now turn to the case of alternating r-linear functions, for any $r \leq n$. We know that the sum of two linear functions is a linear function. From this fact and the definition of multilinear function, the sum $M + N$ of two r-linear functions M and N is r-linear. If M and N are alternating, then $M + N$ is alternating. Similarly, if c is a scalar then cM is r-linear when M is r-linear and alternating when M is alternating. Thus, the set of alternating r-linear functions satisfies the axioms for a vector space (Appendix A.1). In later sections this space is denoted by $(E_r^n)^*$.

To obtain a representation of any alternating r-linear function M, in Proposition 7.2, we need to introduce some more notation. The letter λ denotes an r-tuple of integers,

$$\lambda = (i_1, \ldots, i_r),$$

where $1 \leq i_k \leq n$ for each $k = 1, \ldots, r$. There are n^r such r-tuples of integers. If $i_1 < \cdots < i_r$, then λ is called an *increasing* r-tuple. There are $\binom{n}{r}$ increasing r-tuples, where $\binom{n}{r} = n!/r!(n-r)!$ is a binomial coefficient. We write $\sum_\lambda$ for a sum over all r-tuples and $\sum_{[\lambda]}$ for a sum over all increasing r-tuples.

The following generalization of the Kronecker symbol δ_j^i (Section 1.2) is used. Let $\lambda = (i_1, \ldots, i_r)$, $\mu = (j_1, \ldots, j_r)$ be r-tuples of integers. Then $\delta_{j_l}^{i_k}$ is an element of an $r \times r$ matrix; and is 1 if $i_k = j_l$, 0 otherwise. Let

$$\delta_\mu^\lambda = \det(\delta_{j_l}^{i_k}).$$

The important properties of δ_μ^λ are:

(1) *If no integer is repeated in the r-tuple* λ *and* $\mu = \lambda$, *then* $\delta_\mu^\lambda = 1$.

In this case $i_k = j_l$ if and only if $k = l$. Hence $\delta_\mu^\lambda = \det(\delta_l^k) = 1$.

(2) *If no integer is repeated in the r-tuple* μ *and* λ *is obtained from* μ *by p interchanges, then* $\delta_\mu^\lambda = (-1)^p$.

Each interchange of elements of μ interchanges two column vectors of the matrix $(\delta_{j_l}^{i_k})$ and changes the sign of the determinant. Therefore (2) follows from (1).

(3) *In all other cases,* $\delta_\mu^\lambda = 0$.

If some integer is repeated in μ, then two column vectors of the matrix are the same and the determinant is 0. If the integers $j_1, \ldots, j_r$ are distinct and some i_k does not appear among them, then the kth row covector of the matrix is $\mathbf{0}$ and the determinant is 0.

Now let M be an alternating r-linear function. For brevity let us set

$$\omega_\lambda = \omega_{i_1 \cdots i_r}.$$

Sometimes we will still write $\omega_{i_1 \cdots i_r}$ rather than ω_λ, particularly when $r \leq 3$ or $r = n$. If λ is obtained from μ by one interchange, then $\omega_\lambda = -\omega_\mu$. In particular, $\omega_\lambda = 0$ if any integer is repeated. If λ is obtained from μ by p interchanges, then $\omega_\mu = (-1)^p \omega_\lambda = \delta_\mu^\lambda \omega_\lambda$.

If μ has no repetitions, then exactly one increasing λ is obtained from μ by interchanges. Hence for every μ,

$$\omega_\mu = \sum_{[\lambda]} \omega_\lambda \delta_\mu^\lambda, \tag{7.13}$$

where at most one term on the right-hand side is different from 0.

EXAMPLE 2. Let $n = 5, r = 4$. Then $\omega_{1231} = 0$ since 1 is repeated in the 4-tuple $\lambda = (1, 2, 3, 1)$. Since $(2, 3, 4, 5)$ is obtained from $(5, 4, 2, 3)$ by an odd number of interchanges, $\omega_{2345} = -\omega_{5423}$.

Let us now consider some particular alternating r-linear functions, which play an important role in later sections. For each r-tuple $\lambda = (i_1, \ldots, i_r)$ let $\mathbf{e}^\lambda$ be the function such that

$$\mathbf{e}^\lambda(\mathbf{h}_1, \ldots, \mathbf{h}_r) = \det(h_l^{i_k}) \tag{7.14}$$

for every r-tuple of vectors $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$. Note that the $r \times r$ matrix $(h_l^{i_k})$ is formed from rows $i_1, \ldots, i_r$ of the $n \times r$ matrix (h_l^i) that has $\mathbf{h}_1, \ldots, \mathbf{h}_r$ as

column vectors. By properties of determinants, $\mathbf{e}^\lambda$ is r-linear and alternating. When $r = 2$ and $\lambda = (i, j)$, then $\mathbf{e}^{ij}(\mathbf{h}, \mathbf{k}) = h^i k^j - h^j k^i$. Formula (7.14) agrees with the definition of $\mathbf{e}^{ij}$ in Section 7.1.

Taking, in particular, $\mathbf{h}_1, \ldots, \mathbf{h}_r$ to be standard basis vectors, $\mathbf{h}_l = \mathbf{e}_{j_l}$ for $l = 1, \ldots, r$, we obtain in (7.14) the matrix $(\delta^{i_k}_{j_l})$ whose determinant is δ^λ_μ. Thus

$$\mathbf{e}^\lambda(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r}) = \delta^\lambda_\mu. \tag{7.15}$$

If λ is obtained from μ by an interchange, then two row covectors of the matrix $(h^{i_k}_l)$ in (7.14) are interchanged. The determinant changes sign. Hence

$$\mathbf{e}^\lambda(\mathbf{h}_1, \ldots, \mathbf{h}_r) = -\mathbf{e}^\mu(\mathbf{h}_1, \ldots, \mathbf{h}_r)$$

for every r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$, which means that $\mathbf{e}^\lambda = -\mathbf{e}^\mu$. In particular, $\mathbf{e}^\lambda = \mathbf{0}$ if λ has any repetitions. If λ is obtained from μ by p interchanges, then $\mathbf{e}^\lambda = (-1)^p \mathbf{e}^\mu$.

Let us make the convention that $\mathbf{e}^\lambda = \mathbf{0}$ in case $r > n$. This is useful in defining the exterior product in Section 7.3.

Proposition 7.2 ($r \leq n$). *Let M be alternating and r-linear. Then*

$$M = \sum_{[\lambda]} \omega_\lambda \mathbf{e}^\lambda, \tag{7.16}$$

where the numbers ω_λ are defined by (7.11) *and the alternating r-linear functions $\mathbf{e}^\lambda$ by* (7.14). *The notation $[\lambda]$ indicates a sum over all increasing r-tuples $\lambda = (i_1, \ldots, i_r)$.*

PROOF. Let $\tilde{M}$ equal the right-hand side of (7.16). For each $\mu = (j_1, \ldots, j_r)$,

$$\tilde{M}(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r}) = \sum_{[\lambda]} \omega_\lambda \mathbf{e}^\lambda(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r}).$$

From (7.13) and (7.15),

$$\tilde{M}(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r}) = \sum_{[\lambda]} \omega_\lambda \delta^\lambda_\mu = \omega_\mu.$$

But M and $\tilde{M}$ are r-linear and have the same value ω_μ at $(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r})$ for each μ. By (7.10) $M = \tilde{M}$. □

Let us return to the special case $r = n$ mentioned earlier. The only increasing n-tuple of integers between 1 and n is $(1, 2, \ldots, n)$. By Proposition 7.2, any alternating, n-linear function has the form $M = c\mathbf{e}^{1 \cdots n}$, where $c = \omega_{1 \cdots n}$ is a scalar. In the notation of (7.12), $\mathbf{e}^{1 \cdots n} = D$.

PROBLEMS

1. Let $n = 5$. Find

$$\delta^{15}_{51}, \quad \delta^{25}_{51}, \quad \delta^{142}_{214}, \quad \delta^{525}_{255}, \quad \delta^{345}_{425}.$$

2. Let $n = 4$, $r = 3$, $\omega_{123} = 2$, $\omega_{134} = -1$, and $\omega_\lambda = 0$ for every other increasing triple λ. Find $M(\mathbf{e}_4, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3)$.

3. Show that:

(a) $\delta^\lambda_\mu = \delta^\mu_\lambda$.

(b) $\delta^\lambda_\nu = \sum_{[\mu]} \delta^\lambda_\mu \delta^\mu_\nu = \dfrac{1}{r!} \sum_\mu \delta^\lambda_\mu \delta^\mu_\nu$.

(c) $\mathbf{e}^\lambda = \sum_{[\mu]} \delta^\lambda_\mu \mathbf{e}^\mu = \dfrac{1}{r!} \sum_\mu \delta^\lambda_\mu \mathbf{e}^\mu$.

[*Hint for* (c): Use (b) and (7.15).]

4. Let M be r-linear, not necessarily alternating. Let ω_λ be as in (7.11) and $\tilde{\omega}_\mu = (1/r!) \sum_\lambda \omega_\lambda \delta^\lambda_\mu$. The function $M_1 = \sum_{[\lambda]} \tilde{\omega}_\lambda \mathbf{e}^\lambda$ is r-linear and alternating.
(a) Show that $M_1 = (1/r!) \sum_\mu \tilde{\omega}_\mu \mathbf{e}^\mu$. [*Hint*: $\tilde{\omega}_\mu = \sum_{[\lambda]} \tilde{\omega}_\lambda \delta^\lambda_\mu$; use Problem 3(c).]
(b) Show that if M is alternating, then $\tilde{\omega}_\mu = \omega_\mu$ and hence $M_1 = M$.

7.3 Multicovectors

Let us now introduce a different name and a different notation for alternating, multilinear functions.

Definition. A *multicovector of degree r* is an alternating r-linear function with domain the r-fold cartesian product $E^n \times \cdots \times E^n$.

For brevity, multicovectors of degree r are called *r-covectors*. From now on multicovectors are ordinarily denoted by the Greek letters $\boldsymbol{\omega}$ or $\boldsymbol{\zeta}$ rather than M as in Section 7.2.

We observed in Section 7.2 that the set of all r-covectors satisfies the axioms for a vector space. Let us denote this space by $(E^n_r)^*$. When $r > n$, its only element is $\mathbf{0}$ by Proposition 7.1. When $1 \leq r \leq n$, Proposition 7.2 states that if $\boldsymbol{\omega}$ is any r-covector, then

$$\boldsymbol{\omega} = \sum_{[\lambda]} \omega_\lambda \mathbf{e}^\lambda. \tag{7.17}$$

Therefore the r-covectors $\mathbf{e}^\lambda$ with λ increasing span $(E^n_r)^*$. These r-covectors form a linearly independent set (Problem 4), which is therefore a basis for $(E^n_r)^*$. It is called the *standard basis*. The number ω_λ is the *component* of $\boldsymbol{\omega}$ with respect to the basis element $\mathbf{e}^\lambda$. Since there are $\binom{n}{r}$ increasing r-tuples of integers between 1 and n, $(E^n_r)^*$ has dimension $\binom{n}{r}$.

Every 1-linear function is alternating. Thus a 1-covector is just a covector, and $(E^n_1)^* = (E^n)^*$ is the dual space of E^n. If we identify the 1-tuple (i) with i, then the standard basis 1-covectors $\mathbf{e}^1, \ldots, \mathbf{e}^n$ are just those introduced in Section 3.2. As in previous chapters, we use the letters $\mathbf{a}$, $\mathbf{b}$ to denote 1-covectors, and $\mathbf{a} \cdot \mathbf{h} = \sum_{i=1}^n a_i h^i$ for the value of the 1-covector (i.e., linear function) $\mathbf{a}$ at $\mathbf{h}$. For $r = 2$, Formula (7.17) becomes (7.5), already derived in Section 7.1. We define in Section 7.5 a space E^n_r, whose dual turns out to be $(E^n_r)^*$. The elements of E^n_r are called *multivectors*.

If $r = n$, then we showed in Section 7.2 that the n-covector $\mathbf{e}^{1\cdots n}$ is essentially the determinant function. Its value at $(\mathbf{h}_1, \ldots, \mathbf{h}_n)$ is $\det(h_i^k)$, which is the determinant of the $n \times n$ matrix with column vectors $\mathbf{h}_1, \ldots, \mathbf{h}_n$. Since $(E_n^n)^*$ is one-dimensional, every n-covector has the form $\boldsymbol{\omega} = c\mathbf{e}^{1\cdots n}$ where $c = \boldsymbol{\omega}_{1\cdots n}$.

EXAMPLE 1. Let $n = 5, r = 3$, and $\boldsymbol{\omega} = 6\mathbf{e}^{145} - 2\mathbf{e}^{431} - \mathbf{e}^{514}$. The increasing triple (1, 4, 5) is obtained from (5, 1, 4) by an even number of interchanges. Hence $\mathbf{e}^{514} = \mathbf{e}^{145}$. The increasing triple (1, 3, 4) is obtained from (4, 3, 1) by one interchange. Hence $\mathbf{e}^{431} = -\mathbf{e}^{134}$, and $\boldsymbol{\omega} = 2\mathbf{e}^{134} + 5\mathbf{e}^{145}$. This expresses $\boldsymbol{\omega}$ as a linear combination of the standard basis 3-covectors. The components of $\boldsymbol{\omega}$ are $\omega_{134} = 2$, $\omega_{145} = 5$, and $\omega_\lambda = 0$ for every other increasing triple λ.

Products

In $(E_r^n)^*$ we define the *euclidean inner product*

$$\boldsymbol{\omega} \cdot \boldsymbol{\zeta} = \sum_{[\lambda]} \omega_\lambda \zeta_\lambda$$

and set $|\boldsymbol{\omega}|^2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega}$. The standard basis elements are orthonormal with respect to this inner product.

Another important product is the *exterior product*, denoted by the symbol $\wedge$. The exterior product of an r-covector and an s-covector is an $(r + s)$-covector, defined as follows: If

$$\lambda = (i_1, \ldots, i_r), \qquad \nu = (j_1, \ldots, j_s),$$

let us write λ, ν for the $(r + s)$-tuple

$$(i_1, \ldots, i_r, j_1, \ldots, j_s).$$

Definition. Let $1 \leq r \leq n$, $1 \leq s \leq n$. If λ and ν are increasing, then

$$\mathbf{e}^\lambda \wedge \mathbf{e}^\nu = \mathbf{e}^{\lambda,\nu}. \tag{7.18}$$

If $\boldsymbol{\omega}$ is an r-covector and $\boldsymbol{\zeta}$ is an s-covector, with respective components $\omega_\lambda, \zeta_\nu$, then

$$\boldsymbol{\omega} \wedge \boldsymbol{\zeta} = \sum_{[\lambda][\nu]} \omega_\lambda \zeta_\nu \mathbf{e}^\lambda \wedge \mathbf{e}^\nu.$$

Note that if $r + s > n$ then $\boldsymbol{\omega} \wedge \boldsymbol{\zeta}$, being an $(r + s)$-covector, must be $\mathbf{0}$.

EXAMPLE 2. Let $n = 4$. Then $\mathbf{e}^{12} \wedge \mathbf{e}^{34} = \mathbf{e}^{1234}$.

$$\mathbf{e}^3 \wedge \mathbf{e}^{124} = \mathbf{e}^{3124} = \mathbf{e}^{1234}, \qquad \mathbf{e}^{14} \wedge \mathbf{e}^{24} = \mathbf{e}^{1424} = \mathbf{0},$$

since the integer 4 is repeated.

Proposition 7.3. *The exterior product has the following properties:*

(1) $(\omega + \zeta) \wedge \eta = (\omega \wedge \eta) + (\zeta \wedge \eta)$.
(2) $(c\omega) \wedge \zeta = c(\omega \wedge \zeta)$.
(3) $\zeta \wedge \omega = (-1)^{rs}\omega \wedge \zeta$, *if* ω *has degree* r *and* ζ *has degree* s.
(4) $(\zeta \wedge \omega) \wedge \eta = \zeta \wedge (\omega \wedge \eta)$.

PROOF. The proof of (1) and (2) is almost immediate from the definition and is left to the reader (Problem 5). To prove (3),

$$v, \lambda = (j_1, \ldots, j_s, i_1, \ldots, i_r).$$

By s interchanges we may bring i_1 to the left past $j_1, \ldots, j_s$. Similarly, s interchanges bring each of $i_2, \ldots, i_r$ in turn past $j_1, \ldots, j_s$. Thus λ, v is obtained from v, λ by rs interchanges, and $\mathbf{e}^{v,\lambda} = (-1)^{rs}\mathbf{e}^{\lambda,v}$. Hence

$$\zeta \wedge \omega = \sum_{[v][\lambda]} \zeta_v \omega_\lambda \mathbf{e}^{v,\lambda} = (-1)^{rs} \sum_{[\lambda][v]} \omega_\lambda \zeta_v \mathbf{e}^{\lambda,v},$$

which proves (3).

Let us first prove the associative law (4) for basis elements. Let $\lambda = (i_1, \ldots, i_r)$, $v = (j_1, \ldots, j_s)$, and $\rho = (k_1, \ldots, k_t)$ be increasing r-, s-, and t-tuples, respectively. Let

$$\lambda, v, \rho = (i_1, \ldots, i_r, j_1, \ldots, j_s, k_1, \ldots, k_t).$$

Let us show that

$$\mathbf{e}^{\lambda,v,\rho} = (\mathbf{e}^\lambda \wedge \mathbf{e}^v) \wedge \mathbf{e}^\rho.$$

If some integer is repeated in the $(r + s)$-tuple λ, v, then both sides are $\mathbf{0}$. If no integer is repeated, then

$$(\mathbf{e}^\lambda \wedge \mathbf{e}^v) \wedge \mathbf{e}^\rho = \mathbf{e}^{\lambda,v} \wedge \mathbf{e}^\rho = (-1)^p \mathbf{e}^\tau \wedge \mathbf{e}^\rho = (-1)^p \mathbf{e}^{\tau,\rho},$$

where τ is an increasing $(r + s)$-tuple obtained from λ, v by p interchanges. These same p interchanges change the $(r + s + t)$-tuple λ, v, ρ into τ, ρ. Hence

$$\mathbf{e}^{\lambda,v,\rho} = (-1)^p \mathbf{e}^{\tau,\rho} = (\mathbf{e}^\lambda \wedge \mathbf{e}^v) \wedge \mathbf{e}^\rho.$$

Similarly $\mathbf{e}^{\lambda,v,\rho} = \mathbf{e}^\lambda \wedge (\mathbf{e}^v \wedge \mathbf{e}^\rho)$, and hence

$$(\mathbf{e}^\lambda \wedge \mathbf{e}^v) \wedge \mathbf{e}^\rho = \mathbf{e}^\lambda \wedge (\mathbf{e}^v \wedge \mathbf{e}^\rho). \tag{7.19}$$

From this formula it is a straightforward matter to obtain (4) (Problem 6). □

If either r or s is even, then the exterior product is commutative. If $r = s = 1$, we have $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$.

The exterior product of any finite number of multicovectors is defined by induction. Using (7.18) repeatedly, we find that if λ is increasing,

$$\mathbf{e}^\lambda = \mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_r}. \tag{7.20}$$

Since both sides of (7.18) change sign under interchanges in λ or ν, formula (7.18) also holds for nonincreasing r-tuples. Thus (7.20) is valid regardless of whether λ is increasing.

EXAMPLE 3. Let $n = 5$. Then

$$(\mathbf{e}^1 + 3\mathbf{e}^4) \wedge (\mathbf{e}^{24} - 2\mathbf{e}^{15}) = \mathbf{e}^{124} + 3\mathbf{e}^{424} - 2\mathbf{e}^{115} - 6\mathbf{e}^{415} = \mathbf{e}^{124} + 6\mathbf{e}^{145}.$$
$$\mathbf{e}^2 \wedge (3\mathbf{e}^1 - 2\mathbf{e}^3) \wedge \mathbf{e}^5 \wedge \mathbf{e}^3 = (3\mathbf{e}^{21} - 2\mathbf{e}^{23}) \wedge \mathbf{e}^{53} = 3\mathbf{e}^{2153} = 3\mathbf{e}^{1235}.$$

Let us next give two useful formulas, which hold if ω is the exterior product of 1-covectors.

Proposition 7.4. *Let* $\omega = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$. *Then*

$$\omega(\mathbf{h}_1, \ldots, \mathbf{h}_r) = \det(\mathbf{a}^k \cdot \mathbf{h}_l) \tag{7.21}$$

for every r-tuple of vectors $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$. *For every r-tuple of integers* $\mu = (j_1, \ldots, j_r)$, $1 \leq j_l \leq n$,

$$\omega_\mu = \det(a^k_{j_l}). \tag{7.22}$$

Note that the matrices whose determinants appear on the right side of (7.21) and (7.22) are $r \times r$. The second of them is obtained by deleting all columns except $j_1, \ldots, j_r$ in the $r \times n$ matrix whose row covectors are $\mathbf{a}^1, \ldots, \mathbf{a}^r$.

PROOF. By (7.14) and (7.20), Formula (7.21) is valid if $\mathbf{a}^1, \ldots, \mathbf{a}^r$ are standard basis covectors, $\mathbf{a}^k = \mathbf{e}^{i_k}$. Formula (7.21) then holds in general, since both sides of (7.21) are multilinear in $\mathbf{a}^1, \ldots, \mathbf{a}^r$. If $\mathbf{h}_l = \mathbf{e}_{j_l}$, then $\omega_\mu = \omega(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r})$ and $\mathbf{a}^k \cdot \mathbf{h}_l = a^k_{j_l}$. Thus (7.22) follows from (7.21). □

**Remarks.* The exterior product has been defined in terms of the standard bases. It is not clear that it is "coordinate free," in other words, that the same exterior product would be obtained starting from different bases. However, let us add Formula (7.21) to the list [(1)–(4)] of properties in Proposition 7.3. Then $\wedge$ is the only product with these five properties. To see this, we first observe that Formula (7.20) is a special case of (7.21), upon taking $\mathbf{a}^k = \mathbf{e}^{i_k}$ and using (7.14). Moreover, (7.18) is a consequence of (7.20) and the associative law (4). Once the product is known for basis elements, Properties (1) and (2) determine it in general. Thus $\wedge$ is the only product with Properties (1) through (4) and (7.21). In fact, (3) can be omitted from the list since it follows from the other four. Since none of these five properties refers to bases, the exterior product is coordinate free.

**Note about terminology.* A multilinear function M of degree r and domain $E^n \times \cdots \times E^n$ is often called a *covariant tensor of rank r*. An r-covector is then called an *alternating covariant tensor of rank r*.

The sum of an r-covector and an s-covector has been defined only when $r = s$. However, one may form the direct sum

$$(A^n)^* = (E_0^n)^* \oplus (E_1^n)^* \oplus \cdots \oplus (E_r^n)^* \oplus \ldots,$$

where we agree that $(E_0^n)^*$ is the scalar field. The exterior product induces a product in $(A^n)^*$, which is then an algebra over the real numbers. This algebra is called the *exterior algebra* of $(E^n)^*$ [4]. $(A^n)^*$ is sometimes called the *covariant Grassmann algebra* or the *covariant alternating tensor algebra* of E^n.

PROBLEMS

1. Write down the standard basis for $(E_r^4)^*$ for each $r = 1, 2, 3, 4$. Find all products $\mathbf{e}^\lambda \wedge \mathbf{e}^\nu$ where $\lambda = (i)$ and $\nu = (j, k, l)$ is an increasing triple.

2. Let $n = 3$. Simplify:
   (a) $(2\mathbf{e}^1 - \mathbf{e}^2) \wedge (3\mathbf{e}^2 + \mathbf{e}^3)$.
   (b) $\mathbf{e}^{21} \wedge \mathbf{e}^{23}$.
   (c) $(\mathbf{e}^1 - \mathbf{e}^2 + 3\mathbf{e}^3) \wedge \mathbf{e}^{21}$.
   (d) $(\mathbf{e}^{23} + \mathbf{e}^{31}) \wedge (5\mathbf{e}^1 - \mathbf{e}^2)$.

3. Let $n = 5$. Simplify:
   (a) $\mathbf{e}^{253} \wedge (\mathbf{e}^{14} + \mathbf{e}^{42})$.
   (b) $(\mathbf{e}^2 + \mathbf{e}^5) \wedge \mathbf{e}^{31} \wedge (\mathbf{e}^5 - \mathbf{e}^4)$.

4. Show that if $\sum_{[\lambda]} c_\lambda \mathbf{e}^\lambda = \mathbf{0}$, then $c_\lambda = 0$ for every increasing λ. [*Hint*: See (7.15).]

5. Prove (1) and (2) of Proposition 7.3.

6. Prove the associative law (4) of Proposition 7.3, using (1), (2), and (7.19).

7. Show that $\boldsymbol{\omega} \wedge \boldsymbol{\zeta} \wedge \boldsymbol{\eta} = -\boldsymbol{\eta} \wedge \boldsymbol{\zeta} \wedge \boldsymbol{\omega}$ if $\boldsymbol{\omega}$ has degree r, $\boldsymbol{\eta}$ has degree t, and both r, t are odd.

7.4 Differential forms

In Section 6.3 a differential form of degree 1 was defined as a covector-valued function. It was shown that any such differential form $\boldsymbol{\omega}$ is a linear combination of $dx^1, \ldots, dx^n$,

$$\boldsymbol{\omega} = \sum_{i=1}^{n} \omega_i \, dx^i,$$

where the coefficients $\omega_1, \ldots, \omega_n$ are real-valued functions. For $r > 2$ a differential form of degree r is supposed to be an alternating polynomial of degree r in $dx^1, \ldots, dx^n$ with coefficients ω_λ which are real valued functions. This idea expressed more precisely by the following definition.

Definition. A *differential form of degree r* is a function $\boldsymbol{\omega}$ with domain $D \subset E^n$ and values in $(E_r^n)^*$. The value of $\boldsymbol{\omega}$ at $\mathbf{x}$ is denoted by $\boldsymbol{\omega}(\mathbf{x})$.

The values of $\boldsymbol{\omega}$ are r-covectors. The Greek letters $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$ used in Section 7.3 to denote r-covectors are now used to denote differential forms. The context indicates clearly which is intended.

For brevity we say "*r-form*" instead of "differential form of degree r." It is convenient to call any real valued function f a *0-form*. If $r > n$, then the only r-form is the one which has the value $\mathbf{0}$ for every $\mathbf{x} \in D$. We also use $\mathbf{0}$ to denote this r-form.

Let $\boldsymbol{\omega}$ be an r-form and $\boldsymbol{\zeta}$ be an s-form, with the same domain D. The *exterior product* $\boldsymbol{\omega} \wedge \boldsymbol{\zeta}$ is the $(r + s)$-form defined by

$$(\boldsymbol{\omega} \wedge \boldsymbol{\zeta})(\mathbf{x}) = \boldsymbol{\omega}(\mathbf{x}) \wedge \boldsymbol{\zeta}(\mathbf{x})$$

for every $\mathbf{x} \in D$. Similarly, if f is a real valued function, then $f\boldsymbol{\omega}$ is the r-form such that

$$(f\boldsymbol{\omega})(\mathbf{x}) = f(\mathbf{x})\boldsymbol{\omega}(\mathbf{x})$$

for every $\mathbf{x} \in D$.

Theorem 7.1. *Let $\boldsymbol{\omega}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\eta}$ be differential forms, and f a real valued function, all with the same domain D. Then*

(1) $(\boldsymbol{\omega} + \boldsymbol{\zeta}) \wedge \boldsymbol{\eta} = \boldsymbol{\omega} \wedge \boldsymbol{\eta} + \boldsymbol{\zeta} \wedge \boldsymbol{\eta}$.
(2) $(f\boldsymbol{\omega}) \wedge \boldsymbol{\zeta} = f(\boldsymbol{\omega} \wedge \boldsymbol{\zeta})$.
(3) $\boldsymbol{\zeta} \wedge \boldsymbol{\omega} = (-1)^{rs}\boldsymbol{\omega} \wedge \boldsymbol{\zeta}$, *if $\boldsymbol{\omega}$ has degree r and $\boldsymbol{\zeta}$ has degree s.*
(4) $(\boldsymbol{\zeta} \wedge \boldsymbol{\omega}) \wedge \boldsymbol{\eta} = \boldsymbol{\zeta} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\eta})$.

PROOF. This is immediate from the definitions above, and Proposition 7.3. □

We recall that dx^i is the 1-form with constant value $\mathbf{e}^i$. Similarly, for any r-tuple $\lambda = (i_1, \ldots, i_r)$ the r-form $dx^{i_1} \wedge \cdots \wedge dx^{i_r}$ has constant value $\mathbf{e}^\lambda$. Hence if $\boldsymbol{\omega}$ is an r-form, then

$$\boldsymbol{\omega} = \sum_{[\lambda]} \omega_\lambda \, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \tag{7.23}$$

where the value of ω_λ at $\mathbf{x}$ is $\omega_\lambda(\mathbf{x})$. Using Problem 4, Section 7.2, one can also write

$$\boldsymbol{\omega} = \frac{1}{r!} \sum_{\lambda} \omega_\lambda \, dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

We say that an r-form $\boldsymbol{\omega}$ is of *class* $C^{(q)}$ if the functions ω_λ in (7.23) are of class $C^{(q)}$.

We recall that if f is a 0-form of class $C^{(1)}$, then df is the 1-form

$$df = f_1 \, dx^1 + \cdots + f_n \, dx^n,$$

where $f_1, \ldots, f_n$ are the partial derivatives. In particular, if $\boldsymbol{\omega}$ is an r-form of class $C^{(1)}$, then ω_λ is a 0-form of class $C^{(1)}$ and $d\omega_\lambda$ is defined.

Definition. Let $\boldsymbol{\omega}$ be an r-form of class $C^{(1)}$. The *exterior differential* $d\boldsymbol{\omega}$ is the $(r + 1)$-form defined by the formula

$$d\boldsymbol{\omega} = \sum_{[\lambda]} d\omega_\lambda \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}. \tag{7.24}$$

EXAMPLE 1. Let $n = 3, r = 2$, and $\omega = f\,dx \wedge dy$, where f is of class $C^{(1)}$ on $D \subset E^3$. Then

$$d\omega = df \wedge dx \wedge dy = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) \wedge dx \wedge dy,$$

$$d\omega = \frac{\partial f}{\partial z}dz \wedge dx \wedge dy = \frac{\partial f}{\partial z}dx \wedge dy \wedge dz.$$

Problem 2 generalizes this result.

EXAMPLE 2. If $r = n$, then

$$\omega = f\,dx^1 \wedge \cdots \wedge dx^n$$

where $f = \omega_{1\ldots n}$. Since $d\omega$ is an $(n+1)$-form, $d\omega = \mathbf{0}$.

Theorem 7.2. *The exterior differential has the following properties:*

(1) $d(\omega + \zeta) = d\omega + d\zeta$, *if* ω *and* ζ *are r-forms of class* $C^{(1)}$.
(2) $d(\omega \wedge \zeta) = d\omega \wedge \zeta + (-1)^r\omega \wedge d\zeta$, *if* ω *is an r-form and* ζ *is an s-form, both of class* $C^{(1)}$.
(3) $d(d\omega) = \mathbf{0}$ *if* ω *is an r-form of class* $C^{(2)}$.

[If $r = 0$, we agree that $f \wedge \zeta = f\zeta$. Similarly, if $s = 0$ then $\omega \wedge f = f\omega$. The theorem remains true if $r = 0$ or $s = 0$.]

PROOF. The coefficients of $\omega + \zeta$ in (7.23) are $\omega_\lambda + \zeta_\lambda$ and $d(\omega_\lambda + \zeta_\lambda) = d\omega_\lambda + d\zeta_\lambda$. Therefore (1) holds. Similarly, $d(c\omega) = c\,d\omega$.

To prove (2) let us for brevity set

$$\mathbf{E}^\lambda = dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

This is the constant differential form such that $\mathbf{E}^\lambda(\mathbf{x}) = \mathbf{e}^\lambda$ for all $\mathbf{x}$. Let us first show that

$$(*) \qquad d(f\mathbf{E}^\lambda \wedge \mathbf{E}^\nu) = df \wedge \mathbf{E}^\lambda \wedge \mathbf{E}^\nu.$$

If any integer is repeated in the $(r+s)$-tuple λ, ν then both sides are $\mathbf{0}$. Otherwise, $\mathbf{E}^\lambda \wedge \mathbf{E}^\nu = (-1)^p\mathbf{E}^\tau$ where τ is increasing. By definition, $d(f\mathbf{E}^\tau) = df \wedge \mathbf{E}^\tau$. Multiplying both sides by $(-1)^p$ we get (*). Now

$$\omega \wedge \zeta = \sum_{[\lambda][\nu]} \omega_\lambda \zeta_\nu \mathbf{E}^\lambda \wedge \mathbf{E}^\nu.$$

By the ordinary product rule

$$d(\omega_\lambda \zeta_\nu) = \zeta_\nu\,d\omega_\lambda + \omega_\lambda\,d\zeta_\nu.$$

By (*) with $f = \omega_\lambda\zeta_\nu$,

$$d(\omega_\lambda\zeta_\nu\mathbf{E}^\lambda \wedge \mathbf{E}^\nu) = (\zeta_\nu\,d\omega_\lambda + \omega_\lambda\,d\zeta_\nu) \wedge \mathbf{E}^\lambda \wedge \mathbf{E}^\nu$$

Since $d\zeta_\nu$ has degree 1 and $\mathbf{E}^\lambda$ degree r, by (3) of Theorem 7.1

$$d\zeta_\nu \wedge \mathbf{E}^\lambda = (-1)^r\mathbf{E}^\lambda \wedge d\zeta_\nu.$$

The scalar-valued function ζ_ν commutes with any differential form. Hence

(**) $d(\omega_\lambda \zeta_\nu \mathbf{E}^\lambda \wedge \mathbf{E}^\nu) = (d\omega_\lambda \wedge \mathbf{E}^\lambda) \wedge (\zeta_\nu \mathbf{E}^\nu) + (-1)^r(\omega_\lambda \mathbf{E}^\lambda) \wedge (d\zeta_\nu \wedge \mathbf{E}^\nu)$.

Using (1),

$$d(\boldsymbol{\omega} \wedge \boldsymbol{\zeta}) = \sum_{[\lambda][\nu]} d(\omega_\lambda \zeta_\nu \mathbf{E}^\lambda \wedge \mathbf{E}^\nu),$$

while

$$\sum_{[\lambda][\nu]} (d\omega_\lambda \wedge \mathbf{E}^\lambda) \wedge (\zeta_\nu \mathbf{E}^\nu) = \left[\sum_{[\lambda]} d\omega_\lambda \wedge \mathbf{E}^\lambda\right] \wedge \left[\sum_{[\nu]} \zeta_\nu \mathbf{E}^\nu\right] = d\boldsymbol{\omega} \wedge \boldsymbol{\zeta}.$$

Similarly,

$$(-1)^r \sum_{[\lambda][\nu]} (\omega_\lambda \mathbf{E}^\lambda) \wedge (d\zeta_\nu \wedge \mathbf{E}^\nu) = (-1)^r \boldsymbol{\omega} \wedge d\boldsymbol{\zeta},$$

which proves (2).

If f is of class $C^{(2)}$, then from (7.8)

$$d(df) = \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j = \mathbf{0}.$$

The form $\mathbf{E}^\lambda$ has constant coefficients and hence $d\mathbf{E}^\lambda = \mathbf{0}$. Using the product rule (2), $d(df \wedge \mathbf{E}^\lambda) = \mathbf{0}$. Taking $f = \omega_\lambda$ and using (1),

$$d(d\boldsymbol{\omega}) = d\left(\sum_{[\lambda]} d\omega_\lambda \wedge \mathbf{E}^\lambda\right) = \sum_{[\lambda]} d(d\omega_\lambda \wedge \mathbf{E}^\lambda) = \mathbf{0}. \qquad \square$$

Definition. An r-form $\boldsymbol{\omega}$ is *closed* if $d\boldsymbol{\omega} = \mathbf{0}$. If $\boldsymbol{\omega} = d\boldsymbol{\zeta}$ for some $(r-1)$-form $\boldsymbol{\zeta}$, then $\boldsymbol{\omega}$ is an *exact* r-form.

If $r = 1$, these definitions agree with those given in Section 6.3. If $\boldsymbol{\omega}$ is exact and $\boldsymbol{\zeta}$ can be chosen to be of class $C^{(2)}$, then $d\boldsymbol{\omega} = d(d\boldsymbol{\zeta}) = \mathbf{0}$. Hence $\boldsymbol{\omega}$ is closed. Poincaré's lemma states that if domain D is star-shaped, then conversely any closed form $\boldsymbol{\omega}$ is exact. This is proved in Section 8.10.

**Remark.* The exterior differential d is uniquely determined by Properties (1), (2), (3), and the following property.

(4) *For $r = 0$, df agrees with its definition in Section* 6.3.

Let d' also have these four properties. Then $d'\mathbf{E}^\lambda = d'(d'x^{i_1} \wedge \mathbf{E}^\mu)$, where $\mu = (i_2, \ldots, i_r)$. But $dx^i = d'x^i$ by (4) since dx^i stands for the differential of the coordinate function X^i. Using (2), (3), and induction on r, $d'\mathbf{E}^\lambda = \mathbf{0}$. Using (2) and (4), $d'(f\mathbf{E}^\lambda) = df \wedge \mathbf{E}^\lambda$. Using (1),

$$\begin{aligned} d'\boldsymbol{\omega} &= d'\left(\sum_{[\lambda]} \omega_\lambda \mathbf{E}^\lambda\right) = \sum_{[\lambda]} d'(\omega_\lambda \mathbf{E}^\lambda) \\ &= \sum_{[\lambda]} d\omega_\lambda \wedge \mathbf{E}^\lambda. \end{aligned}$$

Thus $d'\omega = d\omega$ for every ω of class $C^{(1)}$. In particular, this proves that the exterior differential d is "coordinate free."

PROBLEMS

Assume that all forms that appear are of class $C^{(1)}$.

1. Find the exterior differential of:
   (a) $\cos(xy^2)dx \wedge dz$.
   (b) $x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$.

2. Let P, Q, R have domain $D \subset E^3$. Show that

$$d(P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy) = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dx \wedge dy \wedge dz.$$

3. (a) Generalize Problem 1(b) to find an $(n-1)$-form ζ such that $d\zeta = dx^1 \wedge \cdots \wedge dx^n$.
   (b) Find an $(r-1)$-form ζ^λ such that $d\zeta^\lambda = \mathbf{E}^\lambda$.
   (c) Show that if the coefficients ω_λ in (7.23) are constant functions, then ω is exact.

4. (a) Show that if ω and ζ are closed differential forms, then $\omega \wedge \zeta$ is closed.
   (b) Show that if ω is closed and ζ is exact, then $\omega \wedge \zeta$ is exact.

5. Find the exterior differential of:
   (a) $d\omega \wedge \zeta - \omega \wedge d\zeta$.
   (b) $d\omega \wedge \zeta \wedge \eta + \omega \wedge d\zeta \wedge \eta + \omega \wedge \zeta \wedge d\eta$, if ω and ζ are of even degree.

6. A function f is an *integrating factor* for a 1-form ω if $f(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in D$ and $f\omega$ is exact. Show that if ω has an integrating factor then $\omega \wedge d\omega = \mathbf{0}$.

7. Show that if ω is a 2-form, then

$$d\omega = \sum_{i<j<k}\left(\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{ki}}{\partial x^j} + \frac{\partial \omega_{jk}}{\partial x^i}\right)dx^i \wedge dx^j \wedge dx^k.$$

8. Let $\omega^1, \ldots, \omega^p$ be 1-forms such that $\omega^i = \sum_{j=1}^p f^i_j\, dg^j$, $i = 1, \ldots, p$. Assume that the functions f^i_j are of class $C^{(1)}$, the g^j are of class $C^{(2)}$, and that the 1-covectors $\omega^1(\mathbf{x}), \ldots, \omega^p(\mathbf{x})$ are linearly independent for every $\mathbf{x} \in D$. Find 1-forms θ^i_j such that $d\omega^i = \sum_{j=1}^p \theta^i_j \wedge \omega^j$. [*Hint*: The $p \times p$ matrix $(f^i_j(\mathbf{x}))$ must be nonsingular.]

Note. Conversely if $d\omega^i$ is a linear combination of $\omega^1, \ldots, \omega^p$ with coefficients 1-forms θ^i_j, then locally functions f^i_j, g^j as above can be found. This result is called the *Frobenius integration theorem* and has important applications in geometry and differential equations [8, p. 97]. We give a proof for $p = 1$ in Section 7.10.

7.5 Multivectors

Let us now define objects called *multivectors of degree r*, or, for brevity, *r-vectors*. They have a role dual to the r-covectors. Formally, one passes from multicovectors to multicovectors by merely exchanging subscripts and superscripts. Multivectors are usually denoted by $\alpha, \beta, \ldots$. We are interested

mainly in multivectors which can be written as the exterior product of vectors, namely, $\alpha = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$. Such multivectors are called *decomposable*. The concept of orientation is defined in terms of decomposable multivectors. Moreover, there are simple formulas for the measure of an r-parallelepiped or an r-simplex, in terms of the norm of a corresponding decomposable r-vector.

The case $r = 2$

By 1-*vector* we mean a vector $\mathbf{h} \in E^n$. To motivate the general treatment of r-vectors, let us begin with a discussion of 2-vectors. They are defined as alternating bilinear functions on $(E^n)^* \times (E^n)^*$. This is a special case of the definition of r-vector given below. Let E_2^n denote the space of 2-vectors. In exactly the same way as in Section 7.1, one introduces 2-vectors $\mathbf{e}_{ij}$ such that $\mathbf{e}_{ji} = -\mathbf{e}_{ij}$, $\mathbf{e}_{ii} = \mathbf{0}$, and the $n(n-1)/2$ 2-vectors $\mathbf{e}_{ij}$ for $i < j$ form a basis for E_2^n. This is called the *standard basis for* E_2^n; we see below that the basis for $(E_2^n)^*$ of 2-covectors $\mathbf{e}^{ij}$ is dual to it.

Let us denote the component of a 2-vector α with respect to $\mathbf{e}_{ij}$ by α^{ij}. Thus

$$\alpha = \sum_{i<j} \alpha^{ij} \mathbf{e}_{ij}. \tag{7.25}$$

This formula is dual to (7.5); note that subscripts and superscripts have been exchanged.

The exterior product $\mathbf{h} \wedge \mathbf{k}$ of vectors $\mathbf{h}, \mathbf{k}$ is defined, as in Section 7.1, by agreeing that $\mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_{ij}$, and that $\wedge$ distributes with addition and scalar multiplication. We are particularly interested in decomposable 2-vectors, namely, those that can be written as $\alpha = \mathbf{h} \wedge \mathbf{k}$ for suitable vectors $\mathbf{h}, \mathbf{k}$. The reason is that several interesting geometric quantities are associated with a decomposable 2-vector.

Let $\mathbf{h}$ and $\mathbf{k}$ be linearly independent vectors, and P the 2-dimensional vector subspace of E^n that they span:

$$P = \{\mathbf{x} = s\mathbf{h} + t\mathbf{k}, s, t \text{ real}\}.$$

We call the pair $(\mathbf{h}, \mathbf{k})$, in the given order, a *frame* for P. Suppose that $(\mathbf{h}', \mathbf{k}')$ is another frame for P. Then, for suitable real numbers s, t, u, v

$$\mathbf{h}' = s\mathbf{h} + t\mathbf{k}, \qquad \mathbf{k}' = u\mathbf{h} + v\mathbf{k},$$

$$\mathbf{h}' \wedge \mathbf{k}' = (s\mathbf{h} + t\mathbf{k}) \wedge (u\mathbf{h} + v\mathbf{k}).$$

Since $\mathbf{k} \wedge \mathbf{h} = -\mathbf{h} \wedge \mathbf{k}$, and $\mathbf{h} \wedge \mathbf{h} = \mathbf{k} \wedge \mathbf{k} = \mathbf{0}$,

$$\mathbf{h}' \wedge \mathbf{k}' = c\mathbf{h} \wedge \mathbf{k}, \tag{7.26}$$

where $c = sv - tu$. Thus the 2-vector $\mathbf{h}' \wedge \mathbf{k}'$ is a scalar multiple of $\mathbf{h} \wedge \mathbf{k}$. We must have $c \neq 0$, since $\mathbf{h}', \mathbf{k}'$ are linearly independent. It is shown below (Theorem 7.3) that, conversely, if (7.26) holds for some $c \neq 0$, then $(\mathbf{h}, \mathbf{k})$ and

$(\mathbf{h}', \mathbf{k}')$ are frames for the same 2-dimensional subspace P. Thus, although P has many frames $(\mathbf{h}, \mathbf{k})$, the associated 2-vectors $\mathbf{h} \wedge \mathbf{k}$ are determined by P up to a scalar multiple.

The concept of orientation is essential in defining integrals of differential forms in Section 8.7. As a preliminary step, we define orientation for vector subspaces of E^n in the present section. Any frame $(\mathbf{h}, \mathbf{k})$ determines an orientation for the 2-dimensional subspace P spanned by $(\mathbf{h}, \mathbf{k})$. Let us agree that another frame $(\mathbf{h}', \mathbf{k}')$ determines the same orientation for P as $(\mathbf{h}, \mathbf{k})$ if $c > 0$ in (7.26), and the opposite orientation if $c < 0$. This is a somewhat clumsy way to define orientation. We give in Formula (7.29) below an equivalent definition in terms of 2-vectors, which is the one used in this book.

EXAMPLE 1. Show that $(\mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3)$ is a frame for the same 2-dimensional vector subspace of E^3 as $(2\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3, 4\mathbf{e}_1 + \mathbf{e}_2 + 11\mathbf{e}_3)$. Calculating the exterior products, we get

$$(\mathbf{e}_1 + 3\mathbf{e}_3) \wedge (\mathbf{e}_2 - \mathbf{e}_3) = \mathbf{e}_{12} - \mathbf{e}_{13} - 3\mathbf{e}_{23},$$
$$(2\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3) \wedge (4\mathbf{e}_1 + \mathbf{e}_2 + 11\mathbf{e}_3) = -2\mathbf{e}_{12} + 2\mathbf{e}_{13} + 6\mathbf{e}_{23}.$$

The second 2-vector is -2 times the first. These two frames determine opposite orientations for the 2-dimensional subspace P that they span.

There is a convenient formula for areas of parallelograms and triangles, in terms of 2-vectors. In E_2^n we introduce the euclidean inner product and norm:

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \sum_{i<j} \alpha^{ij}\beta^{ij},$$

$$|\boldsymbol{\alpha}| = (\boldsymbol{\alpha} \cdot \boldsymbol{\alpha})^{1/2} = \left[\sum_{i<j} (\alpha^{ij})^2\right]^{1/2}.$$

Let K be a parallelogram with vertices $\mathbf{x}_0, \mathbf{x}_0 + \mathbf{h}, \mathbf{x}_0 + \mathbf{k}, \mathbf{x}_0 + \mathbf{h} + \mathbf{k}$ (see Figure 7.1), Section 7.1. Then the area of K is defined as

$$V_2(K) = |\mathbf{h} \wedge \mathbf{k}|. \tag{7.27}$$

To see that this agrees with the usual definition of area when $K \subset E^2$, by translating and rotating axes we may take $\mathbf{x}_0 = \mathbf{0}, \mathbf{h} = h^1\mathbf{e}_1, \mathbf{k} = k^1\mathbf{e}_1 + k^2\mathbf{e}_2$. Then

$$\mathbf{h} \wedge \mathbf{k} = h^1k^2\mathbf{e}_{12}, |\mathbf{h} \wedge \mathbf{k}| = |h^1||k^2||\mathbf{e}_{12}|.$$

Since $|\mathbf{e}_{12}| = 1$, we have $|\mathbf{h} \wedge \mathbf{k}| = |h^1||k^2|$, which is the elementary formula for area of a parallelogram.

Let S be the triangle with vertices $\mathbf{x}_0, \mathbf{x}_0 + \mathbf{h}, \mathbf{x}_0 + \mathbf{k}$. Its area must be half that of the parallelogram K. Thus

$$V_2(S) = \tfrac{1}{2}|\mathbf{h} \wedge \mathbf{k}|. \tag{7.28}$$

EXAMPLE 2. The area of the triangle in E^3 with vertices $\mathbf{0}$, $3\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_3 - \mathbf{e}_2$ is $\frac{1}{2}|(3\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2)|$. Since

$$(3\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2) = -3\mathbf{e}_{12} + 3\mathbf{e}_{13} + \mathbf{e}_{23},$$

the components are $\alpha^{12} = -3$, $\alpha^{13} = 3$, $\alpha^{23} = 1$. Since $|\boldsymbol{\alpha}|^2 = \sum_{[\lambda]} (\alpha^\lambda)^2$, the area is $\sqrt{19}/2$. The area can also be calculated from Formula (7.36) below.

We now define orientations for a 2-dimensional vector subspace P of E^n as follows. Let $(\mathbf{h}, \mathbf{k})$ be a frame for P, and set

$$\boldsymbol{\alpha}_0 = \frac{\mathbf{h} \wedge \mathbf{k}}{|\mathbf{h} \wedge \mathbf{k}|}. \tag{7.29}$$

Then $\boldsymbol{\alpha}_0$ is a 2-vector and $|\boldsymbol{\alpha}_0| = 1$. We call $\boldsymbol{\alpha}_0$ an *orientation* for P. Since all 2-vectors $\boldsymbol{\alpha}$ associated with P are scalar multiples of each other, only two of them have norm $|\boldsymbol{\alpha}| = 1$. They are $\boldsymbol{\alpha}_0$ and $-\boldsymbol{\alpha}_0$. These are the two possible orientations for P.

EXAMPLE 1 (*continued*). Since $|\mathbf{e}_{12} - \mathbf{e}_{13} - 3\mathbf{e}_{23}| = \sqrt{11}$, the orientations are $\pm(11)^{-1/2}(\mathbf{e}_{12} - \mathbf{e}_{13} - 3\mathbf{e}_{23})$.

Multivectors of any degree r

These are defined as follows. If $\mathscr{V}$ is any vector space, then alternating r-linear functions on $\mathscr{V} \times \cdots \times \mathscr{V}$ can be defined just as in Section 7.2 where we took $\mathscr{V} = E^n$. Let us now take $\mathscr{V} = (E^n)^*$, the dual space to E^n.

Definition. A *multivector of degree r* is an alternating r-linear function with domain the r-fold cartesian product $(E^n)^* \times \cdots \times (E^n)^*$.

For brevity, multivectors of degree r are called *r-vectors.*

For every statement about multicovectors in Sections 7.2 and 7.3, there is a dual statement about multivectors obtained by everywhere exchanging the words "vector" and "covector." For instance, if $\boldsymbol{\alpha}$ is an r-vector and $\lambda = (i_1, \ldots, i_r)$, let

$$\alpha^\lambda = \boldsymbol{\alpha}(\mathbf{e}^{i_1}, \ldots, \mathbf{e}^{i_r}). \tag{7.30}$$

This is dual to the following formula [see (7.11) with $M = \boldsymbol{\omega}$]:

$$\omega_\lambda = \boldsymbol{\omega}(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_r}).$$

Let $\mathbf{e}_\lambda$ be the r-vector defined by the following formula dual to (7.14):

$$\mathbf{e}_\lambda(\mathbf{a}^1, \ldots, \mathbf{a}^r) = \det(a^k_{i_l}) \tag{7.31}$$

for every r-tuple $(\mathbf{a}^1, \ldots, \mathbf{a}^r)$ of covectors.

Let E^n_r denote the set of all r-vectors. Then E^n_r satisfies the axioms for a vector space. It consists of $\mathbf{0}$ only if $r > n$. For $1 \leq r \leq n$ the r-vectors

Table 7.1

	r-vectors	r-covectors
Elements of	E_r^n	$(E_r^n)^*$
Standard basis elements (λ increasing)	$\mathbf{e}_\lambda = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r}$	$\mathbf{e}^\lambda = \mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_r}$
	$\boldsymbol{\alpha} = \sum_{[\lambda]} \alpha^\lambda \mathbf{e}_\lambda$	$\boldsymbol{\omega} = \sum_{[\lambda]} \omega_\lambda \mathbf{e}^\lambda$
Euclidean inner product	$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \sum_{[\lambda]} \alpha^\lambda \beta^\lambda$	$\boldsymbol{\omega} \cdot \boldsymbol{\zeta} = \sum_{[\lambda]} \omega_\lambda \zeta_\lambda$
Euclidean norm	$\lvert\boldsymbol{\alpha}\rvert^2 = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}$	$\lvert\boldsymbol{\omega}\rvert^2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega}$
Scalar product	$\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \sum_{[\lambda]} \omega_\lambda \alpha^\lambda$	
	$\mathbf{e}^\lambda \cdot \mathbf{e}_\mu = \delta_\mu^\lambda$	
	$\mathbf{e}^\lambda \cdot \boldsymbol{\alpha} = \alpha^\lambda$	$\boldsymbol{\omega} \cdot \mathbf{e}_j = \omega_\lambda$

$\mathbf{e}_\lambda$ with λ increasing form the *standard basis* for E_r^n. The number α^λ is the *component* of $\boldsymbol{\alpha}$ with respect to $\mathbf{e}_\lambda$.

The inner product $\boldsymbol{\alpha} \cdot \boldsymbol{\beta}$ of two r-vectors, and the exterior product $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ of an r-vector $\boldsymbol{\alpha}$ and an s-vector $\boldsymbol{\beta}$ are defined by the formulas dual to those in Section 7.3. In each instance subscripts are replaced by superscripts and vice versa. The exterior product of multivectors has the same properties listed in Proposition 7.3. The scalar product $\boldsymbol{\omega} \cdot \boldsymbol{\alpha}$ of an r-covector $\boldsymbol{\omega}$ and an r-vector $\boldsymbol{\alpha}$ is defined in the third from last line of the Table 7.1. The last two lines of the table are particular cases of the formula for $\boldsymbol{\omega} \cdot \boldsymbol{\alpha}$.

The formulas in the second line and in the last two lines of Table 7.1 are valid regardless of whether λ and μ are increasing, since they are known to be valid for increasing r-tuples, and both sides of each formula change sign under interchanges.

The reader should compare this table with the corresponding table for $r = 1$ (Section 3.2). According to the definition (Section A.1), the dual space of E_r^n consists of all real valued linear functions F with domain E_r^n. The dual space may be identified with $(E_r^n)^*$ in the following way. Given an r-covector $\boldsymbol{\omega}$, let $F(\boldsymbol{\alpha}) = \boldsymbol{\omega} \cdot \boldsymbol{\alpha}$ for every $\boldsymbol{\alpha} \in E_r^n$. This establishes an isomorphism between $(E_r^n)^*$ and the dual space of E_r^n. The next to last line of the table implies that the standard bases for E_r^n and $(E_r^n)^*$ are dual.

Note about terminology

Multivectors of degree r are also called *alternating contravariant tensors of rank r*. The exterior algebra A^n of E^n can be introduced in the way indicated at the end of Section 7.3.

Definition. An r-vector $\boldsymbol{\alpha}$ is *decomposable* if there exist vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ such that $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$.

Similarly, an r-covector ω is decomposable if there exist covectors $\mathbf{a}^1, \ldots, \mathbf{a}^r$ such that $\omega = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$.

A 1-vector is simply a vector $\mathbf{h}$. Hence every 1-vector is decomposable. If α is an n-vector, then

$$\alpha = c\mathbf{e}_{1\ldots n} = (c\mathbf{e}_1) \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n,$$

where $c = \alpha^{1\ldots n}$. Hence every n-vector is decomposable. In Section 7.8 it is shown that any $(n-1)$-vector is decomposable. However, for $2 \leq r \leq n-2$ there are nondecomposable r-vectors (see Problem 9). Since $\mathbf{e}_\lambda = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r}$, the standard basis r-vectors are decomposable.

It is not correct to identify a decomposable r-vector α with the r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$, since there are many ways to write α as an exterior product of vectors. The corollary to Theorem 7.3 below furnishes a geometric description of all possible such decompositions of α.

Proposition 7.5. *If* $\alpha = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$, *then for every r-covector* ω,

$$\omega \cdot \alpha = \omega(\mathbf{h}_1, \ldots, \mathbf{h}_r).$$

PROOF. Let us define a function $\tilde{\omega}$ on $E^n \times \cdots \times E^n$ by $\tilde{\omega}(\mathbf{h}_1, \ldots, \mathbf{h}) = \omega \cdot (\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r)$ for every r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$. Then $\tilde{\omega}$ is an alternating r-linear function. Moreover,

$$\tilde{\omega}(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r}) = \omega \cdot \mathbf{e}_\mu = \omega_\mu$$

for every $\mu = (j_1, \ldots, j_r)$. Hence $\tilde{\omega} = \omega$. □

If $\omega = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$ is also decomposable, then by (7.21) and Proposition 7.5,

$$(\mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r) \cdot (\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r) = \det(\mathbf{a}^k \cdot \mathbf{h}_l). \tag{7.32}$$

Proposition 7.6. *If* $\omega \cdot \alpha = \omega \cdot \beta$ *for every decomposable r-covector* ω, *then* $\alpha = \beta$.

PROOF. The standard basis r-covectors $\mathbf{e}^\lambda$ are decomposable. Hence for every increasing λ,

$$\alpha^\lambda = \mathbf{e}^\lambda \cdot \alpha = \mathbf{e}^\lambda \cdot \beta = \beta^\lambda.$$ □

If $\alpha = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$, and $\lambda = (i_1, \ldots, i_r)$, then the component α^λ satisfies the formula dual to (7.22):

$$\alpha^\lambda = \det(h_l^{i_k}). \tag{7.33}$$

The decomposable r-vectors have an important geometric significance, which is described next.

First we recall the following results from linear algebra:

(1) Any linearly independent set $\{\mathbf{h}_1, \ldots, \mathbf{h}_r\}$ is a basis for the vector subspace $P \subset E^n$ spanned by these vectors (definition).
(2) Given any such set there exist $\mathbf{h}_{r+1}, \ldots, \mathbf{h}_n$ such that $\{\mathbf{h}_1, \ldots, \mathbf{h}_n\}$ is a basis for E^n.
(3) For every basis $\{\mathbf{h}_1, \ldots, \mathbf{h}_n\}$ for E^n there is a dual basis $\{\mathbf{a}^1, \ldots, \mathbf{a}^n\}$ for $(E^n)^*$, $\mathbf{a}^k \cdot \mathbf{h}_l = \delta_l^k$ for $k, l = 1, \ldots, n$ (Section A.1).

Definition. A linearly independent r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is called a *frame* for the vector subspace P spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_r$.

The only difference between the notions of basis and frame is that the latter takes into account the order in which the basis vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ are written.

Theorem 7.3

(a) *An r-tuple* $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ *is linearly dependent if and only if* $\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r = \mathbf{0}$.
(b) *Let* $P \subset E^n$ *be an r-dimensional vector subspace and* $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$, $(\mathbf{h}'_1, \ldots, \mathbf{h}'_r)$ *be any two frames for P. Then there is a scalar c such that*

$$\mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_r = c\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r. \tag{7.34}$$

(c) *Conversely, if* $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ *and* $(\mathbf{h}'_1, \ldots, \mathbf{h}'_r)$ *are frames which satisfy* (7.34) *for some scalar c, then they are frames for the same vector subspace P. (see Figure* 7.2).

PROOF. To prove (a), let $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ be linearly dependent and $\alpha = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$. By Propositions 7.1 and 7.5, $\omega \cdot \alpha = 0$ for every ω. By Proposition 7.6, $\alpha = \mathbf{0}$. On the other hand, if $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is linearly independent,

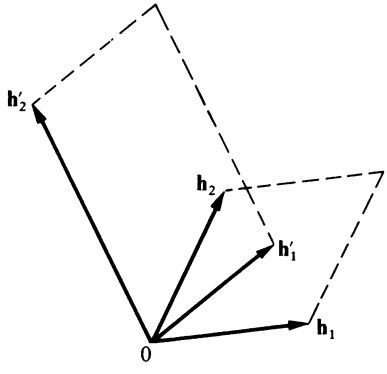


Figure 7.2

let $\mathbf{a}^1, \ldots, \mathbf{a}^r$ be covectors such that $\mathbf{a}^k \cdot \mathbf{h}_l = \delta_l^k$, and let $\boldsymbol{\omega} = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$. By (7.32),

$$\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \det(\delta_l^k) = 1.$$

Hence $\boldsymbol{\alpha} \neq \mathbf{0}$.

To prove (b), let each $\mathbf{h}'_l$ be a linear combination of $\mathbf{h}_1, \ldots, \mathbf{h}_r$,

$$\mathbf{h}'_l = \sum_{m=1}^{r} c_l^m \mathbf{h}_m, \qquad l = 1, \ldots, r.$$

If $\boldsymbol{\omega} = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$ is any decomposable r-covector, then by (7.32),

$$\boldsymbol{\omega} \cdot \boldsymbol{\alpha}' = \det(\mathbf{a}^k \cdot \mathbf{h}'_l) = \det\left(\sum_{m=1}^{r} (\mathbf{a}^k \cdot \mathbf{h}_m) c_l^m \right)$$

where $\boldsymbol{\alpha}' = \mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_r$. The matrix on the right is the product of the matrices $(\mathbf{a}^k \cdot \mathbf{h}_m)$ and (c_l^m). Hence if $c = \det(c_l^m)$,

$$\boldsymbol{\omega} \cdot \boldsymbol{\alpha}' = c \det(\mathbf{a}^k \cdot \mathbf{h}_m) = c\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \boldsymbol{\omega} \cdot (c\boldsymbol{\alpha}).$$

This is true for every decomposable $\boldsymbol{\omega}$; hence $\boldsymbol{\alpha}' = c\boldsymbol{\alpha}$ by Proposition 7.6.

To prove (c), again let $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$, $\boldsymbol{\alpha}' = \mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_r$. By hypothesis, $\{\mathbf{h}_1, \ldots, \mathbf{h}_r\}$ and $\{\mathbf{h}'_1, \ldots, \mathbf{h}'_r\}$ are linearly independent sets. It suffices to show that each $\mathbf{h}'_l$ is a linear combination of $\mathbf{h}_1, \ldots, \mathbf{h}_r$. Suppose that this is false for some l, say for $l = 1$. Then $\{\mathbf{h}_1, \ldots, \mathbf{h}_r, \mathbf{h}'_1\}$ is a linearly independent set. Let $\mathbf{a}^1, \ldots, \mathbf{a}^{r+1}$ be such that $\mathbf{a}^k \cdot \mathbf{h}_m = \delta_m^k$ for $k, m = 1, \ldots, r+1$, where we have set $\mathbf{h}'_1 = \mathbf{h}_{r+1}$. Let $\boldsymbol{\omega} = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$. Then $\boldsymbol{\omega} \cdot \mathbf{a} = 1$, but $\boldsymbol{\omega} \cdot \boldsymbol{\alpha}' = 0$ since the elements $\mathbf{a}^k \cdot \mathbf{h}'_1 = \delta_{r+1}^k$ of the first column of the $r \times r$ matrix $(\mathbf{a}^k \cdot \mathbf{h}'_l)$ are 0. This contradicts the assumption that $\boldsymbol{\alpha}' = c\boldsymbol{\alpha}$, $c \neq 0$. □

EXAMPLE 3. Show that $2\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{e}_1 - 2\mathbf{e}_3$ are linearly dependent. Their exterior product is

$$(2\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3) \wedge (\mathbf{e}_1 + 2\mathbf{e}_2) \wedge (\mathbf{e}_1 - 2\mathbf{e}_3)$$
$$= (\mathbf{e}_{12} - \mathbf{e}_{31} - 2\mathbf{e}_{32}) \wedge (\mathbf{e}_1 - 2\mathbf{e}_3) = -2\mathbf{e}_{123} - 2\mathbf{e}_{321} = \mathbf{0}.$$

Let $\boldsymbol{\alpha} \neq \mathbf{0}$ be a decomposable r-vector. Then $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$, and the vector subspace P spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_r$ is called the *r-space* of $\boldsymbol{\alpha}$. If $\boldsymbol{\alpha} = \mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_r$, then taking $c = 1$ in part (c) of Theorem 7.3, we see that $\mathbf{h}'_1, \ldots, \mathbf{h}'_r$ also span this same vector subspace P. Thus P depends only on $\boldsymbol{\alpha}$ and not on the particular way $\boldsymbol{\alpha}$ is written as the exterior product of vectors.

If $c \neq 0$, then $(c\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_r)$ is another frame for P and $c\boldsymbol{\alpha} = (c\mathbf{h}_1) \wedge \mathbf{h}_2 \wedge \cdots \wedge \mathbf{h}_r$. Thus $\boldsymbol{\alpha}$ and $c\boldsymbol{\alpha}$ have the same r-space P. On the other hand, if $\boldsymbol{\alpha}'$ is not a scalar multiple of $\boldsymbol{\alpha}$, then $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ have different r-spaces.

In much the same way as for the case $r = 2$ above, let us now define r-dimensional measure for parallepipeds and simplices.

Measure for r-parallelepipeds

It is shown in Section 5.7 that if K is an n-parallelepiped spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_n$ with $\mathbf{x}_0$ as vertex, then

$$V_n(K) = |\det(h_l^k)| = |\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_n|.$$

More generally, if $\mathbf{x}_0, \mathbf{h}_1, \ldots, \mathbf{h}_r$ are vectors, then

$$K = \left\{\mathbf{x} : \mathbf{x} = \mathbf{x}_0 + \sum_{k=1}^{r} t^k \mathbf{h}_k, 0 \leq t^k \leq 1, k = 1, \ldots, r\right\}$$

is the *r-parallelepiped* spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_r$ with $\mathbf{x}_0$ as vertex.

Definition. The *r-dimensional measure* of K is

$$V_r(K) = |\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|. \tag{7.35}$$

By part (a) of Theorem 7.3, $V_r(K) = 0$ if and only if $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is linearly dependent.

To show that the definition (7.35) of r-measure for parallelepipeds is reasonable, let us show that $V_r(K)$ is the product of the lengths of the vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ in case these vectors are mutually orthogonal. If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are vectors, let $\mathbf{a}^k$ be the covector with the same components as the vector $\mathbf{v}_k$. Then $\mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$ is the r-covector with the same components as the r-vector $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$. By (7.32),

$$(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r) \cdot (\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r) = \det(\mathbf{v}_k \cdot \mathbf{h}_l),$$

where the $\cdot$ now denotes inner product. In particular, let $\mathbf{v}_k = \mathbf{h}_k$. Then

$$|\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|^2 = \det(\mathbf{h}_k \cdot \mathbf{h}_l). \tag{7.36}$$

Taking square roots, we get a formula for $V_r(K)$. If $\mathbf{h}_1, \ldots, \mathbf{h}_r$ are mutually orthogonal, then $\mathbf{h}_k \cdot \mathbf{h}_l = 0$ for $k \neq l$ and $\det(\mathbf{h}_k \cdot \mathbf{h}_l) = |\mathbf{h}_1|^2 \cdots |\mathbf{h}_r|^2$. In this case $V_r(K) = |\mathbf{h}_1| \cdots |\mathbf{h}_r|$ as required.

Measure for r-simplices

Let S be an r-simplex with vertices $\mathbf{x}_0, \ldots, \mathbf{x}_r$. Let $\mathbf{h}_k = \mathbf{x}_k - \mathbf{x}_0, k = 1, \ldots, r$. Reasoning as in Section 5.7, we have

$$S = \left\{\mathbf{x} : \mathbf{x} = \mathbf{x}_0 + \sum_{k=1}^{r} t^k \mathbf{h}_k, t^k \geq 0 \text{ for } k = 1, \ldots, r, \sum_{k=1}^{r} t^k \leq 1\right\}.$$

The *r-dimensional measure* of S is defined to be

$$V_r(S) = \frac{1}{r!} |\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|. \tag{7.37}$$

Both (7.35) and (7.37) are very special cases of a general formula (8.2) for r-dimensional measure.

Orientations

Let P be an r-dimensional vector subspace of E^n, and $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ a frame for P. The r-vector

$$\alpha_0 = \frac{\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r}{|\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|}$$

has P as its r-space, and $|\alpha_0| = 1$. We call α_0 an *orientation* for P:

Definition. A decomposable r-vector α_0 is an *orientation* for P if $|\alpha_0| = 1$ and P is the r-space of α_0.

There are two orientations for P. If α_0 is one of them, then $-\alpha_0$ is the other. Two frames $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$, $(\mathbf{h}'_1, \ldots, \mathbf{h}'_r)$ for P determine the same orientation for P if and only if $c > 0$ in (7.34). Note that c is the determinant of the $r \times r$ matrix (c_i^m) in the proof of Theorem 7.3(b).

If two vectors in the frame $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ are interchanged, then the exterior product $\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$ changes sign. Thus the orientation of a frame changes under interchanges.

We now have a criterion that shows when two frames lead to the same r-vector.

Corollary. *Let* $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$, $(\mathbf{h}'_1, \ldots, \mathbf{h}'_r)$ *be frames. Then* $\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r = \mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_r$ *if and only if these frames span the same r-space P, have the same orientation, and their parallelepipeds with* $\mathbf{0}$ *as vertex have the same r-measure.*

The case $r = n$

In this case $P = E^n$ and $\pm \mathbf{e}_{1 \ldots n}$ are the two orientations. Let us call $\mathbf{e}_{1 \ldots n}$ the *standard*, or *positive*, orientation of E^n and $-\mathbf{e}_{1 \ldots n}$ the *negative* orientation of E^n. When $r < n$, we do not attempt to call one orientation of P positive and the other negative. If $(\mathbf{h}_1, \ldots, \mathbf{h}_n)$ is a frame for E^n, then by (7.33)

$$\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_n = \det(h_i^k)\mathbf{e}_{1 \ldots n}. \tag{7.38}$$

The frame has positive orientation if $\det(h_i^k) > 0$ and negative orientation if $\det(h_i^k) < 0$. For small values of n, (7.38) sometimes furnishes a convenient way to find the determinant of the matrix (h_i^k), since the exterior product on the left side of (7.38) is readily computed.

EXAMPLE 4. Let $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 + 3\mathbf{e}_3$ be the column vectors of a 3×3 matrix M. Then

$$(\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_1 - \mathbf{e}_3) \wedge (\mathbf{e}_2 + 3\mathbf{e}_3) = -\mathbf{e}_{132} + 3\mathbf{e}_{213} = -2\mathbf{e}_{123}.$$

Therefore $\det M = -2$. Moreover, $(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 + 3\mathbf{e}_3)$ is a frame for E^3, which determines the negative orientation $-\mathbf{e}_{123}$ for E^3.

PROBLEMS

1. Simplify ($n = 6$):
   (a) $\mathbf{e}_3 \wedge \mathbf{e}_5 \wedge \mathbf{e}_{24}$.
   (b) $\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_{62}$.
   (c) $\mathbf{e}_1 \wedge (\mathbf{e}_{14} + \mathbf{e}_{64})$.
   (d) $(\mathbf{e}_1 + 3\mathbf{e}_4 - \mathbf{e}_6) \wedge (2\mathbf{e}_{23} + \mathbf{e}_{36}) \wedge \mathbf{e}_{45}$.
   (e) $(\mathbf{e}_{12} + \mathbf{e}_{13}) \wedge (\mathbf{e}_{34} + \mathbf{e}_{25}) \wedge (\mathbf{e}_{56} + \mathbf{e}_{46})$.
2. Evaluate the indicated scalar products ($n = 4$).
   (a) $(\mathbf{e}^1 + \mathbf{e}^2) \cdot (\mathbf{e}_1 + \mathbf{e}_2)$.
   (b) $\mathbf{e}^{12} \cdot \mathbf{e}_{34}$.
   (c) $\mathbf{e}^{134} \cdot (\mathbf{e}_{431} + 3\mathbf{e}_{124})$.
   (d) $(\mathbf{e}^1 - \mathbf{e}^4) \wedge (\mathbf{e}^2 + \mathbf{e}^4) \cdot (\mathbf{e}_1 + 2\mathbf{e}_4) \wedge (\mathbf{e}_2 - 2\mathbf{e}_4)$.
3. Using Theorem 7.3 show that $(2\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_4, \mathbf{e}_1 + \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_4)$ is a frame for E^4. What is its orientation? Find the determinant of the matrix that has these vectors as columns, using (7.38).
4. Show that $(\mathbf{e}_1 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_5, \mathbf{e}_3 + \mathbf{e}_6, \mathbf{e}_1 + \mathbf{e}_5, \mathbf{e}_2 + \mathbf{e}_6, \mathbf{e}_3 - \mathbf{e}_4)$ is a frame for E^6. What is its orientation?
5. Show that $(\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3)$ and $(3\mathbf{e}_1 - \mathbf{e}_2 - 2\mathbf{e}_3, 2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)$ are frames for the same vector subspace of E^3. Do their orientations agree?
6. Find the area of the triangle with vertices $2\mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3, \mathbf{e}_1 + 3\mathbf{e}_3$.
7. Find the volume of the 3-simplex in E^4 with vertices $\mathbf{0}, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3 + 2\mathbf{e}_4$.
8. Let K be an r-parallelepiped spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_r$ with $\mathbf{0}$ as vertex. For each increasing λ let $K^\lambda = \mathbf{X}^\lambda(K)$ where $\mathbf{X}^\lambda$ is the projection onto the r-space of $\mathbf{e}_\lambda$. ($\mathbf{X}^\lambda$ leaves the components $x^{i_1}, \ldots, x^{i_r}$ of any $\mathbf{x}$ unchanged and replaces each of the other components by 0.) Let $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$. Show that $|\alpha^\lambda| = V_r(K^\lambda)$ and hence $[V_r(K)]^2 = \sum_{[\lambda]} [V_r(K^\lambda)]^2$. Illustrate for $n = 3$ and $r = 1, 2$.
9. Show that:
   (a) If $\boldsymbol{\alpha}$ is decomposable, then $\boldsymbol{\alpha} \wedge \boldsymbol{\alpha} = \mathbf{0}$.
   (b) If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are decomposable r-vectors, then $(\boldsymbol{\alpha} + \boldsymbol{\beta}) \wedge (\boldsymbol{\alpha} + \boldsymbol{\beta}) = 2\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ if r is even and is $\mathbf{0}$ if r is odd.
   (c) The 2-vector $\mathbf{e}_{12} + \mathbf{e}_{34}$ is not decomposable. [*Hint*: Use (a).]
10. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be decomposable nonzero 2-vectors, and P, Q be their respective 2-spaces. Show that if $P \cap Q = \{\mathbf{0}\}$, then $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is not decomposable; and if $P \cap Q$ is a line through $\mathbf{0}$, then $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is decomposable and $\boldsymbol{\alpha} \neq c\boldsymbol{\beta}$. [*Hints*: In the first instance $\boldsymbol{\alpha} = \mathbf{h} \wedge \mathbf{k}$, $\boldsymbol{\beta} = \mathbf{h}' \wedge \mathbf{k}'$, where $\{\mathbf{h}, \mathbf{k}, \mathbf{h}', \mathbf{k}'\}$ is a linearly independent set. In the second $\boldsymbol{\alpha} = \mathbf{h} \wedge \mathbf{k}$, $\boldsymbol{\beta} = \mathbf{h} \wedge \mathbf{k}'$, where $\mathbf{h} \in P \cap Q$.]
11. Let $\boldsymbol{\alpha} = \mathbf{h} \wedge \mathbf{k}$, $\boldsymbol{\alpha} \neq \mathbf{0}$. Show that the matrix (α^{ij}) has rank 2. [*Hint*: Show that each column vector of the matrix is a linear combination of $\mathbf{h}$ and $\mathbf{k}$.]
12. Let $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r)$ be an $(r + 1)$-tuple such that the vectors $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_r - \mathbf{x}_0$ are linearly independent. Such an $(r + 1)$-tuple defines an *oriented* r-simplex. Its *r-vector* is $1/r!(\mathbf{x}_1 - \mathbf{x}_0) \wedge \cdots \wedge (\mathbf{x}_r - \mathbf{x}_0)$. Let $\boldsymbol{\beta}_i$ be the $(r - 1)$-vector of the ith oriented face $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_r)$. Show that

$$\sum_{i=0}^{r} (-1)^i \boldsymbol{\beta}_i = \mathbf{0}.$$

7.6 Induced linear transformations

Let m and n be positive integers. With any linear transformation $\mathbf{L}$ from E^m into E^n is associated for each $r = 1, 2, \ldots$ a linear transformation $\mathbf{L}_r$ from E_r^m into E_r^n with the following property. If $(\mathbf{k}_1, \ldots, \mathbf{k}_r)$ is any r-tuple of vectors in E^m, then we require that

$$\mathbf{L}_r(\mathbf{k}_1 \wedge \cdots \wedge \mathbf{k}_r) = \mathbf{L}(\mathbf{k}_1) \wedge \cdots \wedge \mathbf{L}(\mathbf{k}_r). \tag{7.39}$$

For $r = 2$ this is illustrated in Figure 7.3.

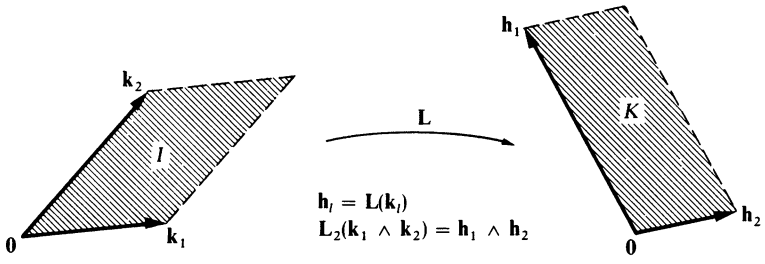


Figure 7.3

At the end of the section we show that $\mathbf{L}_r$ is well defined by Formula (7.39). The linear transformations $\mathbf{L}_r$ are said to be *induced* by $\mathbf{L}$.

There is a geometric way to think about $\mathbf{L}_r$. Let Q be the vector subspace of E^m spanned by $\{\mathbf{k}_1, \ldots, \mathbf{k}_r\}$, and let $\mathbf{h}_i = \mathbf{L}(\mathbf{k}_i)$, $P = \mathbf{L}(Q)$. Then P is the vector subspace of E^n spanned by $\{\mathbf{h}_1, \ldots, \mathbf{h}_r\}$. Let us suppose that Q and P are both of dimension r. [Note: This holds in particular if $r = m$, $m \leq n$, and $\mathbf{L}$ has maximum possible rank r. This case is encountered repeatedly in Chapter 8.] When Q and P have dimension r, $(\mathbf{k}_1, \ldots, \mathbf{k}_r)$ is a frame for Q and $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ a frame for P. Moreover, Q is the r-space of the r-vector $\boldsymbol{\beta} = \mathbf{k}_1 \wedge \cdots \wedge \mathbf{k}_r$; and P is the r-space of the r-vector $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r = \mathbf{L}_r(\boldsymbol{\beta})$ according to (7.39).

Let I be the r-parallelepiped spanned by $(\mathbf{k}_1, \ldots, \mathbf{k}_r)$ with $\mathbf{0}$ as vertex, and let $K = \mathbf{L}(I)$. Then K is the r-parallelepiped spanned by $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$, with $\mathbf{0}$ as vertex (see Figure 7.3). Consider the ratio of r-measures:

$$\mathscr{J} = \frac{V_r(K)}{V_r(I)}. \tag{7.40}$$

By (7.35), $\mathscr{J} = |\boldsymbol{\alpha}||\boldsymbol{\beta}|^{-1}$. Let us show that $\mathscr{J}$ depends only on the r-space Q and on $\mathbf{L}$. Let $(\mathbf{k}_1', \ldots, \mathbf{k}_r')$ be another frame for Q, and $\mathbf{h}_i' = \mathbf{L}(\mathbf{k}_i')$. Then $(\mathbf{h}_1', \ldots, \mathbf{h}_r')$ is a frame for P. Let $\boldsymbol{\beta}' = \mathbf{k}_1' \wedge \cdots \wedge \mathbf{k}_r'$, $\boldsymbol{\alpha}' = \mathbf{h}_1' \wedge \cdots \wedge \mathbf{h}_r' = \mathbf{L}_r(\boldsymbol{\beta}')$. By Theorem 7.3, $\boldsymbol{\beta}' = c\boldsymbol{\beta}$ for some real $c \neq 0$. Since L_r is linear,

$$\boldsymbol{\alpha}' = \mathbf{L}_r(c\boldsymbol{\beta}) = c\mathbf{L}_r(\boldsymbol{\beta}) = c\boldsymbol{\alpha}.$$

Therefore

$$|\alpha'||\beta'|^{-1} = |c\alpha||c\beta|^{-1} = |\alpha||\beta|^{-1} = \mathscr{J}.$$

The case $r = m$ will be of particular interest in Section 8.1. If $\mathbf{L}$ is linear from E^r into E^n, then we may in particular take $\mathbf{k}_j = \boldsymbol{\varepsilon}_j$, where $\{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_r\}$ is the standard basis for E^r. Then $\mathbf{h}_j = \mathbf{v}_j = \mathbf{L}(\boldsymbol{\varepsilon}_j)$, where $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are the column vectors of the matrix of $\mathbf{L}$. We have $\beta = \boldsymbol{\varepsilon}_{1\ldots r}$, $\alpha = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$. Since $|\beta| = 1$, we have

$$\mathscr{J} = |\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r|, \qquad \text{if } r = m. \tag{7.40a}$$

The dual transformation

We recall from Section 4.1 that the dual $\mathbf{L}^*$ of $\mathbf{L}$ is a linear transformation from $(E^n)^*$ into $(E^m)^*$. It has the property (Problem 5b, Section 4.1) that $\mathbf{a} \cdot \mathbf{L}(\mathbf{k}) = \mathbf{L}^*(\mathbf{a}) \cdot \mathbf{k}$ for every covector $\mathbf{a} \in (E^n)^*$ and vector $\mathbf{k} \in E^m$. In a similar way $\mathbf{L}_r$ has a dual $\mathbf{L}_r^*$, which is a linear transformation from $(E_r^n)^*$ into $(E_r^m)^*$. It is required to have the property that

$$\omega \cdot \mathbf{L}_r(\beta) = \mathbf{L}_r^*(\omega) \cdot \beta \tag{7.41}$$

for every $\beta \in E_r^m$, $\omega \in (E_r^n)^*$.

Let us turn to give a definition of $\mathbf{L}_r$ (postponed in the preceding discussion) and of $\mathbf{L}_r^*$. The linear transformation $\mathbf{L}_r$ is determined once its values are given for standard basis r-vectors. Let $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_m$ be the standard basis vectors of E^m. Let $\mathbf{v}_j = \mathbf{L}(\boldsymbol{\varepsilon}_j)$; then $\mathbf{v}_j$ is the jth column vector of the matrix of $\mathbf{L}$ (Section 4.1). The standard basis elements of E_r^m are $\boldsymbol{\varepsilon}_\mu = \boldsymbol{\varepsilon}_{j_i} \wedge \cdots \wedge \boldsymbol{\varepsilon}_{j_r}$, with $\mu = (j_i, \ldots, j_r)$ increasing.

Keeping in mind that we want (7.39) to be correct, we set

$$\mathbf{L}_r(\boldsymbol{\varepsilon}_\mu) = \mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r}. \tag{7.42}$$

Since $\mathbf{L}_r$ is to be linear, its value at any β is determined once the values at the basis elements $\boldsymbol{\varepsilon}_\mu$ are known. For any $\beta = \sum_{[\mu]} \beta^\mu \boldsymbol{\varepsilon}_\mu$

$$\mathbf{L}_r(\beta) = \sum_{[\mu]} \beta^\mu \mathbf{L}_r(\boldsymbol{\varepsilon}_\mu). \tag{7.43}$$

This defines the induced linear transformation $\mathbf{L}_r$ for $1 < r \leq m$. We set $\mathbf{L}_1 = \mathbf{L}$. If $r > m$, then E_r^m has the single element $\mathbf{0}$ and $\mathbf{L}_r(\mathbf{0}) = \mathbf{0}$.

Let us show that for every $\beta \in E_r^m$, $\gamma \in E_s^m$,

$$\mathbf{L}_{r+s}(\beta \wedge \gamma) = \mathbf{L}_r(\beta) \wedge \mathbf{L}_s(\gamma). \tag{7.44}$$

Let $\mu = (j_1, \ldots, j_r)$ and $\nu = (k_1, \ldots, k_s)$ be increasing. Then

$$\mathbf{L}_r(\boldsymbol{\varepsilon}_\mu) \wedge \mathbf{L}_s(\boldsymbol{\varepsilon}_\nu) = \mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r} \wedge \mathbf{v}_{k_1} \wedge \cdots \wedge \mathbf{v}_{k_s}.$$

If any integer is repeated in the $(r + s)$-tuple (μ, ν), then this is $\mathbf{0}$. Otherwise, the right-hand side is $(-1)^p \mathbf{L}_{r+s}(\boldsymbol{\varepsilon}_\tau)$ where τ is the increasing $(r + s)$-tuple obtained from (μ, ν) by p interchanges. Since $\boldsymbol{\varepsilon}_{\mu,\nu} = (-1)^p \boldsymbol{\varepsilon}_\tau$ and $\mathbf{L}_{r+s}$ is

linear, $(-1)^p \mathbf{L}_{r+s}(\boldsymbol{\varepsilon}_\tau) = \mathbf{L}_{r+s}(\boldsymbol{\varepsilon}_{\mu,\nu})$. Thus

$$\mathbf{L}_r(\boldsymbol{\varepsilon}_\mu) \wedge \mathbf{L}_s(\boldsymbol{\varepsilon}_\nu) = \mathbf{L}_{r+s}(\boldsymbol{\varepsilon}_{\mu,\nu}).$$

Therefore (7.44) is correct for basis of elements of E_r^m and E_s^m. Since each of these transformations is linear, (7.44) then holds in general.

By induction there is a generalization of (7.44) for products of any number of multivectors. In particular, in this way we get the required formula (7.39) for products of vectors.

Let $\boldsymbol{\beta} \in E_r^m$ and $\boldsymbol{\alpha} = \mathbf{L}_r(\boldsymbol{\beta})$. Let us find a formula for the components α^λ in terms of the components of $\boldsymbol{\beta}$. If $\lambda = (i_1, \ldots, i_r)$, $\mu = (j_1, \ldots, j_r)$, and (c_j^i) is the matrix of $\mathbf{L}$, let

$$c_\mu^\lambda = \det(c_{j_l}^{i_k}).$$

By (7.33), c_μ^λ is the λth component of $\mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r}$. By (7.42) and (7.43)

$$\boldsymbol{\alpha} = \sum_{[\mu]} \beta^\mu \mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r}.$$

Since both sides have the same components,

$$\alpha^\lambda = \sum_{[\mu]} c_\mu^\lambda \beta^\mu, \qquad \text{if } \boldsymbol{\alpha} = \mathbf{L}_r(\boldsymbol{\beta}). \tag{7.45}$$

When $r = 1$, this becomes (4.2).

Let us next define the dual linear transformation $\mathbf{L}_r^*$. The standard basis covectors for $(E^n)^*$ are $\mathbf{e}^1, \ldots, \mathbf{e}^n$. Let $\mathbf{w}^i = \mathbf{L}^*(\mathbf{e}^i)$. We recall (Problem 5a, Section 4.1) that $\mathbf{w}^i$ is the ith row covector of the matrix of $\mathbf{L}$. Consider any standard basis r-covector $\mathbf{e}^\lambda = \mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_r}$ for $(E_r^n)^*$. We set

$$\mathbf{L}_r^*(\mathbf{e}^\lambda) = \mathbf{w}^{i_1} \wedge \cdots \wedge \mathbf{w}^{i_r} \tag{7.46}$$

Since $\mathbf{L}_r^*$ is to be linear, we set for any r-covector $\boldsymbol{\omega} = \sum_{[\lambda]} \omega_\lambda \mathbf{e}^\lambda$ in $(E_r^n)^*$

$$\mathbf{L}_r^*(\boldsymbol{\omega}) = \sum_{[\lambda]} \omega_\lambda \mathbf{L}_r^*(\mathbf{e}^\lambda).$$

This defines $\mathbf{L}_r^*$.

Let us establish the relation (7.41) between the induced linear transformation $\mathbf{L}_r$ and its dual $\mathbf{L}_r^*$. Since both $\mathbf{L}_r$ and $\mathbf{L}_r^*$ are linear, it suffices to show that (7.41) is correct for standard basis elements, namely, for $\boldsymbol{\beta} = \boldsymbol{\varepsilon}_\mu$ and $\boldsymbol{\omega} = \mathbf{e}^\lambda$. We recall that $\mathbf{e}^i \cdot \mathbf{L}(\boldsymbol{\varepsilon}_j) = \mathbf{L}^*(\mathbf{e}^i) \cdot \boldsymbol{\varepsilon}_j$. In other words, $\mathbf{e}^i \cdot \mathbf{v}_j = \mathbf{w}^i \cdot \boldsymbol{\varepsilon}_j$. By (7.32),

$$\mathbf{e}^\lambda \cdot \mathbf{L}_r(\boldsymbol{\varepsilon}_\mu) = (\mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_r}) \cdot (\mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r}) = \det(\mathbf{e}^{i_k} \cdot \mathbf{v}_{j_l})$$

$$\mathbf{L}_r^*(\mathbf{e}^\lambda) \cdot \boldsymbol{\varepsilon}_\mu = (\mathbf{w}^{i_1} \wedge \cdots \wedge \mathbf{w}^{i_r}) \cdot (\boldsymbol{\varepsilon}_{j_1} \wedge \cdots \wedge \boldsymbol{\varepsilon}_{j_r}) = \det(\mathbf{w}^{i_k} \cdot \boldsymbol{\varepsilon}_{j_l}).$$

Since the right-hand sides are equal, we have $\mathbf{e}^\lambda \cdot \mathbf{L}_r(\boldsymbol{\varepsilon}_\mu) = \mathbf{L}_r^*(\mathbf{e}^\lambda) \cdot \boldsymbol{\varepsilon}_\mu$ and hence (7.41).

The formulas dual to (7.44) and (7.45) are correct (Problems 2, 3).

PROBLEMS

1. Let $m = 2$, $n = 3$, $\mathbf{L}(s, t) = (s - 2t)\mathbf{e}_1 - s\mathbf{e}_2 + (2s + 3t)\mathbf{e}_3$. Find:
   (a) $\mathbf{L}^*(\mathbf{a})$. (b) $\mathbf{L}_2(\boldsymbol{\beta})$.
   (c) $\mathbf{L}_2^*(\boldsymbol{\omega})$. (d) $\mathbf{L}_3^*(\mathbf{e}^{123})$.
   (e) $\mathscr{J}$, if $r = 2$.

2. Prove the dual of (7.44):

   (7.47) $$\mathbf{L}_{r+s}^*(\boldsymbol{\omega} \wedge \boldsymbol{\zeta}) = \mathbf{L}_r^*(\boldsymbol{\omega}) \wedge \mathbf{L}_s^*(\boldsymbol{\zeta})$$

   if $\boldsymbol{\omega} \in (E_r^n)^*$, $\boldsymbol{\zeta} \in (E_s^n)^*$.

3. Prove the dual of (7.45)

   (7.48) $$\zeta_\mu = \sum_{[\lambda]} \omega_\lambda c_\mu^\lambda, \quad \text{if } \boldsymbol{\zeta} = \mathbf{L}_r^*(\boldsymbol{\omega}).$$

4. Let $\mathbf{L}$ be an orthogonal transformation of E^n. Show that $|\mathbf{L}_r(\boldsymbol{\alpha})| = |\boldsymbol{\alpha}|$:
   (a) If $\boldsymbol{\alpha}$ is decomposable. [*Hint*: (7.36).]
   (b) For any r-vector $\boldsymbol{\alpha}$.

7.7 Transformation law for differential forms

Let $\mathbf{g}$ be a transformation of class $C^{(1)}$ from an open set $\Delta \subset E^m$ into E^n. Let D be an open set containing the image $\mathbf{g}(\Delta)$ (see Figure 7.4). If $\boldsymbol{\omega}$ is any r-form with domain D, then there is a corresponding r-form denoted by $\boldsymbol{\omega}^\sharp$ with domain Δ. Formally, $\boldsymbol{\omega}^\sharp$ is obtained by merely substituting $\mathbf{g}(\mathbf{t})$ for $\mathbf{x}$ and dg^i for dx^i with g^i the ith component of $\mathbf{g}$. The precise definition of $\boldsymbol{\omega}^\sharp$ is in terms of induced linear transformations. For each $\mathbf{t} \in \Delta$, the differential $\mathbf{L} = D\mathbf{g}(\mathbf{t})$ is a linear transformation from E^m into E^n (Section 4.3). The usefulness of the induced linear transformation $\mathbf{L}_r$ becomes apparent in Chapter 8. Let us consider the dual induced linear transformation $\mathbf{L}_r^*$. By definition of r-form, $\boldsymbol{\omega}(\mathbf{x})$ is an r-covector in $(E_r^n)^*$. Then $\mathbf{L}_r^*[\boldsymbol{\omega}(\mathbf{x})]$ is an r-covector in $(E_r^m)^*$, which we take as $\boldsymbol{\omega}^\sharp(\mathbf{t})$.

Definition. For each $\mathbf{t} \in \Delta$, the value of $\boldsymbol{\omega}^\sharp$ at $\mathbf{t}$ is

(7.49) $$\boldsymbol{\omega}^\sharp(\mathbf{t}) = \mathbf{L}_r^*[\boldsymbol{\omega}(\mathbf{x})],$$

where

$$\mathbf{x} = \mathbf{g}(\mathbf{t}), \qquad \mathbf{L} = D\mathbf{g}(\mathbf{t}),$$

In case $r = 0$ we agree that $f^\sharp = f \circ \mathbf{g}$. (The notation $^\sharp$ is used for brevity even though it does not indicate the dependence on $\mathbf{g}$ and the degree r.)

Theorem 7.4. *The operation $^\sharp$ has the following properties:*

(1) $(\boldsymbol{\omega} + \boldsymbol{\zeta})^\sharp = \boldsymbol{\omega}^\sharp + \boldsymbol{\zeta}^\sharp$, *if $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$ are of degree r.*
(2) $(\boldsymbol{\omega} \wedge \boldsymbol{\zeta})^\sharp = \boldsymbol{\omega}^\sharp \wedge \boldsymbol{\zeta}^\sharp$, *if $\boldsymbol{\omega}$ is of degree r and $\boldsymbol{\zeta}$ of degree s.*
(3) $(df)^\sharp = d(f \circ \mathbf{g})$, *if f is of class $C^{(1)}$.*
(4) $(dx^{i_1} \wedge \cdots \wedge dx^{i_r})^\sharp = dg^{i_1} \wedge \cdots \wedge dg^{i_r}$.
(5) $d\boldsymbol{\omega}^\sharp = (d\boldsymbol{\omega})^\sharp$, *if $\boldsymbol{\omega}$ is of class $C^{(1)}$ and $\mathbf{g}$ of class $C^{(2)}$.*

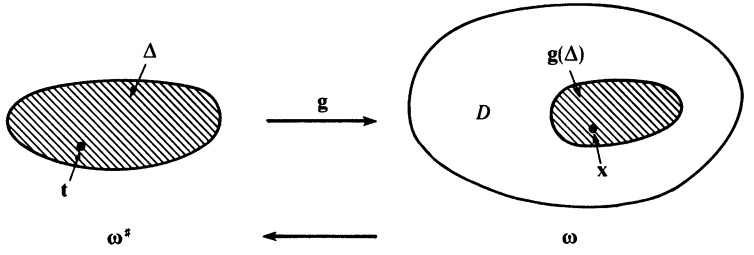


Figure 7.4

PROOF. Since $\mathbf{L}_r^*$ is linear,

$$(\omega + \zeta)^\#(\mathbf{t}) = \omega^\#(\mathbf{t}) + \zeta^\#(\mathbf{t}).$$

Since this is true for every $\mathbf{t} \in \Delta$, this proves (1). Using (7.47), we get

$$(\omega \wedge \zeta)^\#(\mathbf{t}) = \omega^\#(\mathbf{t}) \wedge \zeta^\#(\mathbf{t})$$

for every $\mathbf{t} \in \Delta$, which proves (2). By the chain rule (Section 4.3), $d(f \circ \mathbf{g})(\mathbf{t}) = \mathbf{L}_1^*[df(\mathbf{x})]$. By (7.49) the right-hand side is $(df)^\#(\mathbf{t})$. Thus (3) holds. Recall that dx^i stands for dX^i, where $X^i(\mathbf{x}) = x^i$ for each $\mathbf{x}$. Then $g^i = X^i \circ \mathbf{g}$ and from (3) with $f = X^i$,

$$(dx^i)^\# = dg^i. \tag{7.50}$$

Then (4) follows from this and (2). To prove (5) we have, from (1) through (4);

$$\omega^\# = \sum_{[\lambda]} \omega_\lambda \circ \mathbf{g}\, dg^{i_1} \wedge \cdots \wedge dg^{i_r},$$

$$d\omega^\# = \sum_{[\lambda]} d(\omega_\lambda \circ \mathbf{g}\, dg^{i_1} \wedge \cdots \wedge dg^{i_r}).$$

By (3), $(d\omega_\lambda)^\# = d(\omega_\lambda \circ \mathbf{g})$. Since $\mathbf{g}$ is of class $C^{(2)}$, $d(dg^i) = \mathbf{0}$. Therefore, by the product rule

$$d(dg^{i_1} \wedge \cdots \wedge dg^{i_r}) = \mathbf{0}.$$

Using the product rule again, and with (7.24), we have

$$d\omega^\# = \sum_{[\lambda]} (d\omega_\lambda)^\# \wedge dg^{i_1} \wedge \cdots \wedge dg^{i_r} = (d\omega)^\#. \qquad \square$$

As in Chapter 4, let g^i_j denote the jth partial derivative of the component g^i. Let

$$g^\lambda_\mu = \det(g^{i_k}_{j_l}) = \frac{\partial(g^{i_1}, \ldots, g^{i_r})}{\partial(t^{j_1}, \ldots, t^{j_r})}.$$

The matrix of $D\mathbf{g}(\mathbf{t})$ is $(g^i_j(\mathbf{t}))$, and the row covectors are $dg^1(\mathbf{t}), \ldots, dg^n(\mathbf{t})$. By (7.22), the μth component of $dg^{i_1} \wedge \cdots \wedge dg^{i_r}$ is g^λ_μ. Therefore

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_r})^\# = \sum_{[\mu]} g^\lambda_\mu\, dt^{j_1} \wedge \cdots \wedge dt^{j_r}. \tag{7.51}$$

In applying this formula in Chapter 8 we usually take $r = m$. In that case the only increasing r-tuple is $\mu = (1, 2, \ldots, r)$ and the right-hand side of (7.51) has just one term.

EXAMPLE 1. Let $n = 3, r = m = 2, (x, y, z) = \mathbf{g}(s, t)$. Then

$$(dx \wedge dz)^\sharp = \frac{\partial(g^1, g^3)}{\partial(s, t)} ds \wedge dt.$$

If $\omega = f\, dx \wedge dz$, then $\omega^\sharp = f \circ \mathbf{g}(dx \wedge dz)^\sharp$.

EXAMPLE 2. Let $m = n = r$. Then, writing $f = \omega_{1 \ldots n}$, we have

$$\omega^\sharp = (f\, dx^1 \wedge \cdots \wedge dx^n)^\sharp = f \circ \mathbf{g}(dx^1 \wedge \cdots \wedge dx^n)^\sharp$$
$$= f \circ \mathbf{g} \frac{\partial(g^1, \ldots, g^n)}{\partial(t^1, \ldots, t^n)} dt^1 \wedge \cdots \wedge dt^n.$$

EXAMPLE 3. Let $m = r = 1$. Then $\omega^\sharp(t) = \omega[\mathbf{g}(t)] \cdot \mathbf{g}'(t)$, and the definition (6.12a) of the line integral can be rewritten

$$\int_\gamma \omega = \int_a^b \omega^\sharp.$$

PROBLEMS

1. Let $m = 2, n = 3$, and $\mathbf{g}(s, t) = (st)\mathbf{e}_1 + (s \cos t)\mathbf{e}_2 + (\exp t)\mathbf{e}_3$. Find $\omega^\sharp$ if:
   (a) $\omega = x\, dy$. (b) $\omega = y\, dz \wedge dx$.
   (c) $\omega = dx \wedge dy \wedge dz$. (d) $\omega = f$ (a 0-form).

2. (a) Let $\zeta = \omega^\sharp$. Show that the components satisfy
   (7.52) $$\zeta_\mu = \sum_{[\lambda]} (\omega_\lambda \circ \mathbf{g}) g^\lambda_\mu.$$
   (b) Show that $\omega^\sharp = \mathbf{0}$, if $D\mathbf{g}(\mathbf{t})$ has rank $< r$ for each $\mathbf{t} \in \Delta$ and ω is of degree r.
   [*Note*: In tensor language a differential form of degree r, being an r-covector-valued function, is called a covariant alternating *tensor field* of rank r. Formula (7.52) is the transformation law for the components of such a tensor field.]

3. Let $n = m = 2, r = 1, \omega = M\, dx + N\, dy$. Find explicitly $d\omega$ and $\omega^\sharp$ and verify that $(d\omega)^\sharp = d\omega^\sharp$.

4. Let $n = m = 3, \mathbf{g}(s, t, u) = (s \cos t)\mathbf{e}_1 + (s \sin t)\mathbf{e}_2 + u\mathbf{e}_3$. Find:
   (a) $(f\, dx \wedge dy \wedge dz)^\sharp$. (b) $(x\, dy \wedge dz)^\sharp$.

7.8 The adjoint and codifferential

To each r-vector α we now assign a certain $(n - r)$-vector, which is called the *adjoint* of α and is denoted by $*\alpha$. Let us begin with the special dimension $r = n - 1$, which is the only one needed in connection with the divergence theorem in Chapter 8.

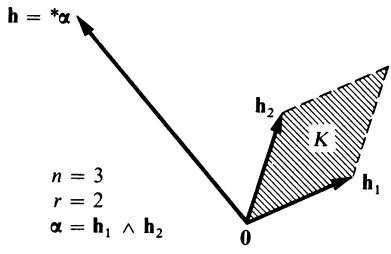


Figure 7.5

Let $\alpha = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_{n-1}$. If $\alpha = \mathbf{0}$, then we set $*\alpha = \mathbf{0}$. If $\alpha \neq \mathbf{0}$, then $*\alpha$ will turn out to be the vector $\mathbf{h}$ with the following three properties: (1) $\mathbf{h}$ is a vector normal to the $(n-1)$-space P spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_{n-1}$; (2) $(\mathbf{h}, \mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$ is a positively oriented frame for E^n; (3) $|\mathbf{h}| = |\alpha|$. Condition (3) says that the length of $\mathbf{h}$ equals $V_{n-1}(K)$, where K is the $(n-1)$-parallelepiped with vertex $\mathbf{0}$ spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_{n-1}$ (see Figure 7.5).

With this in mind, let us define * first for the standard basis $(n-1)$-vectors. Let $i' = (1, 2, \ldots, i-1, i+1, \ldots, n)$. Since $i-1$ interchanges will change the n-tuple (i, i') into the increasing n-tuple $(1, \ldots, n)$,

$$(-1)^{i-1}\mathbf{e}_i \wedge \mathbf{e}_{i'} = \mathbf{e}_{1\cdots n}.$$

Therefore we set

$$*\mathbf{e}_{i'} = (-1)^{i-1}\mathbf{e}_i. \tag{7.53}$$

We want the operation * to behave linearly.

For any $(n-1)$-vector $\alpha = \sum_{i=1}^n \alpha^{i'}\mathbf{e}_{i'}$, let $*\alpha$ be the vector $\mathbf{h} = \sum_{i=1}^n \alpha^{i'}(*\mathbf{e}_{i'})$. Its components are

$$h^i = (-1)^{i-1}\alpha^{i'}, \qquad i = 1, \ldots, n. \tag{7.54}$$

EXAMPLE 1. Let $n = 3$. In this particular dimension it is useful to consider, instead of the standard basis $\{\mathbf{e}_{23}, \mathbf{e}_{13}, \mathbf{e}_{12}\}$ for E_2^3, the basis $\{\mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}\}$, where $\mathbf{e}_{31} = -\mathbf{e}_{13}$. Then any 2-vector α can be written $\alpha = \alpha^{23}\mathbf{e}_{23} + \alpha^{31}\mathbf{e}_{31} + \alpha^{12}\mathbf{e}_{12}$, where $\alpha^{31} = -\alpha^{13}$; and

$$*\mathbf{e}_{23} = \mathbf{e}_1, \qquad *\mathbf{e}_{31} = \mathbf{e}_2, \qquad *\mathbf{e}_{12} = \mathbf{e}_3,$$
$$\alpha^{23} = h^1, \qquad \alpha^{31} = h^2, \qquad \alpha^{12} = h^3 \quad \text{if } \mathbf{h} = *\alpha.$$

Proposition 7.7. *The vector* $\mathbf{h} = *\alpha$ *defined by* (7.54) *has properties* (1), (2), (3) *above.*

PROOF. Given a frame $(\mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$, these three properties determine a vector, which we denote temporarily by $\tilde{\mathbf{h}}$. Let $(\mathbf{h}'_1, \ldots, \mathbf{h}'_{n-1})$ be an orthogonal frame for P, $\mathbf{h}'_k \cdot \mathbf{h}'_l = 0$ if $k \neq l$. Then α is a scalar multiple b of $\mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_{n-1}$, and replacing $\mathbf{h}'_1$ by $b\mathbf{h}'_1$ we may assume that $\alpha = \mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_{n-1}$. By (1), $\tilde{\mathbf{h}} \cdot \mathbf{h}'_k = 0$ for each $k = 1, \ldots, n-1$. Therefore

$$|\tilde{\mathbf{h}} \wedge \alpha| = |\tilde{\mathbf{h}}||\mathbf{h}'_1| \cdots |\mathbf{h}'_{n-1}| = |\tilde{\mathbf{h}}||\alpha|.$$

By (2), $\tilde{\mathbf{h}} \wedge \boldsymbol{\alpha} = c\mathbf{e}_{1\ldots n}$ where $c > 0$. In fact, $c = |\tilde{\mathbf{h}} \wedge \boldsymbol{\alpha}|$. Let $\mathbf{h} = {}^*\boldsymbol{\alpha}$. From (7.54), $|\mathbf{h}| = |\boldsymbol{\alpha}|$ and consequently $c = |\tilde{\mathbf{h}}||\mathbf{h}|$.

On the other hand,

$$\tilde{\mathbf{h}} \wedge \boldsymbol{\alpha} = \left(\sum_{i=1}^{n} \tilde{h}^i \mathbf{e}_i\right) \wedge \left(\sum_{j=1}^{n} \alpha^{j'} \mathbf{e}_{j'}\right) = \left[\sum_{i=1}^{n} \tilde{h}^i \alpha^{i'} (-1)^{i-1}\right] \mathbf{e}_{1\ldots n},$$

since $\mathbf{e}_i \wedge \mathbf{e}_{j'} = \mathbf{0}$ unless $i = j$. By (7.54)

$$\tilde{\mathbf{h}} \wedge \boldsymbol{\alpha} = (\tilde{\mathbf{h}} \cdot \mathbf{h})\mathbf{e}_{1\ldots n},$$

and hence $\tilde{\mathbf{h}} \cdot \mathbf{h} = c = |\tilde{\mathbf{h}}||\mathbf{h}|$. Equality holds in Cauchy's inequality (Section 1.2), and therefore $\tilde{\mathbf{h}}$ is a positive scalar multiple of $\mathbf{h}$. By (3), $|\tilde{\mathbf{h}}| = |\mathbf{h}|$, and hence $\tilde{\mathbf{h}} = \mathbf{h}$ as required. □

Corollary 1. *Every* $(n-1)$*-vector* $\boldsymbol{\alpha}$ *is decomposable.*

PROOF. Let $\boldsymbol{\alpha} \neq \mathbf{0}$, and let $\mathbf{h} = {}^*\boldsymbol{\alpha}$. The vector $\mathbf{h}$ is normal to an $(n-1)$-dimensional subspace P. Let $\tilde{\boldsymbol{\alpha}}$ be an $(n-1)$-vector of P whose orientation and norm are chosen such that $\tilde{\boldsymbol{\alpha}}$ and $\mathbf{h}$ are related by (1)—(3). Then $\tilde{\boldsymbol{\alpha}}$ is decomposable and $\mathbf{h} = {}^*\tilde{\boldsymbol{\alpha}}$. Thus ${}^*\tilde{\boldsymbol{\alpha}} = {}^*\boldsymbol{\alpha}$, which by (7.54) implies that $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$. □

The adjoint ${}^*\boldsymbol{\omega}$ of an $(n-1)$-covector $\boldsymbol{\omega}$ is defined in a similar way. We set ${}^*\boldsymbol{\omega} = \boldsymbol{\zeta}$, where $\boldsymbol{\zeta}$ is the 1-covector whose components satisfy the formula dual to (7.54):

$$\zeta_i = (-1)^{i-1}\omega_{i'}, \qquad i = 1, \ldots, n. \tag{7.55}$$

If $\boldsymbol{\omega}$ is an $(n-1)$-covector and $\boldsymbol{\alpha}$ an $(n-1)$-vector, then

$$({}^*\boldsymbol{\omega}) \cdot ({}^*\boldsymbol{\alpha}) = \boldsymbol{\omega} \cdot \boldsymbol{\alpha}, \tag{7.56}$$

since the two sides equal $\sum_{i=1}^{n} (-1)^{2i} \omega_{i'} \alpha_{i'} = \sum_{i=1}^{n} \omega_{i'} \alpha_{i'}$.

If $\boldsymbol{\omega}$ is an $(n-1)$-form, then ${}^*\boldsymbol{\omega}$ is defined as the 1-form $\boldsymbol{\zeta}$ whose components ζ_i satisfy (7.55). Thus $\boldsymbol{\zeta}(\mathbf{x}) = {}^*\boldsymbol{\omega}(\mathbf{x})$ for each $\mathbf{x}$ in the domain of $\boldsymbol{\omega}$. We have, in particular,

$${}^*(dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n) = (-1)^{i-1}\, dx^i. \tag{7.57}$$

EXAMPLE 1 (*continued*). For $n = 3$, (7.57) says that ${}^*(dx \wedge dy) = dz$, ${}^*(dz \wedge dx) = dy$, ${}^*(dy \wedge dz) = dx$. From these formulas one gets ${}^*\boldsymbol{\omega}$ for any 2-form $\boldsymbol{\omega}$ (Problem 2).

If $\boldsymbol{\omega}$ is an $(n-1)$-form of class $C^{(1)}$, then $d\boldsymbol{\omega}$ is an n-form. Consequently, $d\boldsymbol{\omega} = f\, dx^1 \wedge \cdots \wedge dx^n$, where f is a scalar-valued function. To get a convenient expression for f, let $\boldsymbol{\zeta} = {}^*\boldsymbol{\omega}$. Its components are given by (7.55). Let

$$\operatorname{div} \boldsymbol{\zeta} = \frac{\partial \zeta_1}{\partial x^1} + \cdots + \frac{\partial \zeta_n}{\partial x^n}. \tag{7.58}$$

This function is called the *divergence* of the 1-form ζ. By a short calculation (Problem 2), the desired function f is just div ζ. Thus

$$d\omega = \operatorname{div}\zeta\, dx^1 \wedge \cdots \wedge dx^n, \qquad \text{if } \zeta = {}^*\omega. \tag{7.59}$$

When $n = 3$ the divergence has an important physical significance, which is indicated in Section 8.4.

The remainder of this section is not used in Chapter 8. Let us define $^*\alpha$ for any r-vector α when $0 \leq r \leq n$. If $r = 0$ or n, we set

$$^*c = c\mathbf{e}_{1\ldots n}, \qquad {}^*(c\mathbf{e}_{1\ldots n}) = c.$$

If $0 < r < n$, let $\lambda = (i_1, \ldots, i_r)$ be any increasing r-tuple, and let $\lambda' = (j_1, \ldots, j_{n-r})$ be the increasing $(n-r)$-tuple whose entries are those integers j_l between 1 and n which do not appear in λ. Let

$$\varepsilon_\lambda = \delta^{1\cdots n}_{\lambda',\lambda}.$$

It is ± 1, depending on whether an odd or an even number of interchanges puts λ', λ in increasing order. If α is any r-vector, then its *adjoint* is the $(n-r)$-vector $^*\alpha$ whose components satisfy

$$(^*\alpha)^{\lambda'} = \alpha^\lambda \varepsilon_\lambda. \tag{7.60}$$

If $r = n-1$ and $\lambda = i'$, then $\lambda' = (i)$, $\varepsilon_\lambda = (-1)^{i-1}$, and (7.60) agrees with (7.54). From the definition (7.60),

$$^*(\alpha + \beta) = {}^*\alpha + {}^*\beta, \qquad {}^*(c\alpha) = c{}^*\alpha.$$

Moreover, $^*\alpha = \mathbf{0}$ if and only if $\alpha = \mathbf{0}$. Thus the operation * gives an isomorphism between E^n_r and E^n_{n-r}. This isomorphism preserves inner products. In fact, if α and β are r-vectors, then

$$^*\alpha \cdot {}^*\beta = \sum_{[\lambda]} (^*\alpha)^{\lambda'}(^*\beta)^{\lambda'} = \sum_{[\lambda]} (\varepsilon_\lambda)^2 \alpha^\lambda \beta^\lambda.$$

Since $(\varepsilon_\lambda)^2 = 1$, $^*\alpha \cdot {}^*\beta = \alpha \cdot \beta$. Taking $\alpha = \beta$ we have in particular $|{}^*\alpha| = |\alpha|$. Since $\varepsilon_{\lambda'}\varepsilon_\lambda = (-1)^{r(n-r)}$,

$$^{**}\alpha = (-1)^{r(n-r)}\alpha.$$

Now let ν be any increasing $(n-r)$-tuple. Then

$$\mathbf{e}_\nu \wedge \mathbf{e}_\lambda = \delta^{1\cdots n}_{\nu,\lambda}\mathbf{e}_{1\ldots n},$$

which is $\mathbf{0}$ if $\nu \neq \lambda'$ and is $\varepsilon_\lambda \mathbf{e}_{1\ldots n}$ if $\nu = \lambda'$. If β is any $(n-r)$-vector, then

$$\beta \wedge \alpha = \sum_{[\nu][\lambda]} \beta^\nu \alpha^\lambda \mathbf{e}_\nu \wedge \mathbf{e}_\lambda = \left(\sum_{[\lambda]} \beta^{\lambda'}\alpha^\lambda \varepsilon_\lambda\right)\mathbf{e}_{1\ldots n},$$

and

$$\beta \wedge \alpha = (\beta \cdot {}^*\alpha)\mathbf{e}_{1\ldots n}. \tag{7.61}$$

If $\alpha \neq \mathbf{0}$ is decomposable, then $*\alpha$ has the following geometric interpretation. Let $\alpha = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$, and let $\mathbf{h}_{r+1}, \ldots, \mathbf{h}_n$ be vectors such that: (1) $(\mathbf{h}_{r+1}, \ldots, \mathbf{h}_n)$ is a frame for the orthogonal complement of the r-space of α; (2) $(\mathbf{h}_{r+1}, \ldots, \mathbf{h}_n, \mathbf{h}_1, \ldots, \mathbf{h}_r)$ is a positively oriented frame for E^n; (3) $|\mathbf{h}_{r+1} \wedge \cdots \wedge \mathbf{h}_n| = |\alpha|$. Then $*\alpha = \mathbf{h}_{r+1} \wedge \cdots \wedge \mathbf{h}_n$. The proof is similar to that given above for $r = n - 1$.

If ω is an r-covector, then $*\omega$ is the $(n - r)$-covector such that

$$(*\omega)_{\lambda'} = \omega_\lambda \varepsilon_\lambda. \tag{7.62}$$

If ω is a differential form of degree r, then $*\omega$ is the $(n - r)$-form such that $(*\omega)(\mathbf{x}) = *\omega(\mathbf{x})$ for every $\mathbf{x}$ in the domain of ω.

EXAMPLE 2. $*(f\,dx^{i_1} \wedge \cdots \wedge dx^{i_r}) = \varepsilon_\lambda f\,dx^{j_1} \wedge \cdots \wedge dx^{j_{n-r}}$, where $\lambda' = (j_1, \ldots, j_{n-r})$. For $r = 0$ or n,

$$*f = f\,dx^1 \wedge \cdots \wedge dx^n, \qquad *(f\,dx^1 \wedge \cdots \wedge dx^n) = f.$$

Let ω be an r-form of class $C^{(1)}$. Then $d(*\omega)$ is an $(n - r + 1)$-form and $*d(*\omega)$ is an $(r - 1)$-form.

Definition. The *codifferential* of ω is

$$\tilde{d}\omega = (-1)^{r(n-r)} * d(*\omega). \tag{7.63}$$

Since $**\omega = (-1)^{r(n-r)}\omega$, substituting $*\omega$ for ω we get

$$\tilde{d}(*\omega) = *d\omega. \tag{7.64}$$

If $r = 0$, we invent a form $\mathbf{0}$ of degree -1 and agree that $\tilde{d}f = \mathbf{0}$. If ζ is a 1-form, consider the $(n - 1)$-form $\omega = (-1)^{n-1} *\zeta$. Then $*\omega = \zeta$ and by (7.64) $\tilde{d}\zeta = *d\omega$. By (7.59), $\tilde{d}\zeta = \operatorname{div}\zeta$. Thus the codifferential of a 1-form ζ is just the 0-form $\operatorname{div}\zeta$.

Note. Many authors define the adjoint so that in (2) above $(\mathbf{h}_1, \ldots, \mathbf{h}_r, \mathbf{h}_{r+1}, \ldots, \mathbf{h}_n)$ is a positively oriented frame for E^n. When $r(n - r)$ is odd, according to that definition $*\alpha$ has opposite sign to the one here.

The definition of the adjoint involves the euclidean norm. Hence both the adjoint and the codifferential depend on the euclidean structure inherited by E^n_r and $(E^n_r)^*$ from the euclidean inner product in E^n; while the notions of $\wedge$ and d actually depend only on the vector space structure and not the inner product.

In riemannian geometry one is provided at each point $\mathbf{x}$ with an inner product $B_\mathbf{x}$, not necessarily the euclidean inner product. The definition of adjoint must be modified accordingly. The codifferential is again defined by (7.63). However, the formula (7.58) for the divergence and its generalization (Problem 6) must be modified. [8; 21, Chapter V].

PROBLEMS

1. Let $n = 2$. Show that
   (a) $*\mathbf{h} = h^2\mathbf{e}_1 - h^1\mathbf{e}_2$.
   (b) $*(M\,dx + N\,dy) = N\,dx - M\,dy$.
   (c) $*d(N\,dx - M\,dy) = -(\partial M/\partial x + \partial N/\partial y)$.

2. (a) Let $n = 3$, and $\omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$ be a 2-form. Show that

$$*\omega = P\,dx + Q\,dy + R\,dz, d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dx \wedge dy \wedge dz$$

   and that $**\omega = \omega$.
   (b) Let ω be an $(n-1)$-form,

$$\omega = \sum_{i=1}^{n} \omega_i\, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n,$$

   and let $\zeta = *\omega$. Show that $d\omega = \operatorname{div} \zeta\, dx^1 \wedge \cdots \wedge dx^n$.

3. Show that: (a) $\operatorname{div}(df)$ is the Laplacian of f. (b) $\operatorname{div}(f\omega) = f \operatorname{div} \omega + df \cdot \omega$, where $(\zeta \cdot \omega)(\mathbf{x}) = \zeta(\mathbf{x}) \cdot \omega(\mathbf{x})$ for 1-forms ζ, ω.

4. Let $\mathbf{L}$ be a linear transformation from E^n into E^n such that $V_{n-1}(K) = V_{n-1}[\mathbf{L}(K)]$ for every $(n-1)$-parallelepiped K. Show that $\mathbf{L}$ is an isometry.

5. Let α and β be r-vectors. Show that:
   (a) $(*\alpha) \wedge \beta = \alpha \cdot \beta \mathbf{e}_1 \ldots_n$.
   (b) $(*\alpha) \wedge \beta = (-1)^{r(n-r)}\alpha \wedge (*\beta)$.
   (c) $(*\omega) \cdot (*\alpha) = \omega \cdot \alpha$ for any r-covector ω.

6. Show that the components of $\tilde{d}\omega$ satisfy $(\tilde{d}\omega)_v = \sum_{i=1}^{n} \partial\omega_{v,i}/\partial x^i$, where $(v, i) = (i_1, \ldots, i_{r-1}, i)$.

7. Show that the components of $d\omega$ satisfy $(d\omega)_\lambda = \sum_{j=1}^{r+1} (-1)^{j-1}\, \partial\omega_{\lambda_j}/\partial x^{ij}$, where λ_j is the r-tuple $(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{r+1})$.

8. If ζ is a 1-form and ω is an r-form let $\zeta \cdot \omega$ be the $(r-1)$-form such that $*(\zeta \cdot \omega) = (-1)^{n-1}\zeta \wedge (*\omega)$. Show that:
   (a) $\tilde{d}(f\omega) = f\,\tilde{d}\omega + df \cdot \omega$.
   (b) $(\zeta \cdot \omega)_v = \sum_{i=1}^{n} \zeta_i \omega_{v,i}$.

*7.9 Special results for $n = 3$

Vector analysis in E^3 is traditionally based on four operations besides the usual vector addition, scalar multiplication, and inner product. These operations are the cross product, triple scalar product, curl, and divergence. The last of these was defined in the previous section, for any dimension n. The other three are special to three dimensions, and can be expressed in terms of $\wedge$, $*$, and d as follows.

If $\mathbf{h}_1$ and $\mathbf{h}_2$ are vectors, their cross product is denoted by $\mathbf{h}_1 \times \mathbf{h}_2$. It is the vector

$$\mathbf{h}_1 \times \mathbf{h}_2 = *(\mathbf{h}_1 \wedge \mathbf{h}_2). \tag{7.65}$$

(see Figure 7.5). The cross product distributes with vector addition and scalar multiplication, and $\mathbf{h}_2 \times \mathbf{h}_1 = -\mathbf{h}_1 \times \mathbf{h}_2$. However, it is not associative.

The triple scalar product of three vectors is denoted by $[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]$. It is given by

$$[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3] = *(\mathbf{h}_1 \wedge \mathbf{h}_2 \wedge \mathbf{h}_3). \tag{7.66}$$

Its absolute value equals $|\mathbf{h}_1 \wedge \mathbf{h}_2 \wedge \mathbf{h}_3|$, which is the volume of the parallelepiped spanned by $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ with vertex $\mathbf{0}$. The sign of the triple scalar product is positive if $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ is a positively oriented frame for E^3 and negative if this frame is negatively oriented.

When $n = 3, r(n - r)$ is always even and $(-1)^{r(n-r)} = 1$. Then $*(\mathbf{h}_1 \times \mathbf{h}_2) = \mathbf{h}_1 \wedge \mathbf{h}_2$. Using Problem 5(a), Section 7.8,

$$\mathbf{h}_1 \wedge \mathbf{h}_2 \wedge \mathbf{h}_3 = (\mathbf{h}_1 \times \mathbf{h}_2) \cdot \mathbf{h}_3 \mathbf{e}_{123},$$

which gives another formula for the triple scalar product:

$$[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3] = (\mathbf{h}_1 \times \mathbf{h}_2) \cdot \mathbf{h}_3.$$

The cross product of two covectors, or of two 1-forms, is given by

$$\boldsymbol{\omega} \times \boldsymbol{\zeta} = *(\boldsymbol{\omega} \wedge \boldsymbol{\zeta}). \tag{7.67}$$

The curl of a 1-form $\boldsymbol{\omega}$ is the 1-form curl $\boldsymbol{\omega}$ given by

$$\operatorname{curl} \boldsymbol{\omega} = *d\boldsymbol{\omega} \tag{7.68}$$

Its physical significance is indicated in Section 8.8 in connection with Stokes's formula.

EXAMPLE 1. Show that $\operatorname{div}(\operatorname{curl} \boldsymbol{\omega}) = 0$ for every 1-form $\boldsymbol{\omega}$ of class $C^{(2)}$. Using the fact that $\tilde{d}* = *d$ (Formula 7.64),

$$\operatorname{div}(\operatorname{curl} \boldsymbol{\omega}) = \tilde{d}(*d\boldsymbol{\omega}) = *d(d\boldsymbol{\omega}) = *\mathbf{0} = \mathbf{0}.$$

PROBLEMS

Assume that all forms are of class $C^{(2)}$.

1. Show that:
(a) $\mathbf{h} \times \mathbf{k} = -\mathbf{k} \times \mathbf{h}$. (b) $\mathbf{h} \times (\mathbf{k}_1 + \mathbf{k}_2) = \mathbf{h} \times \mathbf{k}_1 + \mathbf{h} \times \mathbf{k}_2$.

2. Let $\boldsymbol{\omega} = M\,dx + N\,dy + O\,dz$. Show that

$$\operatorname{curl} \boldsymbol{\omega} = (\partial O/\partial y - \partial N/\partial z)dx + (\partial M/\partial z - \partial O/\partial x)dy + (\partial N/\partial x - \partial M/\partial y)dz.$$

3. Find $\mathbf{e}_i \times \mathbf{e}_j$ for all pairs $i, j = 1, 2, 3$.

4. With the aid of (7.67) and (7.68), show that:
(a) $\operatorname{div}(\boldsymbol{\zeta} \times \boldsymbol{\omega}) = 0$ if $\boldsymbol{\zeta}$ and $\boldsymbol{\omega}$ are closed.
(b) $\operatorname{curl}(f\boldsymbol{\omega}) = f \operatorname{curl} \boldsymbol{\omega} + df \times \boldsymbol{\omega}$.
(c) $\operatorname{curl}(f\,df) = \mathbf{0}$.
(d) $\operatorname{curl}(\boldsymbol{\zeta} \times \boldsymbol{\omega}) = \tilde{d}(\boldsymbol{\zeta} \wedge \boldsymbol{\omega})$.

(e) $\text{curl}(\text{curl}\ \omega) = d(\text{div}\ \omega) - \text{Lapl}\ \omega$, where $\text{Lapl}(M\,dx + N\,dy + O\,dz) = (\text{Lapl}\ M)dx + (\text{Lapl}\ N)dy + (\text{Lapl}\ O)dz$ and $\text{Lapl}\ f$ is the Laplacian of the function f.

(f) $\zeta \cdot \text{curl}\ \omega - \omega \cdot \text{curl}\ \zeta = \text{div}(\omega \times \zeta)$. [*Hint*: By the dual to (7.61), $\zeta \cdot {*d\omega} = {*}(\zeta \wedge d\omega)$.]

5. Show that

(a) $\mathbf{h}_1 \times (\mathbf{h}_2 \times \mathbf{h}_3) = (\mathbf{h}_1 \cdot \mathbf{h}_3)\mathbf{h}_2 - (\mathbf{h}_1 \cdot \mathbf{h}_2)\mathbf{h}_3$. [*Hint*: Since both sides are trilinear in $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$, it suffices to prove this when $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ are standard basis vectors. Use Problem 3.]

(b) The cross product is not associative.

(c) $(\mathbf{h}_1 \times \mathbf{h}_2) \times (\mathbf{h}_3 \times \mathbf{h}_4) = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_4]\mathbf{h}_3 - [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]\mathbf{h}_4$.

6. Let $\omega = (E_1\,dx^1 + E_2\,dx^2 + E_3\,dx^3) \wedge dx^4 + B_1\,dx^2 \wedge dx^3 + B_2\,dx^3 \wedge dx^1 + B_3\,dx^1 \wedge dx^2$, where the functions B_i, E_i are of class $C^{(1)}$ on an open subset of E^4. Show that $d\omega = \mathbf{0}$ if and only if $\text{curl}\ \mathbf{E} + \partial B/\partial x^4 = \mathbf{0}$, $\text{div}\ \mathbf{B} = 0$. Here curl and div are taken in the variables (x^1, x^2, x^3). {*Note*: The equation $d\omega = \mathbf{0}$ represents one-half of Maxwell's equations for an electromagnetic field in free space. The functions E_1, E_2, E_3 represent the electrical components of the field and B_1, B_2, B_3 the components of a magnetic induction vector. There is a similar equation representing the other half of Maxwell's equations [8, p. 45], T. Frankel, 1974, *Amer. Math. Monthly*, **81**, 343–348]}.

*7.10 Integrating factors (*continued*)

Let ω be a 1-form with domain D. We assume that ω is of class $C^{(1)}$, with $\omega(\mathbf{x}) \neq \mathbf{0}$. In Section 6.6 we gave necessary and sufficient conditions for ω to have an integrating factor. The following theorem gives another set of conditions, which are easier to apply.

Theorem 7.5

(a) *If ω has an integrating factor in D, then $\omega \wedge d\omega = \mathbf{0}$.*

(b) *Suppose that $\omega \wedge d\omega = \mathbf{0}$. Then, for any $\mathbf{x}^* \in D$, there exists an open set Δ such that $\mathbf{x}^* \in \Delta$ and ω has an integrating factor in Δ.*

The proof of (a) is an easy exercise, which we leave to the reader (see Problem 6, Section 7.4). As already mentioned (Section 6.6), in proving (b) it suffices to suppose that ω has the special form

$$\omega = dx^1 + \sum_{i=2}^{n} \omega_i\,dx^i. \tag{7.69}$$

In preparation for the proof of (b), we first prove the following.

Lemma 1. *Let ζ be a 2-form such that ζ is continuous on D and $\omega \wedge \zeta = \mathbf{0}$. Let $\mathbf{x}^* \in D$. Then there exist a neighborhood Ω of $\mathbf{x}^*$ and a 1-form θ continuous on Ω, such that $\zeta = \theta \wedge \omega$ in Ω.*

PROOF. For sufficiently small Ω there exist 1-forms $\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^n$ continuous on Ω such that $\{\boldsymbol{\eta}^1(\mathbf{x}), \ldots, \boldsymbol{\eta}^n(\mathbf{x})\}$ is an orthonormal basis for $(E^n)^*$ and $\boldsymbol{\eta}^1 = |\boldsymbol{\omega}|^{-1}\boldsymbol{\omega}$ (see Problem 1). Then

$$\boldsymbol{\zeta}(\mathbf{x}) = \sum_{i<j} \zeta_{ij}(\mathbf{x})\boldsymbol{\eta}^i(\mathbf{x}) \wedge \boldsymbol{\eta}^j(\mathbf{x}),$$

where $\zeta_{ij}(\mathbf{x})$ is the component of $\boldsymbol{\zeta}(\mathbf{x})$ with respect to the (i,j)th element of the corresponding orthonormal basis $\{\boldsymbol{\eta}^i(\mathbf{x}) \wedge \boldsymbol{\eta}^j(\mathbf{x}), i<j\}$ for $(E_2^n)^*$. Since $\boldsymbol{\omega} \wedge \boldsymbol{\zeta} = \mathbf{0}$,

$$\mathbf{0} = \boldsymbol{\eta}^1(\mathbf{x}) \wedge \boldsymbol{\zeta}(\mathbf{x}) = \sum_{2 \le i<j} \zeta_{ij}(\mathbf{x})\boldsymbol{\eta}^1(\mathbf{x}) \wedge \boldsymbol{\eta}^i(\mathbf{x}) \wedge \boldsymbol{\eta}^j(\mathbf{x}).$$

Hence $\zeta_{ij} = 0$ for $2 \le i < j$, and

$$\boldsymbol{\zeta}(\mathbf{x}) = \sum_{j=2}^{n} \zeta_{1j}(\mathbf{x})\boldsymbol{\eta}^1(\mathbf{x}) \wedge \boldsymbol{\eta}^j(\mathbf{x}).$$

We take $\boldsymbol{\theta} = -\sum_{j=2}^{n} \zeta_{1j}\boldsymbol{\eta}^j/|\boldsymbol{\omega}|$. Then $\boldsymbol{\zeta} = \boldsymbol{\theta} \wedge \boldsymbol{\omega}$ in Ω. □

PROOF OF THEOREM 7.5(b). Suppose that $\boldsymbol{\omega}$ is as in (7.69) with $\boldsymbol{\omega} \wedge d\boldsymbol{\omega} = \mathbf{0}$. We may suppose that $\mathbf{x}^* = \mathbf{0}$. By Lemma 1 we may also suppose that $d\boldsymbol{\omega} = \boldsymbol{\theta} \wedge \boldsymbol{\omega}$, for some 1-form $\boldsymbol{\theta}$.

In the proof we use a construction similar to that used in proving Theorem 6.2(b). We define $F(u, \mathbf{z})$ below in a way similar to the proof of that theorem, and show that (6.21) holds under the present hypotheses. This implies that an integrating factor exists in some open set Δ containing $\mathbf{0}$, by the last step of the proof of Theorem 6.2(b).

As in Section 6.6, we define $g^1(t)$ as the solution of the differential equation

$$\frac{dg^1}{dt} = -\sum_{i=2}^{n} \omega_i[g^1(t), t\mathbf{z}]x^i, \qquad \mathbf{z} = (x^2, \ldots, x^n) \tag{7.70}$$

with $g^1(0) = u$, for $0 \le t \le 1$, $(u, \mathbf{z}) \in \Omega_2$, where Ω_2 is some neighborhood of $\mathbf{0}$. By definition, $g^1(1) = F(u, \mathbf{z})$. Moreover, $g^1(t) = F(u, t\mathbf{z})$ (see Problem 2). Hence

$$\frac{dg^1}{dt} = \sum_{i=2}^{n} F_i(u, t\mathbf{z})x^i \tag{7.71}$$

where $F_i = \partial F/\partial x^i$. Let $\mathbf{H}(t, u, \mathbf{z}) = (F(u, t\mathbf{z}), t\mathbf{z})$. By (7.70) and (7.71) we have

$$0 = \sum_{i=2}^{n} (F_i + \omega_i \circ \mathbf{H})x^i \tag{7.72}$$

where F_i is evaluated at $(u, t\mathbf{z})$. Let $\#$ denote the transformation of differential forms induced by the transformation $\mathbf{H}$. Then from (7.69),

$$\boldsymbol{\omega}^\# = F_1\, du + \sum_{i=2}^{n} F_i(x^i\, dt + t\, dx^i) + \sum_{i=2}^{n} \omega_i \circ \mathbf{H}(x^i\, dt + t\, dx^i),$$

where $F_1 = (\partial F/\partial u)$. Let

$$\psi_i(t, u, \mathbf{z}) = \{F_i(u, t\mathbf{z}) + \omega_i[\mathbf{H}(t, u, \mathbf{z})]\}t.$$

By (7.72), the terms multiplying dt are zero, and

$$\boldsymbol{\omega}^\sharp = F_1\, du + \sum_{i=2}^{n} \psi_i\, dx^i.$$

By the fact that $d\boldsymbol{\omega} = \boldsymbol{\theta} \wedge \boldsymbol{\omega}$ and Theorem 7.4,

$$d\boldsymbol{\omega}^\sharp = \boldsymbol{\theta}^\sharp \wedge \boldsymbol{\omega}^\sharp.$$

Hence the coefficient of $dt \wedge dx^i$, $i \geq 2$, must be the same on both sides. On the left side this coefficient is $\partial\psi_i/\partial t$, and on the right side $\phi\psi_i$, where

$$\boldsymbol{\theta}^\sharp = \phi\, dt + \text{other terms}.$$

The precise expression for the coefficient $\phi(t, u, \mathbf{z})$ is unimportant. For fixed $u, \mathbf{z}$, the function ψ_i satisfies the linear differential equation

$$\frac{\partial \psi_i}{\partial t} = \phi\psi_i.$$

Since $\psi_i = 0$ when $t = 0$, we must have $\psi_i = 0$ for $0 \leq t \leq 1$. In particular, when $t = 1$ we obtain

$$0 = F_i(u, \mathbf{z}) + \omega_i[F(u, \mathbf{z}), \mathbf{z}], \qquad i = 2, \ldots, n.$$

This is equivalent to (6.21). □

Problems

1. Let $\boldsymbol{\omega}$ be a continuous 1-form with $\boldsymbol{\omega}(\mathbf{0}) \neq \mathbf{0}$. Show that there exist a neighborhood Ω of $\mathbf{0}$ and 1-forms $\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^n$ continuous on Ω such that $\boldsymbol{\eta}^i(\mathbf{x}) \cdot \boldsymbol{\eta}^j(\mathbf{x}) = \delta^{ij}, i, j = 1, \ldots, n$ and $\boldsymbol{\eta}^1(\mathbf{x}) = |\boldsymbol{\omega}(\mathbf{x})|^{-1}\boldsymbol{\omega}(\mathbf{x})$, for all $\mathbf{x} \in \Omega$. Here $\delta^{ij} = 1$ if $i = j$, $\delta^{ij} = 0$ if $i \neq j$. [*Hint*: Apply the Gram–Schmidt process to $\{\boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^n\}$ where $\boldsymbol{\omega}^1 = \boldsymbol{\omega}$ and $\boldsymbol{\omega}^2, \ldots, \boldsymbol{\omega}^n$ are chosen so that the covectors $\boldsymbol{\omega}^1(\mathbf{0}), \ldots, \boldsymbol{\omega}^n(\mathbf{0})$ are linearly independent.]

2. Let $g^1(t)$ be as in (7.70), and $g^1(1) = F(u, \mathbf{z})$. Show that $g^1(k) = F(u, k\mathbf{z})$ for $0 \leq k \leq 1$. [*Hint*: Consider the change of variable $\tau = kt$.]

Integration on manifolds

The topic of this chapter is integration over subsets of an r-manifold $M \subset E^n$. For this purpose we first study in Section 8.1 regular transformations from E^r into M. Then we find that coordinates can be introduced on portions of M, using the inverses of regular transformations. Such a portion S is called a *coordinate patch on* M. It is not always possible to find a single coordinate system for all of M. However, from the implicit function theorem, coordinates can be introduced locally. Using this fact, together with a device called partition of unity, the integral of a continuous function f over a set $A \subset M$ is defined in Section 8.3.

One of the main results of the chapter is the divergence theorem. If $D \subset E^n$ is an open set satisfying certain regularity conditions, then the divergence theorem equates the integral over the boundary fr D of the normal component of a 1-form ζ and the integral over D of the divergence div ζ. An equivalent version of the divergence theorem states that the integral of an $(n-1)$-form ω over fr D equals the integral of $d\omega$ over D, provided D and fr D are oriented consistently. When $n = 2$ and 3, the divergence theorem is equivalent to theorems in vector analysis commonly attributed to Green and to Gauss.

The divergence theorem is a special case of a result that states that the integral of the differential $d\omega$ of an $(r-1)$-form ω over a portion A of an oriented r-manifold M equals the integral of ω over the suitably oriented boundary of A. This is called Stokes's formula. In Section 8.10 the idea of homotopy between two transformations is introduced and is applied to give sufficient conditions in order that a closed differential form be exact. Some physical applications are mentioned in Section 8.5 and at the end of the chapter.

8.1 Regular transformations

Consider integers r and n with $1 \leq r \leq n$. We recall from Section 4.7 the concept of r-manifold $M \subset E^n$. One may think intuitively of M as being locally "like" E^r. For instance, if $r = 2$ and $n = 3$, then M is a 2-dimensional surface in E^3. In Chapter 4 an r-manifold M is defined as a subset of some euclidean E^n that can locally be described by setting equal to 0 functions $\Phi^1, \ldots, \Phi^{n-r}$ with linearly independent differentials. An r-manifold M has at each $\mathbf{x} \in M$ a tangent space, denoted in the present section by $T_M(\mathbf{x})$.

For purposes of integration it is necessary to consider manifolds from a different point of view. We must show that a manifold can be locally described by a system of coordinates. This idea will be made precise in Section 8.2. In the present section we study transformations from a set $\Delta \subset E^r$ into M that are regular in the sense defined below. By means of such transformations we can find geometric quantities associated with M, such as tangent spaces and the r-dimensional measure of certain subsets of M. We recall from Section 4.3 the concepts of differential $D\mathbf{g}(\mathbf{t})$ of a transformation $\mathbf{g}$, and of the partial derivatives $\mathbf{g}_1(\mathbf{t}), \ldots, \mathbf{g}_r(\mathbf{t})$.

Definition. Let $\mathbf{g}$ be a transformation from an open set $\Delta \subset E^r$ into an r-manifold $M \subset E^n$, where $r \leq n$. Then $\mathbf{g}$ is *regular* if:

(1) $\mathbf{g}$ is of class $C^{(1)}$,
(2) $\mathbf{g}$ is univalent, and
(3) $D\mathbf{g}(\mathbf{t})$ has rank r, for every $\mathbf{t} \in \Delta$.

Since $r \leq n$, Condition (3) states that the differential $D\mathbf{g}(\mathbf{t})$ has maximum possible rank. We see below that (3) can be replaced by an equivalent condition (3′). When $r = 1$, Condition (3) states that $\mathbf{g}'(t) \neq \mathbf{0}$. This is the same condition imposed in Section 6.2 in defining parametric representations of curves. The concepts of Section 6.2, such as equivalence, could be generalized to $r > 1$. However, this is not done in this book.

Proposition 8.1. *Let M be an r-manifold and $\mathbf{g}$ a regular transformation from $\Delta \subset E^r$ into M. Let $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$, $\mathbf{t}_0 \in \Delta$. Then the partial derivatives $\mathbf{g}_1(\mathbf{t}_0), \ldots, \mathbf{g}_r(\mathbf{t}_0)$ form a basis for the tangent space $T_M(\mathbf{x}_0)$.*

PROOF. Let $\{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_r\}$ be the standard basis for E^r. Since Δ is open, it contains the δ-neighborhood of $\mathbf{t}_0$ for some $\delta > 0$. For $|s| < \delta$, let $\boldsymbol{\psi}(s) = \mathbf{g}(\mathbf{t}_0 + s\boldsymbol{\varepsilon}_1)$. Then $\boldsymbol{\psi}$ is of class $C^{(1)}$, and $\boldsymbol{\psi}(s) \in M$. (One may think of $\boldsymbol{\psi}$ as representing a curve on M through $\mathbf{x}_0$, obtained by fixing the components $t^j = t_0^j$ of $\mathbf{t}$ for $j = 2, \ldots, r$.) By definition of tangent vector (Section 4.7), $\boldsymbol{\psi}'(0)$ is a tangent vector to M at $\mathbf{x}_0$. But $\boldsymbol{\psi}'(0) = \mathbf{g}_1(\mathbf{t}_0)$ by definition of partial derivative. Thus $\mathbf{g}_1(\mathbf{t}_0) \in T_M(\mathbf{x}_0)$. Similarly, the partial derivatives $\mathbf{g}_2(\mathbf{t}_0), \ldots,$

$\mathbf{g}_r(\mathbf{t}_0)$ are tangent vectors to M at $\mathbf{x}_0$. Since $D\mathbf{g}(\mathbf{t}_0)$ has rank r, the vectors $\mathbf{g}_1(\mathbf{t}_0), \ldots, \mathbf{g}_r(\mathbf{t}_0)$ are linearly independent. Hence they form a basis for $T_M(\mathbf{x}_0)$, which is an r-dimensional vector subspace of E^n. □

EXAMPLE 1. Let $M = \{(x, y, z): z^2 = x^2 + y^2, z > 0\}$. This 2-manifold $M \subset E^3$ is the upper half of a cone, with the vertex deleted. We take $\Delta = \{(s, t): s > 0, 0 < t < \pi\}$, $\mathbf{g}(s, t) = (s \cos t)\mathbf{e}_1 + (s \sin t)\mathbf{e}_2 + s\mathbf{e}_3$. Then $\mathbf{g}(\Delta) \subset M$. In fact, $\mathbf{g}(\Delta) = \{(x, y, z) \in M : y > 0\}$. Let us denote partial derivatives of $\mathbf{g}$ by $\partial\mathbf{g}/\partial s$, $\partial\mathbf{g}/\partial t$ rather than $\mathbf{g}_1$, $\mathbf{g}_2$. Then

$$\frac{\partial \mathbf{g}}{\partial s} = (\cos t)e_1 + (\sin t)\mathbf{e}_2 + \mathbf{e}_3$$

$$\frac{\partial \mathbf{g}}{\partial t} = (-s \sin t)\mathbf{e}_1 + (s \cos t)\mathbf{e}_2.$$

The vectors $\partial\mathbf{g}/\partial s$, $\partial\mathbf{g}/\partial t$ are linearly independent. By Proposition 8.1 they form a basis for the tangent space at $\mathbf{x} = \mathbf{g}(s, t)$. For instance, at the point $\mathbf{g}(2, \pi/3) = \mathbf{e}_1 + \sqrt{3}\mathbf{e}_2 + 2\mathbf{e}_3$, this basis is

$$\left\{\frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2 + \mathbf{e}_3, -\sqrt{3}\mathbf{e}_1 + \mathbf{e}_2\right\}.$$

By the same method of proof as for Proposition 8.1 one can show that $\mathbf{h} = D\mathbf{g}(\mathbf{t}_0)(\mathbf{k})$ is a tangent vector to M at $\mathbf{x}_0$, for any $\mathbf{k} \in E^r$ (see Problem 6). The special choices $\mathbf{k} = \boldsymbol{\varepsilon}_j$, $j = 1, \ldots, r$, gave the basis in Proposition 8.1 for the space $T_M(\mathbf{x}_0)$ of tangent vectors at $\mathbf{x}_0$. Thus we have:

Corollary. *$T_M(\mathbf{x}_0)$ is the image of E^r under the linear transformation $D\mathbf{g}(\mathbf{t}_0)$.*

The exterior product of the partial derivatives of $\mathbf{g}$ at $\mathbf{t}$ is the r-vector $\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t})$. Let us denote the euclidean norm of this r-vector by $\mathscr{J}\mathbf{g}(\mathbf{t})$. Thus

$$\mathscr{J}\mathbf{g}(\mathbf{t}) = |\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t})| \tag{8.1}$$

By Theorem 7.3, $\mathscr{J}\mathbf{g}(\mathbf{t}) > 0$ if and only if the vectors $\mathbf{g}_1(\mathbf{t}), \ldots, \mathbf{g}_r(\mathbf{t})$ are linearly independent. Since these vectors are the columns of the matrix of $D\mathbf{g}(\mathbf{t})$, their linear independence is equivalent to the fact that $D\mathbf{g}(\mathbf{t})$ has maximum rank r. Thus condition (3) in the definition of regular transformation is equivalent to

$$\mathscr{J}\mathbf{g}(\mathbf{t}) > 0 \quad \text{for every } \mathbf{t} \in \Delta. \tag{3'}$$

The notation $\mathscr{J}\mathbf{g}(\mathbf{t})$ gives a convenient way to define the concept of r-dimensional measure for certain subsets of an r-manifold M.

Definition. Let $\mathbf{g}$ be a regular transformation from Δ into M. Let $A = \mathbf{g}(B)$, where B is a measurable subset of Δ. Then the *r-dimensional measure* of A is

$$V_r(A) = \int_B \mathcal{J}\mathbf{g}(\mathbf{t})dV_r(\mathbf{t}). \tag{8.2}$$

For brevity, we also call $V_r(A)$ the *r-measure* of A. When $r = 1, 2, 3$, we often say length, area, or volume of A as usual, rather than r-measure. It will be shown in Section 8.3 that if $\tilde{\mathbf{g}}$ is another regular transformation and $\tilde{B}$ a set such that $A = \tilde{\mathbf{g}}(\tilde{B})$, then $V_r(A) = \int_{\tilde{B}} \mathcal{J}\tilde{\mathbf{g}}(\tilde{\mathbf{t}})dV_r(\tilde{\mathbf{t}})$. In the language introduced in Section 8.2, this means that $V_r(A)$ does not depend on the particular coordinate system chosen on A.

The motivation for Formula (8.2) is as follows. Consider first a point $\mathbf{t}_0 \in \Delta$, and I an r-cube with $\mathbf{t}_0$ as vertex and side length a, as in Figure 8.1. Let $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$ and K be the r-parallelepiped in E^n spanned by $a\mathbf{g}_1(\mathbf{t}_0), \ldots, a\mathbf{g}_r(\mathbf{t}_0)$ with $\mathbf{x}_0$ as vertex.

By (7.35),

$$V_r(K) = |a\mathbf{g}_1(\mathbf{t}_0) \wedge \cdots \wedge a\mathbf{g}_r(\mathbf{t}_0)| = a^r \mathcal{J}\mathbf{g}(\mathbf{t}_0).$$

Since $V_r(I) = a^r$, we have $V_r(K) = \mathcal{J}\mathbf{g}(\mathbf{t}_0)V_r(I)$. The affine approximation to $\mathbf{g}$ at $\mathbf{t}_0$ is

$$G(\mathbf{t}) = \mathbf{x}_0 + \mathbf{L}(\mathbf{t} - \mathbf{t}_0), \quad \mathbf{L} = D\mathbf{g}(\mathbf{t}_0).$$

Moreover, $K = \mathbf{G}(I)$. This suggests that $V_r(K)$ should be a very good approximation to $V_r[\mathbf{g}(I)]$. If $Z \subset \Delta$ is a figure that represents the union of small nonoverlapping r-cubes $I_1, \ldots, I_p$, then $V_r[\mathbf{g}(Z)]$ should approximately equal

$$\sum_{k=1}^{p} V(K_k) = \sum_{k=1}^{p} \mathcal{J}\mathbf{g}(\mathbf{t}_k)V_r(I_k),$$

where K_k is the r-parallelepiped obtained by affine approximation at an appropriate vertex $\mathbf{t}_k \in I_k$. Since a measurable set B can be approximated by

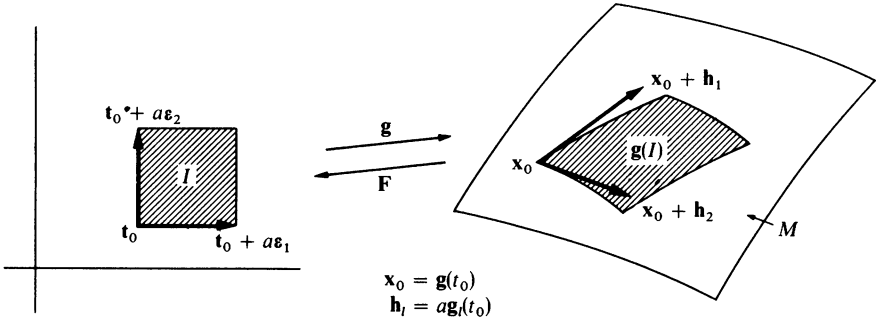


Figure 8.1

figures, this rough argument should make plausible the definition (8.2) of $V_r(A)$.

EXAMPLE 1 (*continued*). Let us find $\mathscr{J}\mathbf{g}(s, t)$. From the rules of exterior multiplication

$$\frac{\partial \mathbf{g}}{\partial s} \wedge \frac{\partial \mathbf{g}}{\partial t} = [(\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + \mathbf{e}_3] \wedge [(-s \sin t)\mathbf{e}_1 + (s \cos t)\mathbf{e}_2]$$

$$\frac{\partial \mathbf{g}}{\partial s} \wedge \frac{\partial \mathbf{g}}{\partial t} = s\mathbf{e}_{12} - (s \sin t)\mathbf{e}_{31} - (s \cos t)\mathbf{e}_{23},$$

$$\mathscr{J}\mathbf{g}(s, t) = |s\mathbf{e}_{12} - (s \sin t)\mathbf{e}_{31} - (s \cos t)\mathbf{e}_{23}|$$

$$\mathscr{J}\mathbf{g}(s, t) = (s^2 + s^2 \sin^2 t + s^2 \cos^2 t)^{1/2} = \sqrt{2}s.$$

Let us take $B = \{(s, t) : a < s < b, 0 < t < \pi\}$. Then $A = \mathbf{g}(B)$ is the portion of the cone $z^2 = x^2 + y^2$ between the planes $z = a$ and $z = b$, for which $y > 0$. We have

$$V_2(A) = \int_B 2s \, dV_2(s, t) = \int_0^\pi dt \int_a^b 2s \, ds = \frac{\pi}{2}(b^2 - a^2).$$

This result can, of course, also be found from elementary geometry.

EXAMPLE 2. Let $S \subset E^3$ be a set such that

$$S = \{(x, y, \phi(x, y)) : (x, y) \in R\},$$

where R is an open subset of E^2 and ϕ is of class $C^{(1)}$ on R. The set S is a 2-manifold. To see this, let $\Phi(x, y, z) = z - \phi(x, y)$, $D = \{(x, y, z) : (x, y) \in R\}$. Then grad $\Phi(\mathbf{x}) \neq \mathbf{0}$ and $S = \{\mathbf{x} \in D : \Phi(\mathbf{x}) = 0\}$. Hence S is the 2-manifold determined by Φ; see Section 4.7.

Let

$$\mathbf{g}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + \phi(x, y)\mathbf{e}_3.$$

Then $\mathbf{g}$ is of class $C^{(1)}$ from R onto S and is univalent. The vectors $\partial\mathbf{g}/\partial x$ and $\partial\mathbf{g}/\partial y$ give a basis for the tangent space $T_S[\mathbf{g}(x, y)]$. By (8.1), $\mathscr{J}\mathbf{g}(x, y) = |\partial\mathbf{g}/\partial x \wedge \partial\mathbf{g}/\partial y|$. (See Figure 8.2).

Calculating these partial derivatives, we get

$$\frac{\partial \mathbf{g}}{\partial x} = \mathbf{e}_1 + \frac{\partial \phi}{\partial x}\mathbf{e}_3, \qquad \frac{\partial \mathbf{g}}{\partial y} = \mathbf{e}_2 + \frac{\partial \phi}{\partial y}\mathbf{e}_3,$$

$$\frac{\partial \mathbf{g}}{\partial x} \wedge \frac{\partial \mathbf{g}}{\partial y} = \mathbf{e}_{12} - \frac{\partial \phi}{\partial y}\mathbf{e}_{31} - \frac{\partial \phi}{\partial x}\mathbf{e}_{23},$$

$$\left|\frac{\partial \mathbf{g}}{\partial x} \wedge \frac{\partial \mathbf{g}}{\partial y}\right| = \left[1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]^{1/2}$$

$$= [1 + |\text{grad } \phi|^2]^{1/2}.$$

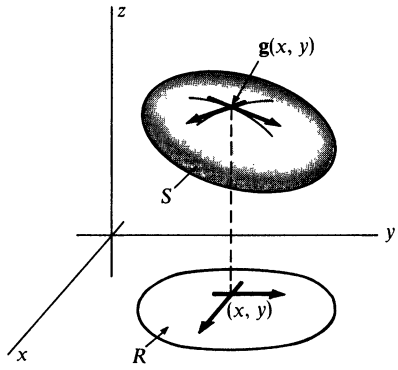


Figure 8.2

Since the last line is positive, $\mathscr{J}\mathbf{g}(x, y) > 0$. Thus $\mathbf{g}$ is a regular transformation from R onto S. The area of a set $A \subset S$ lying above a measurable set $B \subset R$ is

$$V_2(A) = \int_B [1 + |\operatorname{grad} \phi(x, y)|^2]^{1/2}\, dV_2(x, y).$$

Example 2 can be generalized to a set S described by expressing $n - r$ of the components $x^1, \ldots, x^n$ as functions of the remaining r components.

EXAMPLE 3. Let $\lambda = (i_1, \ldots, i_r)$ be an increasing r-tuple of integers, $1 \leq i_k \leq n$ for $k = 1, \ldots, r$; and let $(j_1, \ldots, j_{n-r})$ be the increasing $(n - r)$-tuple complementary to λ. Let R be an open subset of E^r, and $\phi^1, \ldots, \phi^{n-r}$ of class $C^{(1)}$ on R. Let $\mathbf{g}$ be the transformation from R into E^n such that

$$g^{i_k}(\mathbf{x}^\lambda) = x^{i_k}, \qquad k = 1, \ldots, r$$
$$g^{j_l}(\mathbf{x}^\lambda) = \phi^l(\mathbf{x}^\lambda), \qquad l = 1, \ldots, n - r,$$

where

$$\mathbf{x}^\lambda = (x^{i_1}, \ldots, x^{i_r}).$$

Let $S = \mathbf{g}(R)$. Then S is the r-manifold determined by $\mathbf{\Phi} = (\Phi^1, \ldots, \Phi^{n-r})$, where $\Phi^l(\mathbf{x}) = x^l - \phi^l(\mathbf{x}^\lambda)$ for $\mathbf{x} \in D$ with $D = \{\mathbf{x} : \mathbf{x}^\lambda \in R\}$. Then $\mathbf{g}$ is of class $C^{(1)}$ and univalent. The explicit formula for $\mathscr{J}\mathbf{g}(\mathbf{x}^\lambda)$ is complicated. However, we can show that $\mathscr{J}\mathbf{g}(\mathbf{x}^\lambda) > 0$ as follows. Since

$$\frac{\partial \mathbf{g}}{\partial x^{i_k}} = \mathbf{e}_{i_k} + \sum_{l=1}^{n-r} \frac{\partial \phi^l}{\partial x^{i_k}} \mathbf{e}_{j_l}, \qquad \frac{\partial \mathbf{g}}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial \mathbf{g}}{\partial x^{i_r}} = \mathbf{e}_\lambda + \text{other terms.}$$

This r-vector is not $\mathbf{0}$ since its λth component (the coefficient of $\mathbf{e}_\lambda$) is 1. Hence $\mathscr{J}\mathbf{g}(\mathbf{x}_\lambda) > 0$, which shows that $\mathbf{g}$ is regular from R onto S. If we set $\mathbf{X}^\lambda(\mathbf{x}) = \mathbf{x}^\lambda$, then $\mathbf{X}^\lambda$ projects E^n onto E^r and $\mathbf{g} = (\mathbf{X}^\lambda | S)^{-1}$.

The case $r = n$

Let ϕ be a transformation of class $C^{(1)}$ from $\Delta \subset E^r$ into E^r. In this chapter we call ϕ a *flat* transformation. Condition (3) in the definition of regular transformation above is equivalent to $J\phi(\mathbf{t}) \neq 0$ for every $\mathbf{t} \in \Delta$, where $J\phi(\mathbf{t})$ is the Jacobian. For flat transformations, this definition of regularity agrees with that in Section 4.5. Regular flat transformations will be an essential tool in relating overlapping coordinate systems on a manifold M. As a first step in this direction, we show the following.

Proposition 8.2. *Let $\tilde{\mathbf{g}}$ be a regular transformation from $\tilde{\Delta}$ into M, and ϕ a regular flat transformation from Δ into $\tilde{\Delta}$. Then $\mathbf{g} = \tilde{\mathbf{g}} \circ \phi$ is a regular transformation from Δ into M. Moreover,*

$$\mathscr{J}\mathbf{g}(\mathbf{t}) = \mathscr{J}\tilde{\mathbf{g}}[\phi(\mathbf{t})]\,|J\phi(\mathbf{t})|. \tag{8.3}$$

PROOF. Since $\tilde{\mathbf{g}}$ and ϕ are both $C^{(1)}$ and univalent, their composite $\mathbf{g}$ is also of class $C^{(1)}$ and univalent. By the composite function theorem (Section 4.3), $D\mathbf{g}(\mathbf{t}) = D\tilde{\mathbf{g}}[\phi(\mathbf{t})] \circ D\phi(\mathbf{t})$. Since $D\phi(\mathbf{t})$ is a nonsingular linear transformation from E^r onto E^r, and $D\tilde{\mathbf{g}}[\phi(\mathbf{t})]$ is a linear transformation of rank r, $D\mathbf{g}(\mathbf{t})$ also has rank r. Thus $\mathbf{g}$ is regular. It remains to prove (8.3). Given $\mathbf{t}$, let $\tilde{\mathbf{L}} = D\tilde{\mathbf{g}}[\phi(\mathbf{t})]$. If $\mathbf{k}_1, \ldots, \mathbf{k}_r$ are linearly independent vectors in E^r and $\mathbf{h}_j = \tilde{\mathbf{L}}(\mathbf{k}_j)$, $j = 1, \ldots, r$, then

$$\mathscr{J}\tilde{\mathbf{g}}[\phi(\mathbf{t})] = \frac{|\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|}{|\mathbf{k}_1 \wedge \cdots \wedge \mathbf{k}_r|}.$$

This follows by Formulas (7.40) and (7.40a) with $\mathbf{v}_j = \tilde{\mathbf{g}}_j[\phi(\mathbf{t})]$. In particular, let $\mathbf{k}_j = \phi_j(\mathbf{t})$, $j = 1, \ldots, r$, namely, the partial derivatives of ϕ at $\mathbf{t}$. By the composite function theorem, $\mathbf{h}_j = \mathbf{g}_j(\mathbf{t})$. Therefore,

$$\mathscr{J}\tilde{\mathbf{g}}[\phi(\mathbf{t})] = \frac{\mathscr{J}\mathbf{g}(\mathbf{t})}{\mathscr{J}\phi(\mathbf{t})}.$$

By (7.38),

$$\begin{aligned}\phi_1(\mathbf{t}) \wedge \cdots \wedge \phi_r(\mathbf{t}) &= \det \phi_i^k(\mathbf{t})\mathbf{e}_{1\ldots r} \\ &= J\phi(\mathbf{t})\mathbf{e}_{1\ldots r}.\end{aligned}$$

Therefore $\mathscr{J}\phi(\mathbf{t}) = |\phi_1(\mathbf{t}) \wedge \cdots \wedge \phi_r(\mathbf{t})| = |J\phi(\mathbf{t})|$. This proves (8.3). □

We recall that an r-manifold $M \subset E^n$ is a topological space, when given the relative topology (Section 2.6).

Proposition 8.3. *If $\mathbf{g}$ is a regular transformation from Δ into M, then $\mathbf{g}(\Delta)$ is a relatively open subset of M.*

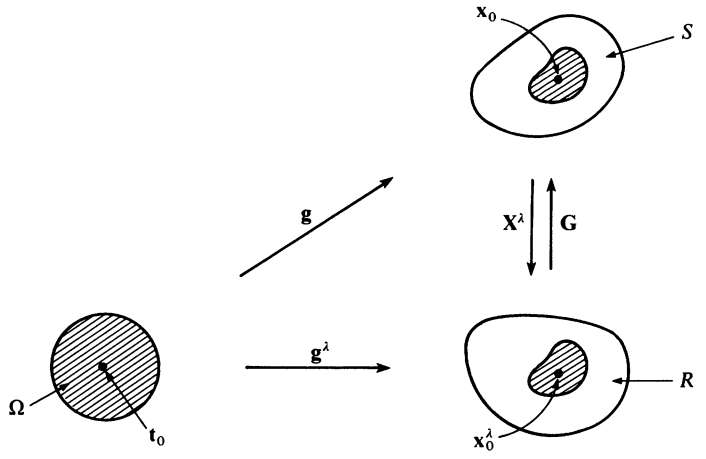


Figure 8.3

PROOF. Consider any $\mathbf{t}_0 \in \Delta$, and let $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$. It suffices to show that a neighborhood Ω of $\mathbf{t}_0$ exists such that $\mathbf{g}(\Omega)$ is a relatively open subset of M containing $\mathbf{x}_0$. If $\lambda = (i_1, \ldots, i_r)$ is an increasing r-tuple, then let $\mathbf{x}^\lambda = (x^{i_1}, \ldots, x^{i_r})$, and $\mathbf{X}^\lambda(\mathbf{x}) = \mathbf{x}^\lambda$ as in Example 3 above.

By the implicit function theorem, there exist an increasing r-tuple λ and a neighborhood U of $\mathbf{x}_0$ with the following properties. Let $S = M \cap U$ and $R = \mathbf{X}^\lambda(S)$. Then R is an open subset of E^r containing $\mathbf{x}_0$, $\mathbf{X}^\lambda | S$ is univalent, and $\mathbf{G} = (\mathbf{X}^\lambda | S)^{-1}$ is a regular transformation from R onto S. Note that $\mathbf{G}$ is the kind of regular transformation denoted by $\mathbf{g}$ in Example 3.

Let $\mathbf{g}^\lambda = \mathbf{X}^\lambda \circ \mathbf{g}$. Note that $\mathbf{g}^\lambda$ is the flat transformation obtained by considering only the components $g^{i_1}, \ldots, g^{i_r}$ of $\mathbf{g}$. Let Ω be a neighborhood of $\mathbf{t}_0$ such that $\mathbf{g}(\Omega) \subset S$. For $\mathbf{t} \in \Omega$, $\mathbf{g}(\mathbf{t}) = \mathbf{G}[\mathbf{g}^\lambda(\mathbf{t})]$ (see Figure 8.3). The transformation $\mathbf{g}^\lambda$ is $C^{(1)}$, and $\mathbf{g}^\lambda | \Omega$ is univalent. By the composite function theorem, $D\mathbf{g}(\mathbf{t}) = D\mathbf{G}(\mathbf{x}^\lambda) \circ D\mathbf{g}^\lambda(\mathbf{t})$. Since $D\mathbf{g}(\mathbf{t})$ has maximum rank r, $D\mathbf{g}^\lambda(\mathbf{t})$ must also have rank r for any $\mathbf{t} \in \Delta$. Thus $\mathbf{g}^\lambda | \Omega$ is a regular flat transformation from Ω into R. By the corollary to the inverse function theorem (Section 4.5), $\mathbf{g}^\lambda(\Omega)$ is an open subset of R. By the way $\mathbf{G}$ is constructed, $\mathbf{G}(R_1)$ is open relative to M, for any open $R_1 \subset R$. But $\mathbf{g}(\Omega) = \mathbf{G}(R_1)$ when $R_1 = \mathbf{g}^\lambda(\Omega)$. Thus $\mathbf{g}(\Omega)$ is a relatively open subset of M containing $\mathbf{x}_0$. □

Remark. In the definition of regular transformation at the beginning of the section, we assumed not only Conditions (1), (2) and (3) but also that the set $\mathbf{g}(\Delta)$ is contained in an r-manifold M. It is shown in Section 8.9 that if $\mathbf{g}$ is any transformation from Δ into E^n, then $\mathbf{g}(\Delta)$ is an r-manifold provided (1), (2), and (3) hold and that $\mathbf{g}(\Delta_1)$ is open relative to $\mathbf{g}(\Delta)$ for any open set $\Delta_1 \subset \Delta$. Problem 3 illustrates the difficulty encountered if simply (1), (2), and (3) are assumed.

PROBLEMS

1. For each of the following transformations from $\Delta \subset E^2$ into E^3, find $\mathscr{J}\mathbf{g}(s, t)$ and $\mathbf{g}(\Delta)$. Show that $\mathbf{g}$ is univalent.
   (a) $\mathbf{g}(s, t) = (s + t)\mathbf{e}_1 + (s - 3t)\mathbf{e}_2 + (-2s + 2t + 2)\mathbf{e}_3$, $\Delta = \{(s, t): 0 < s + t < 1, s > 0, t > 0\}$.
   (b) $\mathbf{g}(\rho, \theta) = (\rho \cos \alpha)\mathbf{e}_1 + (\rho \sin \alpha \cos \theta)\mathbf{e}_2 + (\rho \sin \alpha \sin \theta)\mathbf{e}_3, 0 < \alpha < \pi/2$ (α fixed), $\Delta = (0, \infty) \times (0, 2\pi)$.
   (c) $\mathbf{g}(s, t) = st\mathbf{e}_1 + s\mathbf{e}_2 + t\mathbf{e}_3, \Delta = E^2$.

2. In each case find the tangent space $T_S(\mathbf{x}_0)$ to $S = \mathbf{g}(\Delta)$ at the indicated point $\mathbf{x}_0$.
   (a) In Problem 1(a), take $\mathbf{x}_0 = \frac{2}{3}\mathbf{e}_1 - \frac{2}{3}\mathbf{e}_2 + 2\mathbf{e}_3$.
   (b) In Problem 1(b), take $\mathbf{x}_0 = (\cos \alpha)\mathbf{e}_1 + \sqrt{2}/2 \sin \alpha(\mathbf{e}_2 + \mathbf{e}_3)$.
   (c) In Problem 1(c), take $\mathbf{x}_0 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$.

3. Let $\mathbf{g}(t) = (\cos t)\mathbf{e}_1 + (\sin 2t)\mathbf{e}_2, \Delta = (0, 3\pi/2)$. Sketch $\mathbf{g}(\Delta)$. Show that $\mathbf{g}$ is univalent, that $\mathscr{J}\mathbf{g}(t) > 0$, but that $\mathbf{g}(\Delta)$ is not a 1-manifold.

4. Find the area of $\{(x, y, xy): x^2 + y^2 \leq 1\}$.

5. Let $n = 2, r = 3$. Show that (8.2) becomes

$$V_2(A) = \int_B \left\{ \left[\frac{\partial(g^2, g^3)}{\partial(s, t)} \right]^2 + \left[\frac{\partial(g^3, g^1)}{\partial(s, t)} \right]^2 + \left[\frac{\partial(g^1, g^2)}{\partial(s, t)} \right]^2 \right\}^{1/2} dV_2(s, t).$$

6. Let $\mathbf{g}$ be regular from Δ into M. Show that $D\mathbf{g}(\mathbf{t}_0)(\mathbf{k})$ is a tangent vector to M at $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$, for any vector $\mathbf{k} \in E^r$.

7. Let $n = 4, r = 2, \lambda = (1, 2)$.
   (a) Show that in Example 3,

$$\mathscr{J}\mathbf{g}(x^1, x^2) = [1 + |\operatorname{grad} \phi^1|^2 + |\operatorname{grad} \phi^2|^2 + (\phi_1^1\phi_2^2 - \phi_1^2\phi_2^1)^2]^{1/2}.$$

   (b) Find $V_2(A)$ if

$$A = \{\mathbf{x} \in E^4 : x^3 = (x^1)^2 - (x^2)^2, x^4 = 2x^1x^2, (x^1)^2 + (x^2)^2 \leq 1\}.$$

8.2 Coordinate systems on manifolds

Let M be an r-manifold. Since M is r-dimensional, it should be possible to find, at least locally on M, r functions $F^1, \ldots, F^r$ such that the numbers $F^1(\mathbf{x}), \ldots, F^r(\mathbf{x})$ serve as coordinates of a point $\mathbf{x} \in M$. If S is a portion of M coordinatized in this way, then $\mathbf{F} = (F^1, \ldots, F^r)$ should be a univalent function from S into E^r. We also require that the inverse $\mathbf{g} = \mathbf{F}^{-1}$ be regular, as defined in Section 8.1.

Definition. Let S be a nonempty, relatively open subset of an r-manifold $M \subset E^n$. A univalent transformation $\mathbf{F}$ from S into E^r is a *coordinate system* for S if $\mathbf{F} = \mathbf{g}^{-1}$, where $\mathbf{g}$ is a regular transformation from an open set Δ onto S. The coordinates of a point $\mathbf{x} \in S$ in this system are $F^1(\mathbf{x}), \ldots, F^r(\mathbf{x})$.

One might prefer a criterion in terms of $\mathbf{F}$ rather than its inverse, for $\mathbf{F}$ to be a coordinate system. Such a criterion is given in Problem 5. On the other hand, the inverse $\mathbf{g}$ rather than $\mathbf{F}$ is ordinarily used for calculations in the sections to follow.

When $r = n$, we called in Section 5.9 any regular flat transformation $\mathbf{F} = (F^1, \ldots, F^n)$ a coordinate system. The inverse $\mathbf{g} = \mathbf{F}^{-1}$ of a regular flat transformation is regular. Hence the present definition is consistent with that in Section 5.9, when $r = n$.

EXAMPLE 1. Let $M = \{(x, y, z) : z^2 = k^2(x^2 + y^2), z \neq 0\}$ where $k > 0$. This 2-manifold M is a cone in E^3 with the vertex removed. Let us consider three possible coordinate systems.

(a) Let S be the upper half of M, namely, $S = \{(x, y, z) \in M : z > 0\}$. Let us take x, y as coordinates of a point on S. Thus, $\mathbf{F} = (X \mid S, Y \mid S)$ where X, Y, Z are the standard cartesian coordinate functions for E^3. The inverse is

$$\mathbf{g}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + k(x^2 + y^2)^{1/2}\mathbf{e}_3,$$

and $\Delta = E^2 - \{(0, 0)\}$. The lower half of M is coordinatized similarly, replacing k by $-k$.

(b) Let us use polar coordinates (r, θ) in the (x, y) plane as coordinates of points on the upper half S of the cone M. Let

$$\tilde{\mathbf{g}}(r, \theta) = (r \cos \theta)\mathbf{e}_1 + (r \sin \theta)\mathbf{e}_2 + kr\mathbf{e}_3,$$

for $(r, \theta) \in \tilde{\Delta}$, where $\tilde{\Delta} = (0, \infty) \times (0, 2\pi)$. This coordinatizes $\tilde{S} = \tilde{\mathbf{g}}(\tilde{\Delta})$. The set $\tilde{S}$ is a relatively open subset of S, obtained by removing the ray on S corresponding to $\theta = 0$ or 2π. The restriction $0 < \theta < 2\pi$ is needed to ensure that $\tilde{\mathbf{g}}$ is univalent. When $k = 1$, $\tilde{\mathbf{g}}$ is the same transformation considered in Example 1, Section 8.1.

(c) Spherical coordinates (see Problem 3).

EXAMPLE 2. Let us return to Examples 2 and 3, Section 8.1. In the first of these examples, $F = (X \mid S, Y \mid S)$ is a coordinate system for S. The coordinates of a point $(x, y, \phi(x, y)) \in S$ in this system are x, y. In Example 3, $\mathbf{F} = \mathbf{X}^\lambda \mid S$. The coordinates of a point $\mathbf{x}$ in the coordinate system $\mathbf{F}$ are $x^{i_1}, \ldots, x^{i_r}$. Such a coordinate system is called *cartesian*. The choice of cartesian or some other coordinate system is often determined by convenience.

Proposition 8.4. *If* $\mathbf{F}$ *is a coordinate system for* S, *then* $\mathbf{F}$ *is a homeomorphism from* S *onto* $\Delta = \mathbf{F}(S)$.

PROOF. By definition, $\mathbf{F}$ is univalent and its inverse $\mathbf{g}$ is $C^{(1)}$, hence continuous. We must show that $\mathbf{F}$ is continuous from S onto Δ. By Theorem 2.6, this is valid provided $\mathbf{F}^{-1}(B)$ is open relative to S for any open set $B \subset \Delta$. But

$\mathbf{F}^{-1}(B) = \mathbf{g}(B)$; by Proposition 8.3 (with Δ there replaced by B and M by S), $\mathbf{g}(B)$ is open relative to S. □

It is not ordinarily possible to find a single coordinate system for all of a manifold M. If $S \subset M$ has a coordinate system $\mathbf{F}$, then by Proposition 8.4 S is homeomorphic with an open set $\Delta \subset E^r$. Since Δ cannot be both open and compact, S is not compact. In particular, a compact manifold M (e.g., a sphere or torus) cannot be coordinatized by a single system.

Definition. A relatively open set $S \subset M$ which has a coordinate system is a *coordinate patch* on M.

By the implicit function thorem every point of M lies in some coordinate patch S of the sort in Example 3, Section 8.1. Actually, each point of M lies in an infinite number of coordinate patches. Let us now show how different coordinate systems are related in overlapping patches.

Coordinate changes

We recall from Proposition 8.2 that the composite $\mathbf{g} = \tilde{\mathbf{g}} \circ \boldsymbol{\phi}$ of a regular transformation $\mathbf{g}$ and a regular flat transformation $\boldsymbol{\phi}$ is regular. If $\tilde{\Delta}$ and Δ are the respective domains and if $\tilde{\Delta} = \boldsymbol{\phi}(\Delta)$, then $\tilde{\mathbf{g}}(\tilde{\Delta}) = \mathbf{g}(\Delta)$. If we denote this set by S, then S is a coordinate patch on M. Moreover, $\mathbf{F} = \mathbf{g}^{-1}$ and $\tilde{\mathbf{F}} = \tilde{\mathbf{g}}^{-1}$ are both coordinate systems for S. In this way, any regular flat transformation leads to a change of coordinates.

EXAMPLE 1 (*continued*). In this example, $\tilde{\mathbf{g}}(r, \theta) = \mathbf{g}(r \cos \theta, r \sin \theta)$. Let us replace Δ in part (a) by $\Delta_1 = E^2 - \{(x, 0) : x > 0\}$. Such a change makes no difference in computing integrals over a part of Δ, since the half-line that has been removed has 2-dimensional measure 0. For $(x, y) \in \Delta_1$, let $R(x, y) = (x^2 + y^2)^{1/2}$ and $\Theta(x, y)$ the angle from the positive x-axis to (x, y) with $0 < \Theta(x, y) < 2\pi$. Then $\boldsymbol{\phi} = (R, \Theta)$ and $\mathbf{g} = \tilde{\mathbf{g}} \circ \boldsymbol{\phi}$.

Let us next consider overlapping coordinate patches S and $\tilde{S}$ on M, with respective coordinate systems $\mathbf{F}$ and $\tilde{\mathbf{F}}$. Let us show that on $S \cap \tilde{S}$ these coordinate systems are related by such a regular flat transformation $\boldsymbol{\phi}$. Let

$$\Delta_0 = \mathbf{F}(S \cap \tilde{S}), \qquad \tilde{\Delta}_0 = \tilde{\mathbf{F}}(S \cap \tilde{S}),$$

and $\mathbf{g} = \mathbf{F}^{-1}$, $\tilde{\mathbf{g}} = \tilde{\mathbf{F}}^{-1}$. Figure 8.4 shows the appropriate choice for $\boldsymbol{\phi}$.

Proposition 8.5. *Let* $\boldsymbol{\phi} = \tilde{\mathbf{F}} \circ (\mathbf{g}|\Delta_0)$ *as in Figure* 8.4. *Then* $\boldsymbol{\phi}$ *is a regular flat transformation from* Δ_0 *onto* $\tilde{\Delta}_0$, *and* $\mathbf{g}(\mathbf{t}) = \tilde{\mathbf{g}}[\boldsymbol{\phi}(\mathbf{t})]$ *for all* $\mathbf{t} \in \Delta_0$.

PROOF. The final assertion holds by construction. It remains to show that $\boldsymbol{\phi}$ is regular. Since $\boldsymbol{\phi}$ is univalent by construction, it suffices to show that any

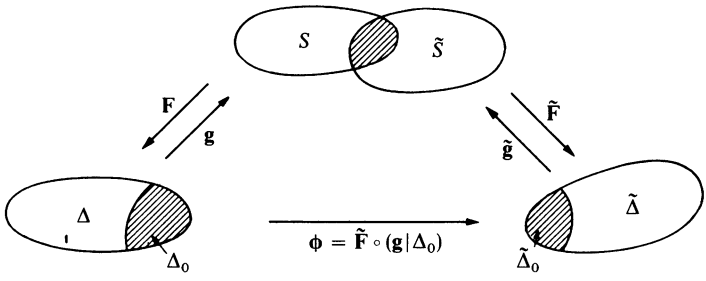


Figure 8.4

$\mathbf{t}_0 \in \Delta_0$ has a neighborhood Ω_1 such that $\boldsymbol{\phi}|\Omega_1$ is regular. Let $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$. Let λ and a neighborhood U of $\mathbf{x}_0$ be determined from the implicit function theorem, as in the proof of Proposition 8.3. It was shown there that $\mathbf{t}_0$ has a neighborhood Ω such that $\mathbf{g}^\lambda|\Omega$ is a regular flat transformation. Similarly, $\tilde{\mathbf{t}}_0 = \boldsymbol{\phi}(\mathbf{t}_0)$ has a neighborhood $\tilde{\Omega}$ such that $\tilde{\mathbf{g}}^\lambda|\tilde{\Omega}$ is regular. By the corollary to the inverse function theorem, $\tilde{R} = \tilde{\mathbf{g}}^\lambda(\tilde{\Omega})$ is an open set. Moreover, $\mathbf{x}_0^\lambda = \mathbf{g}^\lambda(\mathbf{t}_0) \in \tilde{R}$. Let Ω_1 be a neighborhood of $\mathbf{t}_0$ of radius small enough that $\Omega_1 \subset \Omega \cap \Delta_0$ and $\mathbf{g}^\lambda(\Omega_1) \subset \tilde{R}$. Let $\mathbf{f} = (\tilde{\mathbf{g}}^\lambda|\tilde{\Omega})^{-1}$. Then $\boldsymbol{\phi}(\mathbf{t}) = \mathbf{f}[\mathbf{g}^\lambda(\mathbf{t})]$ for all $\mathbf{t} \in \Omega_1$. Since $\mathbf{f}$ is the inverse of a regular flat transformation, $\mathbf{f}$ is regular. Since $\mathbf{g}^\lambda|\Omega_1$ is regular, the composite $\boldsymbol{\phi}|\Omega_1$ is regular. □

*Manifolds defined by coordinate systems

We have seen that an r-manifold $M \subset E^n$ is covered by coordinate patches. Conversely, let M be a subset of E^n with the following property. There is a collection $\mathscr{S}$ of relatively open subsets of M that cover M, and with each $S \in \mathscr{S}$ there is associated a homeomorphism $\mathbf{F}$ from S onto an open set $\Delta \subset E^r$ such that the inverse $\mathbf{g} = \mathbf{F}^{-1}$ is regular from Δ onto S. Since each $S \in \mathscr{S}$ is an r-manifold, M is an r-manifold. This shows that instead of the approach in Section 4.7 we could have defined manifolds in terms of coordinate systems.

We have taken a rather concrete approach to the idea of manifold, considering only manifolds given as subsets of some euclidean space. However, the manifolds encountered in practice often are not given in this way. The approach via coordinate systems allows one to take a more abstract point of view. From this viewpoint the definition of manifold is as follows. An r-manifold of class $C^{(1)}$ is a Hausdorff topological space Z provided with a collection of open subsets S (called coordinate patches) covering Z and for each coordinate patch a homeomorphism $\mathbf{F}$ from S onto an open set $\Delta \subset E^r$. The regularity of the flat transformation $\boldsymbol{\phi}$ in Figure 8.4 is now imposed as an axiom.

If Z is an r-manifold according to this more abstract definition and Z is separable, then Z can be realized as a submanifold of some euclidean space, in fact as a submanifold of E^{2r+1} [24, Chapter IV]. A result of this type is

called an embedding theorem. By separable we mean that Z has a covering either by finitely many coordinate patches or by a sequence of coordinate patches.

PROBLEMS

1. Let $M = \{(x, y, z): x^2 + 2y + z^2 = 3, z > 0, y > |x|\}$. Let $\mathbf{F} = (X|M, Y|M)$ and $\tilde{\mathbf{F}} = (X|M, Z|M)$. Describe Δ, $\tilde{\Delta}$, $\mathbf{g}$, $\tilde{\mathbf{g}}$, and $\boldsymbol{\phi}$ (see Figure 8.4).

2. Let $M = \{(y^2 + z^2, y, z): y > 0\}$ and let $\mathbf{F}(x, y, z) = (y + z, \exp z)$ for $(x, y, z) \in M$. Show that $\mathbf{F}$ is a coordinate system for M and find $\mathbf{F}(M)$. [*Hint*: First take y and z as coordinates on M and then find a suitable coordinate change $\boldsymbol{\phi}$ giving the system $\mathbf{F}$.]

3. In Example 1, take (ρ, θ) as coordinates on $\tilde{S}$, where $\rho = (x^2 + y^2 + z^2)^{1/2}$. Find a regular flat transformation $\boldsymbol{\phi}$ taking (ρ, θ) into (r, θ). Show that $\tilde{\tilde{\mathbf{F}}} = (\tilde{\mathbf{g}} \circ \boldsymbol{\phi})^{-1}$ is such that $\tilde{\tilde{\mathbf{F}}}(x, y, z) = (\rho, \theta)$.

4. Let $\mathbf{g}$ be regular from an open interval $\Delta \subset E^1$ into a 1-manifold M. Let $S = \mathbf{g}(\Delta)$. Describe a coordinate system F for S such that $|F(\mathbf{x}_1) - F(\mathbf{x}_2)|$ is the length of the subarc of S with endpoints $\mathbf{x}_1$ and $\mathbf{x}_2$. [*Hint*: A curve can be parameterized by arc length (Section 6.2).]

5. Let $M \subset E^n$ be an r-manifold and S a relatively open subset of M. Let $\mathbf{F}$ be of class $C^{(1)}$ from an open set containing S into E^r. Suppose that $\mathbf{F}|S$ is univalent and that $D\mathbf{F}(\mathbf{x})|T_M(\mathbf{x})$ is a linear transformation of rank r for each $\mathbf{x} \in S$. Show that $\Delta = \mathbf{F}(S)$ is open and $\mathbf{g} = (\mathbf{F}|S)^{-1}$ is regular. [*Hint*: Consider $\mathbf{G}$ as in the proof of Proposition 8.3. Show that $\mathbf{F} \circ \mathbf{G}$ is a regular flat transformation if R is a sufficiently small open set containing $\mathbf{x}_0^{\lambda}$.]

6. (*Stereographic projection.*) Let M be the sphere $x^2 + y^2 + (z - 1)^2 = 1$. For each $\mathbf{x} = (x, y, z) \in M$ except the "north pole" $2\mathbf{e}_3$, let $(s, t, 0)$ be the point where the line through $2\mathbf{e}_3$ and $\mathbf{x}$ meets the plane $z = 0$. Let $\mathbf{F}(\mathbf{x}) = (s, t)$.
   (a) Show that $\mathbf{F}$ is a coordinate system for $M - \{2\mathbf{e}_3\}$.
   (b) Let $\mathbf{h}_1$, $\mathbf{h}_2$ be tangent vectors to M at $\mathbf{x}$, and let $\mathbf{k}_l = D\mathbf{F}(\mathbf{x})(\mathbf{h}_l)$, $l = 1, 2$. Show that the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$ equals the angle between $\mathbf{h}_1$ and $\mathbf{h}_2$.

7. Let $f^1, \ldots, f^r, \Phi^1, \ldots, \Phi^{n-r}$ be functions of class $C^{(1)}$ on an open set D. Suppose that $\mathbf{F} = (f^1|S, \ldots, f^r|S)$ is a coordinate system for S, that $S = \{\mathbf{x} \in D: \boldsymbol{\Phi}(\mathbf{x}) = \mathbf{0}\}$, and that $D\boldsymbol{\Phi}(\mathbf{x})$ has rank $n - r$ for every $\mathbf{x} \in D$. Show that each $\mathbf{x}_0 \in S$ has a neighborhood U such that $(f^1, \ldots, f^r, \Phi^1, \ldots, \Phi^{n-r})$ restricted to U is a coordinate system for U.

8. Let $1 \leq r < n$, and let $\mathscr{M}(n, r) = \{\boldsymbol{\alpha} \in E_r^n: |\boldsymbol{\alpha}| = 1, \boldsymbol{\alpha} \text{ is decomposable}\}$. Identify E_r^n with $E^{(\eta)}$, and show that $\mathscr{M}(n, r)$ is a manifold of dimension $r(n - r)$. [*Hint*: Given $\boldsymbol{\alpha}_0 \in \mathscr{M}(n, r)$, let $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an orthonormal frame for E^n such that $\boldsymbol{\alpha}_0 = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$. Show that if $\boldsymbol{\alpha}$ is in a small enough neighborhood of $\boldsymbol{\alpha}_0$, then $\boldsymbol{\alpha}$ can be uniquely written in the form

$$\boldsymbol{\alpha} = c\left(\mathbf{v}_1 + \sum_{k=r+1}^{n} t_{1k}\mathbf{v}_k\right) \wedge \cdots \wedge \left(\mathbf{v}_r + \sum_{k=r+1}^{n} t_{rk}\mathbf{v}_k\right).$$

   The $r(n - r)$ numbers t_{jk} can be taken as coordinates of $\boldsymbol{\alpha}$.]

8.3 Measure and integration on manifolds

Let us now define r-dimensional measure for subsets of an r-manifold M and integrals with respect to it. The r-dimensional measure of a set $A \subset M$ is denoted by $V_r(A)$, and the integral of a function f over A by $\int_A f\, dV_r$ or by $\int_A f(\mathbf{x}) dV_r(\mathbf{x})$. When $r = n$, these turn out to have the same meaning as in Chapter 5.

For simplicity, the integral will be defined only for continuous functions. By so doing, we avoid some slightly tedious discussions of measurability of functions, which for present purposes are irrelevant.

Let us first consider the case when A is contained in some coordinate patch S. Then $S = \mathbf{g}(\Delta)$, where $\mathbf{g}$ is a regular transformation from Δ into M, and $A = \mathbf{g}(B)$ for some $B \subset \Delta$. We call A an *r-measurable* subset of S if B is measurable. The requirement that A be r-measurable is a mild one. For example, any relatively open set $A \subset S$ is r-measurable. Since $\mathbf{g}$ is a homeomorphism (by Proposition 8.4), A is relatively open if and only if B is open.

The r-measure $V_r(A)$ of an r-measurable set is defined in Section 8.1. The definition of $\int_A f\, dV_r$ can be motivated by a discussion such as that following the definition (8.2) of $V_r(A)$. Using the notation there, $f[\mathbf{g}(\mathbf{t}_0)]V_r(K)$ should furnish a good approximation to $\int_{\mathbf{g}(I)} f\, dV_r$. If Z is a figure composed of small nonoverlapping r-cubes $I_1, \ldots, I_p$, then $\int_{\mathbf{g}(Z)} f\, dV$ should be approximately the corresponding sum

$$\sum_{k=1}^{p} f[\mathbf{g}(\mathbf{t}_k)] \mathscr{J}\mathbf{g}(\mathbf{t}_k) V_r(I_k).$$

In the exact formula Z is replaced by the set $B = \mathbf{g}^{-1}(A)$, and the sum by an integral.

Definition. Let A be an r-measurable subset of a coordinate patch S, and let f be continuous on A. Then

$$\int_A f(\mathbf{x}) dV_r(\mathbf{x}) = \int_B f[\mathbf{g}(\mathbf{t})] \mathscr{J}\mathbf{g}(\mathbf{t}) dV_r(\mathbf{t}), \qquad A = \mathbf{g}(B), \tag{8.4}$$

provided the function $(f \circ \mathbf{g})\mathscr{J}\mathbf{g}$ is integrable over B.

The integral over B is taken in the sense of Section 5.6. By (8.1), $\mathscr{J}\mathbf{g}(\mathbf{t}) = |\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t})|$. Since $\mathbf{g}$ is of class $C^{(1)}$, $\mathscr{J}\mathbf{g}$ is continuous. Hence $(f \circ \mathbf{g})\mathscr{J}\mathbf{g}$ is continuous. If $f \geq 0$, then the integral over B either exists or diverges to $+\infty$. When the latter occurs we agree that the integral of f over A also diverges to $+\infty$.

We must show that the integral does not depend on the particular choice of coordinate system. Let $\tilde{S}$ be another coordinate patch such that $A \subset \tilde{S}$, and let $\tilde{\mathbf{F}}$ be a coordinate system for $\tilde{S}$. Let us adopt the notation of Figure 8.4. Then $\mathbf{g}(\mathbf{t}) = \tilde{\mathbf{g}}[\boldsymbol{\phi}(\mathbf{t})]$ for all $\mathbf{t} \in \Delta_0$ and $A = \mathbf{g}(B) = \tilde{\mathbf{g}}(\tilde{B})$, where $B \subset \Delta_0$ and

$\tilde{B} = \boldsymbol{\phi}(B) \subset \tilde{\Delta}_0$. By the transformation formula for integrals (Theorem 5.8) and Formula (8.3)

$$\int_{\tilde{B}} f[\tilde{\mathbf{g}}(\tilde{\mathbf{t}})] \mathscr{J}\tilde{\mathbf{g}}(\tilde{\mathbf{t}}) dV_r(\tilde{\mathbf{t}}) = \int_B f[\mathbf{g}(\mathbf{t})] \mathscr{J}\tilde{\mathbf{g}}[\boldsymbol{\phi}(\mathbf{t})] |J\boldsymbol{\phi}(\mathbf{t})| dV_r(\mathbf{t})$$

$$= \int_B f[\mathbf{g}(\mathbf{t})] \mathscr{J}\mathbf{g}[\boldsymbol{\phi}(\mathbf{t})] dV_r(\mathbf{t}),$$

as required.

EXAMPLE 1. Let $M \subset E^3$ be a 2-manifold, and S a relatively open subset of M on which x and y can be taken as coordinates, as in Example 2, Section 8.1. Since $\mathscr{J}\mathbf{g} = [1 + |\text{grad } \phi|^2]^{1/2}$, we have

$$\int_A f \, dV_2 = \int_B f[x, y, \phi(x, y)](1 + |\text{grad } \phi(x, y)|^2)^{1/2} dV_2(x, y).$$

EXAMPLE 2. Let M be a 1-manifold and B a closed interval $[a, b]$. Using the terminology of Section (6.2), A is the trace of the simple arc γ represented on $[a, b]$ by $\mathbf{g}$. Formula (8.4) becomes

$$\int_A f(\mathbf{x}) dV_1(\mathbf{x}) = \int_a^b f[\mathbf{g}(t)] |\mathbf{g}'(t)| dt.$$

The right-hand side is $\int_\gamma f \, ds$, as defined in Section 6.4, Problem 6.

EXAMPLE 3. Let H be the hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$. Introducing spherical coordinates on H, let

$$\mathbf{g}(\phi, \theta) = (\sin \phi \cos \theta)\mathbf{e}_1 + (\sin \phi \sin \theta)\mathbf{e}_2 + (\cos \phi)\mathbf{e}_3.$$

The image of a small square $[\phi, \phi + a] \times [\theta, \theta + a]$ in the (ϕ, θ) plane is a small sector of the hemisphere that is approximately a rectangle of side lengths a and $a \sin \phi$. Since $|\partial\mathbf{g}/\partial\phi \wedge \partial\mathbf{g}/\partial\theta| a^2$ is approximately the area of the sector, this suggests that $\mathscr{J}\mathbf{g}(\phi, \theta) = |\partial\mathbf{g}/\partial\phi \wedge \partial\mathbf{g}/\partial\theta| = \sin \phi$. The reader should check this formula (Problem 2). If we take $B = (0, \pi/2) \times (0, 2\pi)$ then $H - \mathbf{g}(B)$ is an arc of a great circle corresponding to $\theta = 0$. This arc is 2-dimensionally null in the sense defined below, and hence

$$\int_H f(\mathbf{x}) dV_2(\mathbf{x}) = \int_{\mathbf{g}(B)} f(\mathbf{x}) dV_2(\mathbf{x}) = \int_0^{\pi/2} d\phi \int_0^{2\pi} f[\mathbf{g}(\phi, \theta)] \sin \phi \, d\theta.$$

Let us turn to the matter of giving a general definition of the r-dimensional measure $V_r(A)$ and the integral $\int_A f \, dV_r$ when A is not necessarily contained in some coordinate patch. To simplify matters let us at first assume that M is a compact manifold and f is continuous everywhere on M. The traditional way to proceed is to dissect M into a finite number of nonoverlapping pieces $S_1, \ldots, S_m$ each of which has a coordinate system, with fr $S_k \cap$ fr S_l contained

in a finite union of $(r-1)$-manifolds and $M = \operatorname{cl} S_1 \cup \cdots \cup \operatorname{cl} S_m$. Then

$$\int_A f\, dV_r = \sum_{k=1}^{m} \int_{A \cap S_k} f\, dV_r. \tag{8.5}$$

In simple examples it is easy to find such dissections of M. However, the theorem that every compact r-manifold M has such a dissection is a difficult one to prove [24, Chapter IV]. Nor is it evident that the integral is independent of the particular dissection chosen.

The same result can be achieved by a simpler device called partition of unity. The basic difficulty with dissections is that S_k and S_l cannot overlap. With partitions of unity this problem is avoided.

Remark. In examples and homework problems considered in this book, A is usually contained in a single coordinate patch, except perhaps for a subset of r-dimensional measure 0 that contributes nothing to $V_r(A)$ or to the integral. If this is not the case, (8.5) can be used in these examples and homework problems after a rather obvious dissection of M. Thus, the reader should regard partitions of unity as an important theoretical tool rather than as an aid to calculating integrals.

Partition of unity

Let us recall from Section 5.3 that the support of a function ψ is the smallest closed set outside of which $\psi(\mathbf{x}) = 0$. Let us first find for every $\mathbf{x}_0$ and $\delta > 0$ a function ψ of class $C^{(\infty)}$ on E^n such that $\psi(\mathbf{x}) > 0$ on the δ-neighborhood of $\mathbf{x}_0$ and the support of ψ is the closure of that neighborhood. In fact, let

$$h(x) = \exp\left(\frac{-1}{1-x^2}\right), \qquad -1 < x < 1,$$

$$h(x) = 0, \qquad |x| \geq 1.$$

From the example at the end of Section 3.4 and the composite function theorem, h is of class $C^{(\infty)}$ on E^1. Let

$$\psi(\mathbf{x}) = h\left(\frac{|\mathbf{x} - \mathbf{x}_0|}{\delta}\right).$$

Definition. Let M be a compact manifold. A collection of functions $\{\pi_1, \ldots, \pi_m\}$ is a *partition of unity* for M if:

(1) π_k is of class $C^{(\infty)}$ on M and $\pi_k \geq 0$, $k = 1, \ldots, m$;
(2) The support of π_k is a compact subset of some coordinate patch, $k = 1, \ldots, m$; and
(3) $\sum_{k=1}^{m} \pi_k(\mathbf{x}) = 1$ for every $\mathbf{x} \in M$.

Proposition 8.6. *Any compact manifold M has a partition of unity.*

PROOF. Every $\mathbf{x} \in M$ is contained in some coordinate patch S. Since S is relatively open there is a neighborhood U of $\mathbf{x}$ such that $M \cap \text{cl } U \subset S$. Since M is compact, a finite collection $\{U_1, \ldots, U_m\}$ of such neighborhoods covers M. Let $\mathbf{x}_k$ be the center of U_k, δ_k the radius, and ψ_k the function of class $C^{(\infty)}$ constructed above with $\mathbf{x}_0 = \mathbf{x}_k$, $\delta = \delta_k$. The collection of functions $\{\psi_1, \ldots, \psi_m\}$ satisfies (1) and (2) of the definition, but not necessarily (3). However, by construction $\psi_1(\mathbf{x}) + \cdots + \psi_m(\mathbf{x}) > 0$ for every $\mathbf{x} \in M$. Let

$$\pi_k(\mathbf{x}) = \frac{\psi_k(\mathbf{x})}{\psi_1(\mathbf{x}) + \cdots + \psi_m(\mathbf{x})}, \qquad k = 1, \ldots, m, \qquad \mathbf{x} \in M.$$

The collection $\{\pi_1, \ldots, \pi_m\}$ is a partition of unity for M. □

Definition. A set $A \subset M$ is *r-measurable* if $A \cap S$ is r-measurable for any coordinate patch S.

An n-manifold $M \subset E^n$ is just an open subset of E^n. For $r = n$, an n-measurable set A is just a measurable subset of M in the sense of Section 5.2. It can be shown that, for $r < n$, the collection of r-measurable subsets of an r-manifold M has the same properties listed in Theorem 5.1 for measurable sets (Problem 9).

Let f be continuous on M. Since M is compact, f is bounded on M. This ensures that all of the following integrals exist. If the support of f is a compact subset K of a coordinate patch, then we let

$$\int_A f \, dV_r = \int_{A \cap K} f \, dV_r.$$

In particular, if $\{\pi_1, \ldots, \pi_m\}$ is a partition of unity, then for any f the support of $f\pi_k$ is compact and lies in some coordinate patch. Hence the integral of $f\pi_k$ is defined.

Definition. Let A be an r-measurable subset of a compact manifold M, and $\{\pi_1, \ldots, \pi_m\}$ be a partition of unity for M. Then for any f continuous on M

$$\int_A f \, dV_r = \sum_{k=1}^{m} \int_A f\pi_k \, dV_r. \tag{8.6}$$

In case A is contained in some coordinate patch S, this agrees with the earlier definition (8.4), since $\sum \pi_k(\mathbf{x}) = 1$ for every $\mathbf{x} \in A$, We must show that the integral does not depend on the particular partition of unity chosen for M. Let $\{\chi_1, \ldots, \chi_p\}$ be another partition of unity for M. Then for every $\mathbf{x} \in M$

$$f(\mathbf{x})\chi_l(\mathbf{x}) = \sum_{k=1}^{m} f(\mathbf{x})\chi_l(\mathbf{x})\pi_k(\mathbf{x}), \qquad l = 1, \ldots, p.$$

Since the support of $f\chi_l$ is contained in some coordinate patch, its integral over A can be written according to (8.4) as an integral over a set $B \subset E^r$. By Theorem 5.4, the integral over B of a finite sum is the sum of the integrals. Hence

$$\int_A f\chi_l \, dV_r = \sum_{k=1}^{m} \int_A f\chi_l \pi_k \, dV_r, \qquad l = 1, \ldots, p,$$

$$\sum_{l=1}^{p} \int_A f\chi_l \, dV_r = \sum_{l=1}^{p} \sum_{k=1}^{m} \int_A f\chi_l \pi_k \, dV_r.$$

In the same way

$$\sum_{k=1}^{m} \int_A f\pi_k \, dV_r = \sum_{k=1}^{m} \sum_{l=1}^{p} \int_A f\pi_k \chi_l \, dV_r.$$

Since the right-hand sides are equal, the integral of f over A does not depend on which partition of unity is chosen.

If $f(\mathbf{x}) = 1$ for every $\mathbf{x} \in M$, then the integral gives the *r-dimensional measure*

$$V_r(A) = \sum_{k=1}^{m} \int_A \pi_k \, dV_r.$$

When A is a subset of some coordinate patch, this agrees with the previous definition. If $V_r(A) = 0$, then A is called an *r-null* set. The integral has the same elementary properties listed in Theorem 5.4 for $r = n$ (Problem 11). Moreover, V_r is countably additive (Problem 10).

Measure and integration on noncompact manifolds

If M is not compact, the discussion is somewhat more complicated. Partitions of unity consisting of infinite collections $\{\pi_1, \pi_2, \ldots\}$ must be considered. To Conditions (1) through (3) must be added:

(4) If K is any compact subset of M, then the support of π_k meets K for only finitely many k.

The sum in (3) is now an infinite series. However, on any compact set only finitely many terms are different from 0. Every manifold has a partition of unity. This can be proved by an elaboration of the proof of Proposition 8.6, which we do not give.

Let f be continuous on A. Then f is called integrable over A if $\sum_{k=1}^{\infty} \int_A |f| \pi_k \, dV_r$ is finite. If f is integrable over A, then its integral is $\sum_{k=1}^{\infty} \int_A f\pi_k \, dV_r$.

Proposition 8.7. *If A is a subset of M that is an $(r-1)$-manifold, then A is an r-null set.*

This proposition can be proved as follows. By using partitions of unity, it suffices to show that $V_r(A \cap S) = 0$ if S is any coordinate patch. Let $A \cap S = \mathbf{g}(B)$, where $\mathbf{g}$ is regular from an open set $\Delta \subset E^r$ onto S. It follows from Theorem 8.3 (Section 8.9) and Corollary 3 (Section 5.8) that $V_r(B) = 0$. Hence by Equation (8.2), $V_r(A \cap S) = 0$.

Corollary. *If A is contained in a countable union of $(r-1)$-manifolds, then A is an r-null set.*

**Note.* The line integral $\int_\gamma f\,ds$ was defined in Section 6.4 without requiring that γ be simple. If γ is not simple, then its trace $A = \mathbf{g}([a, b])$ need not be contained in a 1-manifold. There is a more general notion of r-dimensional measure and integral for sets that are not necessarily subsets of an r-manifold. The general formula, which becomes for simple arcs the one in Example 2, is

$$\int_A f(\mathbf{x})N(\mathbf{x})dV_1(\mathbf{x}) = \int_a^b f[\mathbf{g}(t)]\,|\mathbf{g}'(t)|\,dt,$$

where $N(\mathbf{x})$ is the multiplicity of the point $\mathbf{x}$. For any $r \geq 1$ there is a similar formula

$$(*) \qquad \int_A f(\mathbf{x})N(\mathbf{x})dV_r(\mathbf{x}) = \int_B f[\mathbf{g}(\mathbf{t})]\mathscr{J}\mathbf{g}(\mathbf{t})dV_r(\mathbf{t}),$$

where B is a measurable subset of E^r, $\mathbf{g}$ is of class $C^{(1)}$ on B, $A = \mathbf{g}(B)$, and again $N(\mathbf{x})$ (= number of points $\mathbf{t} \in B$ such that $\mathbf{g}(\mathbf{t}) = \mathbf{x}$) is the multiplicity of $\mathbf{x}$ [7, p. 244].

In Formula (*) it is not necessary to assume that $D\mathbf{g}(\mathbf{t})$ has maximum rank r. If $B' = \{\mathbf{t} \in B : \text{rank } D\mathbf{g}(\mathbf{t}) < r\}$, then $\mathscr{J}\mathbf{g}(\mathbf{t}) = 0$ for every $\mathbf{t} \in B'$. Therefore B' contributes nothing to the integral on the right-hand side of (*). It turns out that $\mathbf{g}(B')$ has r-dimensional measure 0, and therefore contributes nothing to the integral on the left-hand side of (*).

PROBLEMS

1. Let $A = \{(x, y, z) : x^2 + z^2 = a^2, 0 \leq y \leq b\}$. Find $\int_A yz^2\,dV_2(x, y, z)$. [*Hint*: Cylindrical coordinates.]

2. In Example 3, verify that $\mathscr{J}\mathbf{g}(\phi, \theta) = \sin\phi$.

Moments and centroids. These are defined in the same way as for $r = n$. For example, if A has positive r-dimensional measure, then the components of its centroid are

$$\bar{x}^i = \frac{1}{V_r(A)} \int_A x^i\,dV_r(\mathbf{x}), \qquad i = 1, \ldots, n.$$

If $r = 2$, $n = 3$, and A is thought of as a surface with continuous density $\rho(\mathbf{x})$ (mass per unit of area), then the mass is $\int_A \rho(\mathbf{x})dV_2(\mathbf{x})$ and the components of

the center of mass are

$$\bar{\bar{x}}^i = \int_A x^i \rho(\mathbf{x}) dV_2(\mathbf{x}) \Big/ \int_A \rho(\mathbf{x}) dV_2(\mathbf{x}), \qquad i = 1, 2, 3.$$

3. Show that $\frac{1}{2}\mathbf{e}_3$ is the centroid of the hemisphere H in Example 3. Use spherical coordinates on H.

4. Find the second moment about the z-axis of:
   (a) The sphere $x^2 + y^2 + z^2 = 1$.
   (b) The triangle with vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

5. Let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r$ be the vertices of an r-simplex Σ. Show that the centroid of Σ is $(r + 1)^{-1}(\mathbf{x}_0 + \mathbf{x}_1 + \cdots + \mathbf{x}_r)$.

6. (*Surfaces of revolution.*) Let γ be a simple arc (or simple closed curve) lying in the half $y > 0$ of the (x, y) plane. From Section 6.2, γ has a standard representation $\mathbf{G}$ on $[0, l]$, where l is the length and $|\mathbf{G}'(s)| = 1$ for $0 \leq s \leq l$. Let $\mathbf{g}(s, t) = G^1(s)\mathbf{e}_1 + G^2(s)[(\cos t)\mathbf{e}_2 + (\sin t)\mathbf{e}_3]$, and let $M = \mathbf{g}((0, l) \times [0, 2\pi])$.
   (a) Prove Pappus's theorem: $V_2(M) = 2\pi\bar{y}l$, where $(\bar{x}, \bar{y})$ is the centroid of γ.
   (b) Find the area of a torus (doughnut) of major radius r_1 and minor radius r_2.

7. Let $S = \{\mathbf{x} : |\mathbf{x}| = 1\}$ be the unit $(n - 1)$-sphere in E^n. Show that the $(n - 1)$-measure of the "zone" $\{\mathbf{x} \in S : a < x^n < b\}$ depends only on the difference $b - a$ when $n = 3$, but this is false when $n \neq 3$. Assume that $-1 \leq a < b \leq 1$.

8. Let $v(r) = \alpha_n r^n$ be the n-dimensional measure of a spherical n-ball of radius r. Show that $v'(r)$ is the $(n - 1)$-measure of its boundary. [*Hint*: Spherical coordinates (Section 5.9).] [*Note*: α_n was calculated in (5.46). If $\beta_n = V_{n-1}$ [unit $(n - 1)$-sphere], then $\beta_n = n\alpha_n$.]

9. Let M be a r-manifold. Show that:
   (a) Any set A open relative to M is r-measurable.
   (b) If A is an r-measurable subset of M, then $M - A$ is r-measurable.
   (c) If $A_1, A_2, \ldots$ are r-measurable subsets of M, then $A_1 \cup A_2 \cup \cdots$ is r-measurable.

10. Let M be a compact r-manifold. Let $A = A_1 \cup A_2 \cup \cdots$, where $A_1, A_2, \ldots$ are disjoint r-measurable subsets of M. Show that $V_r(A) = V_r(A_1) + V_r(A_2) + \cdots$ [*Hint*: Use a partition of unity.]

11. Prove that the statements (1)–(7) obtained by replacing n by r everywhere in Theorem 5.4 are true for integrals over r-measurable subsets of a compact r-manifold.

8.4 The divergence theorem

This is an n-dimensional generalization of the fundamental theorem of calculus and has numerous applications in geometry and in physics. We begin by stating the theorem and deriving some consequences of it. A proof is given later in the present section. In Section 8.7 we give another formulation of the divergence theorem, which is then used to obtain Stokes's formula in Section 8.8.

Let $\zeta = \zeta_1\, dx^1 + \cdots + \zeta_n\, dx^n$ be a class $C^{(1)}$ differential form of degree 1. We recall from Formula (7.58) that its divergence is the real valued function

$$\operatorname{div} \zeta = \sum_{i=1}^{n} \frac{\partial \zeta_i}{\partial x^i}.$$

The divergence theorem equates the integral of div ζ over a set D with an integral over the boundary fr D, provided D belongs to a suitably restricted class of sets called *regular domains.*

Definition. A set D is a *regular domain* if:

(1) D is open and bounded; and
(2) For any $\mathbf{x}_0 \in \operatorname{fr} D$ there exist a neighborhood U of $\mathbf{x}_0$ and a function Φ of class $C^{(1)}$ on U, such that grad $\Phi(\mathbf{x}) \neq \mathbf{0}$ and

$$(\operatorname{fr} D) \cap U = \{\mathbf{x} \in U : \Phi(\mathbf{x}) = 0\},$$
$$D \cap U = \{\mathbf{x} \in U : \Phi(\mathbf{x}) < 0\}.$$

If D is a regular domain, then fr D is a compact $(n-1)$-manifold. Condition (2) ensures that D cannot be "on both sides" of its boundary fr D.

EXAMPLE 1. Let $D = \{\mathbf{x} : |\mathbf{x}| < 1 \text{ or } 1 < |\mathbf{x}| < 2\}$. Then fr D is the union of two concentric $(n-1)$-spheres of radii 1 and 2. However, D is on both sides of the inner $(n-1)$-sphere.

Actually, this example is rather artificial. If D is the interior of its closure, D is bounded, and fr D is an $(n-1)$-manifold, then using the implicit function theorem it can be shown that D is a regular domain.

Definition. Let $\mathbf{x} \in \operatorname{fr} D$, and $\mathbf{n} \neq \mathbf{0}$ be a vector normal to fr D at $\mathbf{x}$. Then $\mathbf{n}$ is an *exterior normal* at $\mathbf{x}$ if there exists $\delta > 0$ such that $\mathbf{x} + t\mathbf{n} \in D$ for $-\delta < t < 0$ and $\mathbf{x} + t\mathbf{n} \in (\operatorname{cl} D)^c$ for $0 < t < \delta$.

From the definition, all exterior normals at $\mathbf{x}$ are positive scalar multiples of any particular one. We shall be principally concerned with the unit exterior normal, which will be denoted by $\mathbf{v}(\mathbf{x})(|\mathbf{v}(\mathbf{x})| = 1)$.

Let us show that the exterior unit normal $\mathbf{v}$ is a continuous function on fr D, if D is a regular domain. Let $\mathbf{x}_0$ be any point of fr D, and U, Φ as in the definition above. By Corollary 1, Section 4.7, the vector $\mathbf{n}(\mathbf{x}) = \text{grad } \Phi(\mathbf{x})$ is normal to fr D at $\mathbf{x} \in (\operatorname{fr} D) \cap U$.

Let $\psi(t) = \Phi(\mathbf{x} + t\mathbf{n}(\mathbf{x}))$. Then $\psi(0) = 0$ and

$$\psi'(0) = \text{grad } \Phi(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = |\text{grad } \Phi(\mathbf{x})|^2 > 0.$$

There exists $\delta > 0$ such that $\psi(t) < 0$ for $-\delta < t < 0$ and $\psi(t) > 0$ for $0 < t < \delta$. Therefore grad $\Phi(\mathbf{x})$ is an exterior normal at $\mathbf{x}$. The vector

$$\mathbf{v}(\mathbf{x}) = |\text{grad } \Phi(\mathbf{x})|^{-1} \text{ grad } \Phi(\mathbf{x})$$

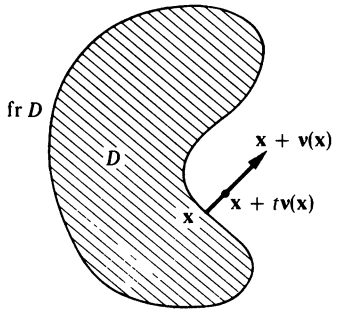


Figure 8.5

is the unit exterior normal to D at $\mathbf{x}$. Since Φ is of class $C^{(1)}$, $\mathbf{v}$ is a continuous function on (fr D) $\cap$ U (see Figure 8.5). Since any $\mathbf{x}_0 \in$ fr D has such a neighborhood U, $\mathbf{v}$ is continuous on fr D.

Divergence theorem. *Let D be a regular domain and ζ a 1-form of class $C^{(1)}$ on* cl D. *Then*

$$\int_{\text{fr}\,D} \zeta(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) dV_{n-1}(\mathbf{x}) = \int_D \operatorname{div} \zeta(\mathbf{x}) dV_n(\mathbf{x}) \tag{8.7}$$

The number $\zeta(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})$ is called the (exterior) *normal component* of the covector $\zeta(\mathbf{x})$. Let us defer the proof until later in the present section. The somewhat restrictive assumption that fr D is a $C^{(1)}$ manifold is made to simplify the proof. The theorem is still true if fr D is not a manifold but instead consists of a finite number of pieces of class $C^{(1)}$ intersecting in sets of dimension $n - 2$. For example, if D is the interior of an n-cube then the pieces are the faces, which are cubes of dimension $n - 1$ and intersect in $(n - 2)$-dimensional cubes. This more general form of the divergence theorem will be precisely stated at the end of the section. For certain special kinds of sets D there is an easy proof of the theorem (Problem 3).

Note. In applying (8.7) we sometimes ignore the distinction between vectors and covectors. If $\mathbf{F}$ is vector-valued with the same components as ζ ($F^i = \zeta_i$, $i = 1, \ldots, n$). then we set div $\mathbf{F}$ = div ζ. On the left side of (8.7), $\zeta \cdot \mathbf{v}$ becomes $\mathbf{F} \cdot \mathbf{v}$, the euclidean inner product of $\mathbf{F}$ and $\mathbf{v}$.

For $n = 3$, the divergence theorem is often called *Gauss's* or *Ostrogradsky's theorem*. It has various interesting physical interpretations. Let ζ be a force field acting in some open set $D_0 \subset E^3$. For each $\mathbf{x} \in D_0$, $\zeta(\mathbf{x})$ is the force acting at $\mathbf{x}$ (Section 6.4). For notational simplicity, let us set $M =$ fr D throughout the discussion to follow. The number $\int_M \zeta(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) dV_2(\mathbf{x})$ is called the *outward flux* across the boundary M. The divergence theorem expresses the outward flux as a volume integral over D. If D has small diameter and contains $\mathbf{x}_0$, then the outward flux is approximately $V_3(D)\operatorname{div} \zeta(\mathbf{x}_0)$. To make this statement more precise let us prove the following.

Lemma 1. *If f is continuous on an open set D_0 containing $\mathbf{x}_0$, then*

$$f(\mathbf{x}_0) = \lim_{\operatorname{diam} D \to 0} \frac{1}{V_n(D)} \int_D f(\mathbf{x}) dV_n(\mathbf{x}).$$

In words, this formula says that given $\varepsilon > 0$ there exists $\delta > 0$ such that if D is any open set of diameter less than δ with $\mathbf{x}_0 \in D$, then

$$\left| V_n(D) f(\mathbf{x}_0) - \int_D f(\mathbf{x}) dV_n(\mathbf{x}) \right| < \varepsilon V_n(D).$$

PROOF. Given $\varepsilon > 0$, let $\delta > 0$ be such that $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \varepsilon$ whenever $|\mathbf{x} - \mathbf{x}_0| < \delta$. If $\mathbf{x}_0 \in D$ and diam $D < \delta$, then

$$\left| V_n(D) f(\mathbf{x}_0) - \int_D f(\mathbf{x}) dV_n \right| = \left| \int_D [f(\mathbf{x}_0) - f(\mathbf{x})] dV_n(\mathbf{x}) \right|$$

$$\leq \int_D |f(\mathbf{x}_0) - f(\mathbf{x})| dV_n(\mathbf{x}) < \varepsilon V_n(D). \qquad \square$$

If in Lemma 1 we take D regular and $f = \operatorname{div} \zeta$, then for any n and $\mathbf{x}_0$ in the domain of ζ

$$\operatorname{div} \zeta(\mathbf{x}_0) = \lim_{\operatorname{diam} D \to 0} \frac{1}{V_n(D)} \int_M \zeta(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) dV_{n-1}(\mathbf{x}). \tag{8.8}$$

If $\operatorname{div} \zeta(\mathbf{x}) = 0$ for every $\mathbf{x}$ in its domain D_0, then ζ is called *divergence free* (or *solenoidal*). The divergence theorem has the following corollary.

Corollary. *Let ζ be of class $C^{(1)}$ on an open set D_0. Then ζ is divergence free if and only if*

$$\int_M \zeta(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) dV_{n-1}(\mathbf{x}) = 0, \qquad M = \operatorname{fr} D, \tag{*}$$

for every regular domain D such that $\operatorname{cl} D \subset D_0$.

PROOF. If ζ is divergence free, then the equation (*) is immediate from (8.7). Conversely if (*) holds for every such D, then by (8.8) $\operatorname{div} \zeta(\mathbf{x}_0) = 0$ for every $\mathbf{x}_0 \in D_0$. $\square$

EXAMPLE 2. Let $n = 3$ and $\zeta = yz\, dx$. Then $\zeta_1(x, y, z) = yz$, $\zeta_2 = \zeta_3 = 0$, and $\zeta \cdot \mathbf{v} = yzv^1$ where $v^1 = v^1(x, y, z)$ is the first component of the exterior unit normal $\mathbf{v}$. Moreover, $\operatorname{div} \zeta = (\partial \zeta_1 / \partial x) = 0$. Thus $\int_{\operatorname{fr} D} yzv^1\, dV_2 = 0$, by the divergence theorem. For special choices of D this could also be checked directly. For instance, if D is a spherical ball with center $\mathbf{0}$ and radius a, then $\nu(\mathbf{x}) = a^{-1}\mathbf{x}$, $\mathbf{x} = (x, y, z)$. The integral over the sphere fr D can be easily evaluated using spherical coordinates.

**Flows in E^n*

Let t denote time, and imagine that points of E^n move in such a way that a point at $\mathbf{x}$ when $t = 0$ moves to another position $\mathbf{y}$ at time t. For instance, the movement might be due to the flow of a fluid in E^3 (Section 8.5). If we set $\mathbf{y} = \mathbf{T}_t(\mathbf{x})$, then $\mathbf{T}_t$ is a flat transformation from some portion of E^n into E^n for each t. Let us require that $\mathbf{T}_{t+s} = \mathbf{T}_t \circ \mathbf{T}_s$. Thus a point initially at $\mathbf{x}$ moves first to $\mathbf{y} = T_s(\mathbf{x})$ at time s, then to $\mathbf{z} = \mathbf{T}_t(\mathbf{y})$ at time $t + s$. We also require that each $\mathbf{T}_t$ be regular and possess continuous second order partial derivatives (see Figure 8.6).

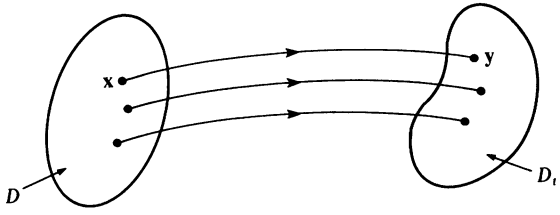


Figure 8.6

Definition. Let $D_0 \subset E^n$ be open, and for each $t \geq 0$, let $\mathbf{T}_t$ be a transformation from D_0 into D_0. Assume that: (1) $\mathbf{T}_t(\mathbf{x})$ is of class $C^{(2)}$ as a function of $(\mathbf{x}, t)$ on $D_0 \times [0, \infty)$; (2) $\mathbf{T}_t$ is regular for each $t \geq 0$; (3) $\mathbf{T}_0(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in D_0$; and (4) $\mathbf{T}_{t+s} = \mathbf{T}_t \circ \mathbf{T}_s$ for each $s, t \geq 0$. Then the collection $\{\mathbf{T}_t\}$ is a *flow*.

If D is an open subset of D_0, then the flow takes points initially in D into points of $D_t = \mathbf{T}_t(D)$ at time t. Let $v(t) = V_n(D_t)$. The derivative $v'(t)$ is the rate of change of the n-dimensional measure of D_t. Let $\mathbf{W}_t = (\partial \mathbf{T}_t/\partial t)$. Then

$$v'(t) = \int_D J\mathbf{T}_t(\mathbf{x}) \operatorname{div} \mathbf{W}_0(\mathbf{x}) dV_n(\mathbf{x}), \tag{8.9}$$

provided cl $D \subset D_0$. We leave the proof to the reader (Problem 4). In particular, (8.9) implies that the n-dimensional measure $v(t)$ is constant if $\operatorname{div} \mathbf{W}_0(\mathbf{x}) = 0$ for all $\mathbf{x} \in D$.

Green's formulas

Formulas (8.10) and (8.11) are consequences of the divergence theorem and are useful in many applications. Let f be of class $C^{(2)}$ on cl D, and let $M = \text{fr } D$ as above. Let $f_\nu(\mathbf{x})$ denote the derivative of f in the direction of the exterior normal at $\mathbf{x} \in M$, namely,

$$f_\nu(\mathbf{x}) = df(\mathbf{x}) \cdot \nu(\mathbf{x}).$$

Let ϕ be another function of class $C^{(2)}$ on cl D, and let $\zeta(\mathbf{x}) = \phi(\mathbf{x})df(\mathbf{x})$. Then

$$\zeta(\mathbf{x}) \cdot \nu(\mathbf{x}) = \phi(\mathbf{x}) f_\nu(\mathbf{x}),$$

$$\operatorname{div} \zeta = \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\left(\phi \frac{\partial f}{\partial x^i}\right) = d\phi \cdot df + \phi \operatorname{Lapl} f.$$

Hence we get the *first Green's formula*:

$$\int_M \phi f_\nu \, dV_{n-1} = \int_D [d\phi \cdot df + \phi \operatorname{Lapl} f] dV_n. \tag{8.10}$$

In the same way

$$\int_M f\phi_\nu \, dV_{n-1} = \int_D [df \cdot d\phi + f \operatorname{Lapl} \phi] dV_n.$$

Subtracting, we get the *second Green's formula*

$$\int_M [\phi f_\nu - f\phi_\nu] dV_{n-1} = \int_D [\phi \operatorname{Lapl} f - f \operatorname{Lapl} \phi] dV_n. \tag{8.11}$$

EXAMPLE 3. A function f is called *harmonic* if $\operatorname{Lapl} f = 0$. Let f be harmonic, and apply the first Green's formula with $\phi = f$. Then

$$\int_M f f_\nu \, dV_{n-1} = \int_D |df|^2 \, dV_n. \tag{8.12}$$

When $n = 3$ the right-hand side often has (except for a suitable multiplicative constant) the physical interpretation of energy.

If f is harmonic and $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in M$, then from (8.12) the integral of the nonnegative continuous function $|df|^2$ is 0. Hence $df(\mathbf{x}) = 0$ for every $\mathbf{x} \in$ cl D. Given $\mathbf{x}_0 \in D$ let $\mathbf{x}_1$ be a point of M nearest $\mathbf{x}_0$. The line joining $\mathbf{x}_0$ and $\mathbf{x}_1$ lies in cl D, and from the mean value theorem f is constant on it. Since $f(\mathbf{x}_1) = 0$, we must have $f(\mathbf{x}_0) = 0$. Thus $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in$ cl D.

Suppose that f and g are both of class $C^{(2)}$ on cl D and harmonic, and that $f(\mathbf{x}) = g(\mathbf{x})$ for every $\mathbf{x} \in M$. Then $\phi(\mathbf{x}) = f(\mathbf{x}) - g(\mathbf{x}) = 0$ for $\mathbf{x} \in M$ and ϕ is harmonic. Hence $\phi(\mathbf{x}) = 0$, and $f(\mathbf{x}) = g(\mathbf{x})$, for every $\mathbf{x} \in$ cl D. This shows that there is at most one harmonic function of class $C^{(2)}$ on cl D with given values on the boundary M. It is more difficult to show that there is in fact a harmonic function f with given boundary values. This is called *Dirichlet's problem*. If the boundary data $f \mid M$ are merely continuous, then f is continuous on cl D and of class $C^{(2)}$ and harmonic on D. [15, Chapter XI]. If the boundary data are smooth enough, then f is of class $C^{(2)}$ and harmonic on cl D. For instance this is true if M is of class $C^{(3)}$ and f is of class $C^{(3)}$ on M.

Let us now turn to the proof of the divergence theorem. The proof proceeds by first considering two particular cases (Lemmas 2 and 3). The general result is reduced to these via a partition of unity. As in Chapter 5, $\int f \, dV_n$ denotes the integral of f over all of E^n.

Lemma 2. *Let ζ be a 1-form of class $C^{(1)}$ on E^n such that ζ has compact support. Then $\int div\, \zeta\, dV_n = 0$.*

PROOF. Let $1 \leq i \leq n$. By the iterated integrals theorem

$$\int \frac{\partial \zeta_i}{\partial x^i} dV_n = \int \left\{ \int \frac{\partial \zeta_i}{\partial x^i} dx^i \right\} dV_{n-1}(\mathbf{x}^{i'}),$$

where $\mathbf{x}^{i'} = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$. Since ζ_i has compact support, the inner integral is 0 by the fundamental theorem of calculus. Therefore $\int \partial \zeta_i / \partial x^i \, dV_n = 0$. Summing from 1 to n we get the lemma. □

In Lemma 3 we write (as in Section 5.5) $\mathbf{x}' = (x^1, \ldots, x^{n-1})$ instead of $\mathbf{x}^{n'}$. We return to the situation depicted in Figure 8.2. Let $R \subset E^{n-1}$ be an open set, and ϕ of class $C^{(1)}$ on R with $\phi(\mathbf{x}) > 0$. Let $S = \{(\mathbf{x}', \phi(\mathbf{x}')) : \mathbf{x}' \in R\}$, and $\mathbf{v}(\mathbf{x}) = (v^1(\mathbf{x}), \ldots, v^n(\mathbf{x}))$ the normal vector to S at $\mathbf{x}$ with $|\mathbf{v}(\mathbf{x})| = 1$ and $v^n(\mathbf{x}) > 0$. Let C be the cylindrical region bounded above by S and below by the hyperplane $x^n = 0$, namely,

$$C = \{(\mathbf{x}', x^n) : 0 < x^n < \phi(\mathbf{x}'), \mathbf{x}' \in R\}.$$

Lemma 3. *Suppose that $\zeta_n(\mathbf{x}', 0) = 0$ for all $\mathbf{x}' \in R$. Then*

$$\int_C \frac{\partial \zeta_n}{\partial x^n} dV_n = \int_S \zeta_n v^n \, dV_{n-1}.$$

PROOF. By the iterated integrals theorem,

$$\int_C \frac{\partial \zeta_n}{\partial x^n} dV_n = \int_R \left\{ \int_0^{\phi(\mathbf{x}')} \frac{\partial \zeta_n}{\partial x^n} dx^n \right\} dV_{n-1}(\mathbf{x}').$$

Since $\zeta_n(\mathbf{x}', 0) = 0$, the fundamental theorem of calculus implies

$$(*) \qquad \int_C \frac{\partial \zeta_n}{\partial x^n} dV_n = \int_R \zeta_n(\mathbf{x}', \phi(\mathbf{x}')) dV_{n-1}(\mathbf{x}').$$

Let us next show that

$$(**) \qquad v^n(\mathbf{x})(1 + |\operatorname{grad} \phi(\mathbf{x}')|^2)^{1/2} = 1$$

for any $\mathbf{x} = (\mathbf{x}', \phi(\mathbf{x}') \in S$. To see this, let $\Phi(\mathbf{x}) = x^n - \phi(\mathbf{x}')$ as in Example 2, Section 8.1. Then

$$\operatorname{grad} \Phi(\mathbf{x}) = \mathbf{e}_n - \operatorname{grad} \phi(\mathbf{x}'),$$
$$|\operatorname{grad} \Phi(\mathbf{x})| = (1 + |\operatorname{grad} \phi(\mathbf{x}')|^2)^{1/2}.$$

Since $\mathbf{v}(\mathbf{x}) = |\operatorname{grad} \Phi(\mathbf{x})|^{-1} \operatorname{grad} \Phi(\mathbf{x})$, we get (**). In the same way as for Example 1, Section 8.3, if f is continuous on S, then

$$\int_S f dV_{n-1} = \int_R f(\mathbf{x}', \phi(\mathbf{x}'))(1 + |\operatorname{grad} \phi(\mathbf{x}')|^2)^{1/2} \, dV_{n-1}(\mathbf{x}').$$

We get Lemma 3 by taking $f = \zeta_n v^n$, and using (*), (**). □

Let us next show that the divergence theorem is correct for 1-forms with sufficiently small support.

Lemma 4. *Let D be a regular domain. Then any $\mathbf{x}_0 \in \operatorname{cl} D$ has a neighborhood U such that* (8.7) *holds, provided ζ is a* 1*-form of class $C^{(1)}$ and the support of ζ is contained in U.*

PROOF. If $\mathbf{x}_0 \in D$, let U be any neighborhood of $\mathbf{x}_0$ of radius less than the distance from $\mathbf{x}_0$ to $\operatorname{fr} D$. Then $\zeta(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \operatorname{fr} D$, since ζ has support contained in U. Thus the left side of (8.7) is 0. By Lemma 2 the right side of (8.7) is 0.

Next suppose that $\mathbf{x}_0 \in \operatorname{fr} D$. By a translation and rotation in E^n we may assume that the components of both $\mathbf{x}_0$ and the exterior unit normal $\nu(\mathbf{x}_0)$ are positive:

$$x_0^i > 0, \nu^i(\mathbf{x}_0) > 0, \qquad i = 1, \ldots, n.$$

(see Problem 9). Let us show that, for each $i = 1, \ldots, n$, there exists a neighborhood U_i of $\mathbf{x}_0$ such that

$$\int_{\operatorname{fr} D} \zeta_i \nu^i \, dV_{n-1} = \int_D \frac{\partial \zeta_i}{\partial x^i} \, dV_n$$

provided ζ has support contained in U_i. One then gets (8.7) by summing from $i = 1$ to n, and taking ζ with support contained in $U = U_1 \cap \cdots \cap U_n$. For notational simplicity, consider $i = n$. The same proof applies when $i < n$. By the implicit function theorem, there is a relatively open set $S \subset \operatorname{fr} D$ with $\mathbf{x}_0 \in S$, such that

$$S = \{(\mathbf{x}', \phi(\mathbf{x}')) : \mathbf{x}' \in R\},$$

with $R \subset E^{n-1}$ an open set and ϕ of class $C^{(1)}$. Since $x_0^n > 0$ and $x_0^n = \phi(\mathbf{x}_0')$, we may choose S small enough that $\phi(\mathbf{x}') > 0$ for all $\mathbf{x} \in R$. Let Φ have the properties in the definition of regular domain. Since grad $\Phi(\mathbf{x}_0)$ is a positive scalar multiple of $\nu(\mathbf{x}_0)$ and $\nu^n(\mathbf{x}_0) > 0$, the partial derivative $\Phi_n(\mathbf{x}_0)$ is positive. Let us choose a neighborhood V of $\mathbf{x}_0$ sufficiently small that $\mathbf{x}' \in R$, $x^n > 0$, and $\Phi_n(\mathbf{x}) > 0$ for any $\mathbf{x} = (\mathbf{x}', x^n) \in V$. Then $\Phi(\mathbf{x}', \cdot)$ is an increasing function on the interval $I_{\mathbf{x}'} = \{x^n : (\mathbf{x}', x^n) \in V\}$ and $\Phi(\mathbf{x}', \phi(\mathbf{x}')) = 0$. Let $U_n \subset V$ be a neighborhood of $\mathbf{x}_0$ small enough that $(\mathbf{x}', \phi(\mathbf{x}')) \in V$ for any $\mathbf{x} \in U_n$. For $\mathbf{x} \in D \cap U_n$, we have $\Phi(\mathbf{x}) < 0$, $\Phi(\mathbf{x}', \phi(\mathbf{x}')) = 0$, and $\Phi(\mathbf{x}', \ )$ increasing on $I_{\mathbf{x}'}$. Thus, $x^n < \phi(\mathbf{x}')$. Therefore, $D \cap U_n \subset C$ with C as in Lemma 3 (see Figure 8.7).

If ζ has support contained in U_n, then

$$\int_{\operatorname{fr} D} \zeta_n \nu^n \, dV_{n-1} = \int_S \zeta_n \nu^n \, dV_{n-1},$$

$$\int_D \frac{\partial \zeta_n}{\partial x^n} \, dV_n = \int_{D \cap U_n} \frac{\partial \zeta_n}{\partial x^n} \, dV_n = \int_C \frac{\partial \zeta_n}{\partial x^n} \, dV_n.$$

By Lemma 3, the right-hand sides are equal. □

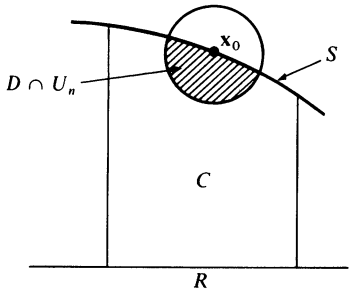


Figure 8.7

Proof of divergence theorem

Let ζ be a 1-form of class $C^{(1)}$ on cl D. Then by definition ζ has a $C^{(1)}$ extension to an open set D_0 containing cl D (Section 3.4). We also denote this extension by ζ. Every $\mathbf{x}_0 \in \text{cl } D$ has a neighborhood $U \subset D_0$ with the property stated in Lemma 4. Since cl D is a compact set, a finite number of such neighborhoods $U_1, \ldots, U_m$ cover cl D. Let $\{\pi_1, \ldots, \pi_m\}$ be a partition of unity of the kind constructed in the proof of Proposition 8.6, with the support of π_k contained in U_k for $k = 1, \ldots, m$. Then $\pi_k \zeta$ has support in U_k, and by the product rule for derivatives,

$$\text{div}(\pi_k \zeta) = \pi_k \text{ div } \zeta + d\pi_k \cdot \zeta.$$

By Lemma 4,

$$\int_{\text{fr } D} (\pi_k \zeta) \cdot \mathbf{v} \, dV_{n-1} = \int_D \text{div}(\pi_k \zeta) dV_n, k = 1, \ldots, m.$$

Since $\sum_{k=1}^m \pi_k = 1$, we have $\sum_{k=1}^m d\pi_k = 0$ and

$$\int_{\text{fr } D} \zeta \cdot \mathbf{v} \, dV_{n-1} = \sum_{k=1}^m \int_{\text{fr } D} (\pi_k \zeta) \cdot \mathbf{v} \, dV_{n-1},$$

$$\int_D \text{div } \zeta \, dV_n = \sum_{k=1}^m \int_D \pi_k \text{ div } \zeta \, dV_n$$

$$0 = \int_D \left(\sum_{k=1}^m d\pi_k \right) \cdot \zeta \, dV_n = \sum_{k=1}^m \int_D d\pi_k \cdot \zeta \, dV_n.$$

This gives (8.7), completing a proof of the divergence theorem. □

The assumption that fr D is a manifold of class $C^{(2)}$ can be considerably weakened. Let us state without proof a somewhat more general version of the divergence theorem. Let D be an open, bounded set. Assume that

$$\text{fr } D = A_1 \cup \cdots \cup A_p \cup B,$$

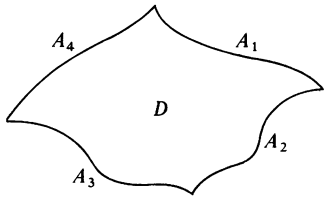


Figure 8.8

where: (a) A_k is a relatively open subset of fr D and cl A_k is a compact subset of an $(n-1)$-manifold $M_k, k = 1, \ldots, p$; and (b) B is a compact set contained in a finite union of $(n-2)$-manifolds and $(\text{cl } A_k) \cap (\text{cl } A_l) \subset B$ whenever $k \neq l$. Moreover, assume that D is the interior of its closure. Then

$$\sum_{k=1}^{p} \int_{A_k} \zeta \cdot \mathbf{v}\, dV_{n-1} = \int_D \operatorname{div} \zeta\, dV_n$$

provided ζ is of class $C^{(1)}$ on cl D (see Figure 8.8). Let us say that such a set D has a boundary which is *piecewise of class* $C^{(1)}$.

EXAMPLE 4. Let D be an n-simplex. Let $A_0, \ldots, A_n$ be its (open) $(n-1)$-dimensional faces, let M_k be the hyperplane containing A_k, and let B be the union of the $(n-2)$-dimensional faces of D.

PROBLEMS

1. Let $n = 2$ and $\zeta = y\,dx - x\,dy$. Verify (8.7) by evaluating the integrals on both sides if:
   (a) $D = \{(x, y) : x^2 + y^2 < a^2\}$.
   (b) D is the triangle with vertices $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2$.

2. Let D be the solid cone in E^3 with vertex $\mathbf{e}_3$ and base $B = \{(x, y, 0) : x^2 + y^2 < 1\}$. Let $\zeta = x^2\,dx + y^2\,dz$. Evaluate
   (a) $\int_B \zeta \cdot \mathbf{v}\, dV_2$.
   (b) $\int_A \zeta \cdot \mathbf{v}\, dV_2$, where $A = \text{fr } D - B$. Use the divergence theorem.

3. Prove the divergence theorem directly from the fundamental theorem of calculus when D is:
   (a) The unit n-cube $\{\mathbf{x} : 0 < x^i < 1, i = 1, \ldots, n\}$.
   (b) The standard n-simplex.

4. Consider a flow $\{\mathbf{T}_t\}$, and define $\mathbf{W}_t$, $v(t) = V_n(D_t)$ as in the discussion preceding (8.9). Show that:
   (a) $J\mathbf{T}_t(\mathbf{x}) > 0$ for all $\mathbf{x} \in D_0$ and $t \geq 0$.
   (b) $(\partial/\partial t)J\mathbf{T}_t = \operatorname{div} \mathbf{W}_0$ when $t = 0$.
   (c) $(\partial/\partial t)J\mathbf{T}_t = J\mathbf{T}_t \operatorname{div} \mathbf{W}_0$.
   (d) Formula (8.9) holds.

5. Let $f(x, y) = \frac{1}{2}\log(x^2 + y^2), (x, y) \neq (0, 0)$. Let $D = \{(x, y) : 0 < x^2 + y^2 < a^2\}$, and let $\zeta = df$. Show that the left side of (8.7) is 2π, but the right side is 0. Why does not this contradict the divergence theorem?

In Problems 6 and 7, let $M = \text{fr } D$.

6. Show that:
(a) $\int_M v^i(\mathbf{x}) dV_{n-1}(\mathbf{x}) = 0$.
(b) $\int_M \mathbf{x} \cdot \mathbf{v}(\mathbf{x}) dV_{n-1}(\mathbf{x}) = nV_n(D)$.
(c) $\int_M f_{\mathbf{v}}(\mathbf{x}) dV_{n-1}(\mathbf{x}) = \int_D \text{Lapl } f(\mathbf{x}) dV_n(\mathbf{x})$.

7. Let D be connected, f harmonic, and $f_{\mathbf{v}}(\mathbf{x}) = 0$ for every $\mathbf{x} \in M$. Show that $f(\mathbf{x})$ is constant on D.

8. Let $D = \{\mathbf{x} : a < |\mathbf{x}| < b\}$, where $0 < a < b$.
(a) Show that if $f(\mathbf{x}) = \psi(|\mathbf{x}|)$, then $f_{\mathbf{v}}(\mathbf{x}) = \psi'(|\mathbf{x}|)$ when $|\mathbf{x}| = b$ and $f_{\mathbf{v}}(\mathbf{x}) = -\psi'(|\mathbf{x}|)$ when $|\mathbf{x}| = a$.
(b) Let $\psi(r) = r^k$, and let $n > 2$. Find the value of k for which the function f in (a) is harmonic.
(c) Let ϕ be harmonic on the n-ball $B = \{\mathbf{x} : |\mathbf{x}| \leq b\}$. Show that

$$\phi(\mathbf{0}) = (\beta_n b^{n-1})^{-1} \int_{\text{fr } B} \phi \, dV_{n-1}.$$

[*Hint*: Apply the second Green's formula with D and f as above and let $a \to 0^+$.]
β_n is as in Problem 8, Section 8.3.

9. Let $\mathbf{g}(\mathbf{t}) = \mathbf{x}_0 + \mathbf{L}(\mathbf{t})$, where $\mathbf{L}$ is a rotation of E^n. Show that both sides of (8.7) equal the corresponding integrals in which ζ is replaced by $\zeta^* = \sum_{i=1}^n (\zeta_i \circ \mathbf{g}) dg^i$ and D by $\tilde{D} = \mathbf{g}^{-1}(D)$.

*8.5 Fluid flow

As another illustration of the role of the divergence theorem in physical applications, consider a fluid flowing in an open set $D_0 \subset E^3$. Let t denote time and $\mathbf{x} = (x, y, z)$. Let $\rho(\mathbf{x}, t)$ be the density and $\mathbf{v}(\mathbf{x}, t)$ the velocity at $\mathbf{x}$ and time t. They are assumed to be of class $C^{(1)}$. Let $m(t)$ denote the mass of the fluid at time t, in a portion D of D_0. Then

$$m(t) = \int_D \rho(\mathbf{x}, t) dV_3(\mathbf{x}).$$

Suppose that D is a regular domain with $\text{cl } D \subset D_0$. The rate at which mass flows out of D is

$$-\frac{dm}{dt} = \int_{\text{fr } D} (\rho \mathbf{v}) \cdot \mathbf{v} \, dV_2.$$

From the divergence theorem,

$$-\frac{dm}{dt} = \int_D \text{div}(\rho \mathbf{v}) dV_3.$$

On the other hand, by differentiating under the integral sign (Section 5.12),

$$\frac{dm}{dt} = \int_D \frac{\partial \rho}{\partial t} dV_3.$$

For each t_0 the functions $-\operatorname{div}(\rho \mathbf{v})$ and $\partial\rho/\partial t$ have the same integral over every regular D with $\operatorname{cl} D \subset D_0$. By Lemma 1, Section 8.4, for every $\mathbf{x}_0 \in D_0$ these functions have the same value at $(\mathbf{x}_0, t_0)$. In other words,

$$\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho \mathbf{v}).$$

If the density ρ is constant, then the fluid is *incompressible*. Thus for incompressible fluids $\operatorname{div} \mathbf{v} = 0$ at every time t.

In the above discussion the only physical principle used is conservation of mass. To proceed further, some additional description of the physical properties of the flow is needed. Let us consider the following simple model of fluid flow. It is assumed that: (1) the fluid is incompressible, (2) the velocity $\mathbf{v}(\mathbf{x})$ at a point $\mathbf{x}$ is not time dependent, and (3) there exists a function f of class $C^{(2)}$ on D_0 such that $\mathbf{v} = \operatorname{grad} f$.

Flows that satisfy (2) are called *steady*. The function f in (3) is called a *velocity potential*. To better understand Condition (3), let us introduce the concept of circulation along a curve γ lying in the region D_0 of flow. Let us choose arc length s as parameter on γ, and let $\mathbf{G}$ be the corresponding standard representation of γ on $[0, l]$, where l is the length of γ (Section 6.2). Then $\mathbf{G}'(s)$ is a unit tangent vector at the point $\mathbf{x} = \mathbf{G}(s)$ on γ. The *tangential component* of the flow is $\mathbf{v}[\mathbf{G}(s)] \cdot \mathbf{G}'(s)$, and the *circulation* along γ is

$$c = \int_0^l \mathbf{v}[\mathbf{G}(s)] \cdot \mathbf{G}'(s) ds.$$

The circulation can be rewritten as follows. Consider the 1-form $\zeta = v^1\,dx + v^2\,dy + v^3\,dz$, where $v^1(\mathbf{x}), v^2(\mathbf{x}), v^3(\mathbf{x})$ are the components of the velocity $\mathbf{v}(\mathbf{x})$. Then

$$c = \int_\gamma \zeta.$$

Condition (3) states that $\zeta = df$ is an exact differential form. By Theorem 6.1, Condition (3) is equivalent to the statement that the circulation along any closed curve γ lying in D is 0.

A weaker assumption than (3) is that: (3′) ζ is a closed differential form ($d\zeta = \mathbf{0}$). The flow is called *irrotational* if (3′) holds, for reasons indicated in Section 8.8. Theorem 8.4 (Section 8.10), implies that (3) and (3′) are equivalent if D_0 is simply connected.

EXAMPLE 1. Let D_0 be E^3 with the z-axis removed, and let $\mathbf{v}(x, y, z) = (x^2 + y^2)^{-1}(-y\mathbf{e}_1 + x\mathbf{e}_2)$. The corresponding 1-form is

$$\zeta = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

By Example 2, Section 6.3, ζ is closed but not exact. Condition (3′) is satisfied, but there is no velocity potential. In this example, z plays no role; the flow is

effectively 2-dimensional. Particles move in circles $x^2 + y^2 = a^2$ about $(0, 0)$, with the same period 2π for all a. The point $(0, 0)$ is called a *vortex* [19, p. 24].

Let us return to fluid flows satisfying (1) through (3). Since

$$0 = \operatorname{div} \zeta = \operatorname{div}(df) = \operatorname{Lapl} f,$$

the velocity potential must be a harmonic function in D_0. Let us assume that $\operatorname{fr} D_0$ is a $C^{(1)}$ manifold and impose the additional condition at the boundary: (4) $f_{\mathbf{v}}(\mathbf{x}) = 0$ at each $\mathbf{x} \in \operatorname{fr} D_0$, where $f_{\mathbf{v}}(\mathbf{x})$ is the derivative in the direction of the exterior unit normal $\mathbf{v}(\mathbf{x})$. Since $\mathbf{f}_{\mathbf{v}}(\mathbf{x}) = \operatorname{grad} f(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})$, Condition (4) states that the component of velocity normal to $\operatorname{fr} D_0$ is 0. This is a reasonable condition since the fluid cannot enter or leave D_0 through $\operatorname{fr} D_0$. In addition, a condition may be imposed on $\mathbf{v}(\mathbf{x})$ for large $|\mathbf{x}|$, if D_0 is unbounded.

EXAMPLE 2. Let us consider 2-dimensional flow past a circular obstacle. Let $D_0 = \{(x, y) : x^2 + y^2 > 1\}$. We require that $\mathbf{v}(x, y) \to v_0 \mathbf{e}_1$ as $x^2 + y^2 \to \infty$. In other words, the flow is nearly in the positive x direction with speed v_0 far from the obstacle. Let us find the velocity potential $f(x, y)$. For this purpose we suppose that

$$f(r \cos \theta, r \sin \theta) = g(r)h(\theta).$$

We assume that $g'(1) = 0$, which amounts to Condition (4). Moreover, we assume that $h(\theta)$ is decreasing for $0 < \theta < \pi/2$, and that $h(-\theta) = h(\theta) = -h(\pi - \theta)$. The last condition imposes reasonable symmetries on the flow. Since $\operatorname{Lapl} f = 0$, one gets (Problem 1):

$$(*) \qquad \begin{aligned} h(\theta) &= \cos \theta \\ r^2 g''(r) + r g'(r) &= g(r). \end{aligned}$$

This second-order linear differential equation for g can be solved by observing that r and r^{-1} are particular solutions. The general solution is $g(r) = C_1 r + C_2 r^{-1}$. The choice $C_1 = C_2 = v_0$ gives $g(r) = v_0(r + r^{-1})$, satisfying $g'(1) = 0$ and $g'(r) \to v_0$ as $r \to \infty$. Then

$$f(r \cos \theta, r \sin \theta) = v_0 \left(r + \frac{1}{r} \right) \cos \theta$$

is the desired velocity potential. It can be shown that f is uniquely determined up to an additive constant. Figure 8.9 indicates the flow. At each point (x, y) of a flow curve, $\mathbf{v}(x, y)$ is a tangent vector. Our solution to this problem depended on solving differential equations for g and h. The theory of complex analytic functions and conformal mapping gives another method for finding velocity potentials for 2-dimensional flows past obstacles of various shapes [19, Chapter 1]. The reader who has observed the flow of a swiftly moving river past obstacles may question the realism of this simple model of fluid

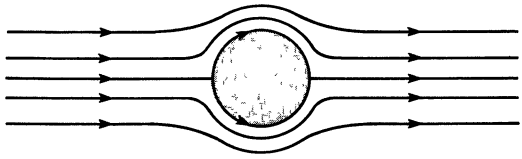


Figure 8.9

flow. In particular, the solution does not account for eddies downstream from the obstacle, in which the velocity vector differs greatly in direction from the general direction of flow.

PROBLEMS

1. In Example 2, show first that $g^{-1}(r^2g'' + rg') = -h^{-1}h'' = C$ where C is a positive constant. Then show that $C = 1$ from the conditions imposed on h.
2. Suppose that the velocity potential has the form $f(r\cos\theta, r\sin\theta) = g(r)$ for all $r > 0$. Show that $g(r) = C_1 \log r + C_2$, where C_1, C_2 are constants. Find the velocity vector $\mathbf{v}(x, y)$ and describe the flow.
3. Consider the flow in E^2 such that $\mathbf{v} = \operatorname{grad} f$, where $f(x, y) = xy^2$. Show that the flow is not incompressible. Find the flow curves and sketch them; each such curve is represented by a solution $\mathbf{g}$ of the differential equation $\mathbf{g}'(t) = \mathbf{v}[\mathbf{g}(t)]$.

8.6 Orientations

Let M be an r-manifold. For each $\mathbf{x} \in M$, the tangent space $T_M(\mathbf{x})$ is an r-dimensional vector subspace of E^n. According to Section 7.5 $T_M(\mathbf{x})$ has two possible orientations, each of which is an r-vector of norm 1. If one of these orientations is denoted by $\mathbf{o}(\mathbf{x})$, then the other is $-\mathbf{o}(\mathbf{x})$. We would like to choose the orientation for $T_M(\mathbf{x})$ consistently on M; in other words, we want the function $\mathbf{o}$ whose value at $\mathbf{x}$ is $\mathbf{o}(\mathbf{x})$ to be continuous on M.

Definition. M is an *orientable* manifold if there exists a continuous r-vector-valued function $\mathbf{o}$ such that $\mathbf{o}(\mathbf{x})$ is an orientation for the tangent space $T_M(\mathbf{x})$ for every $\mathbf{x} \in M$. The function $\mathbf{o}$ is an *orientation* for M.

It can be shown that a connected manifold has at most two orientations. Let us find out what orientability means in the extreme dimensions $r = 1$, $n - 1, n$.

The case $r = 1$

If M is a 1-manifold, then the two orientations for the 1-dimensional vector space $T_M(\mathbf{x})$ are unit tangent vectors at $\mathbf{x}$ pointing in opposite directions. M is oriented by assigning a unit tangent vector $\mathbf{v}(\mathbf{x})$ continuously on M (Figure 8.10). It can be shown that every 1-manifold is orientable.

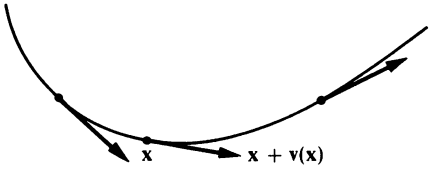


Figure 8.10

The case $r = n$

Here the n-manifold M is an open subset of E^n. The possible values for $\mathbf{o}(\mathbf{x})$ are $\pm\mathbf{e}_{1\ldots n}$. If M is connected, then $\mathbf{o}(\mathbf{x})$ must be constant. If $\mathbf{o}(\mathbf{x}) = \mathbf{e}_{1\ldots n}$ for every $\mathbf{x} \in M$, then M is *positively oriented*; and if $\mathbf{o}(\mathbf{x}) = -\mathbf{e}_{1\ldots n}$ for every $\mathbf{x} \in M$, then M is *negatively oriented.*

The case $r = n - 1$

If M is an $(n-1)$-manifold in E^n, then the adjoint $\mathbf{n}(\mathbf{x}) = {}^*\mathbf{o}(\mathbf{x})$ is a unit normal vector to M at $\mathbf{x}$. The condition that M be orientable is that a unit normal can be chosen continuously on M. If D is an open set that is a regular domain (Section 8.4), then the exterior unit normal orients fr D. We call this the *positive orientation* on fr D. If M is not the boundary of an open set, then M may not be orientable. This is shown by the following famous surface.

EXAMPLE 1 (*Möbius strip*). This is a 2-manifold $M \subset E^3$ that is not orientable. It may be visualized by twisting a strip of paper and pasting together the ends (Figure 8.11). The edge of the strip must be omitted in order that M be locally like E^2. The fact that a unit normal cannot be chosen continuously may be expressed more picturesquely by saying that the Möbius strip is a surface with "only one side."

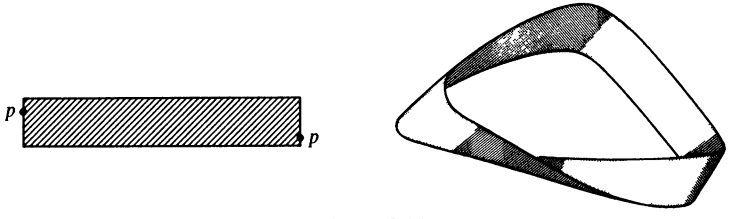


Figure 8.11

EXAMPLE 2. The Möbius strip is not a compact 2-manifold. An example of a compact, nonorientable 2-manifold is the *Klein bottle*, or *twisted torus*. It is obtained by also joining together the lateral edges of the rectangle used to make the Möbius strip, as indicated in Figure 8.12. The Klein bottle cannot be realized as a submanifold of E^3, since it can be proved that any compact $(n-1)$-manifold $M \subset E^n$ is the boundary of an open set and hence is orientable. However, the Klein bottle can be realized as a submanifold of E^4.

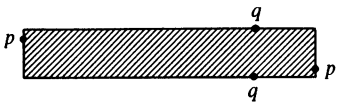


Figure 8.12

Orientation induced by a regular transformation

Let S be a coordinate patch on an r-manifold M, and $\mathbf{g}$ a regular transformation from an open set $\Delta \subset E^r$ onto S. By Proposition 8.1, the partial derivatives of $\mathbf{g}$ give a frame $(\mathbf{g}_1(\mathbf{t}), \ldots, \mathbf{g}_r(\mathbf{t}))$ for the tangent space to M at $\mathbf{x} = \mathbf{g}(\mathbf{t})$. These frames vary continuously with $\mathbf{t}$, and thus assign an orientation α_0 varying continuously on S. In fact, from the discussion of orientations in Section 7.5

$$\alpha_0(\mathbf{x}) = [\mathscr{J}\mathbf{g}(\mathbf{t})]^{-1}\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t}), \qquad \mathbf{x} = \mathbf{g}(\mathbf{t}) \tag{8.13}$$

with $\mathscr{J}\mathbf{g}(\mathbf{t}) = |\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t})|$ as in (8.1).

Definition. α_0 is the orientation *induced* on S by $\mathbf{g}$ from the positive orientation on Δ.

For evaluating integrals of r-forms in later sections, we need to know whether an induced orientation α_0 agrees with a preassigned orientation $\mathbf{o}$. If S is connected, then either $\alpha_0(\mathbf{x}) = \mathbf{o}(\mathbf{x})$ for all $\mathbf{x} \in S$ or else $\alpha_0(\mathbf{x}) = -\mathbf{o}(\mathbf{x})$ for all $\mathbf{x} \in S$. This is true since α_0 and $\mathbf{o}$ are continuous on S, and $\alpha_0(\mathbf{x}) = \pm\mathbf{o}(\mathbf{x})$ for all $\mathbf{x} \in S$.

The following proposition is often useful in this connection. Let $\mathbf{o}$ be an orientation for S. Then

$$\mathbf{o}(\mathbf{x}) = \sum_{[\lambda]} o^\lambda(\mathbf{x})\mathbf{e}_\lambda,$$

where $o^\lambda(\mathbf{x})$ is the component of $\mathbf{o}(\mathbf{x})$ with respect to the standard basis r-vector $\mathbf{e}_\lambda$ and $\lambda = (i_1, \ldots, i_r)$ denotes an increasing r-tuple as in Section 7.5. As in Section 8.1, let $\mathbf{g}^\lambda = (g^{i_1}, \ldots, g^{i_r})$ be the flat transformation obtained by considering only those components of $\mathbf{g}$.

Proposition 8.8. *The induced orientation agrees with* $\mathbf{o}(\mathbf{x})$, *at* $\mathbf{x} = \mathbf{g}(\mathbf{t})$, *if* $o^\lambda(\mathbf{x})$ *and the Jacobian* $J\mathbf{g}^\lambda(\mathbf{t})$ *have the same sign. If* $o^\lambda(\mathbf{x})$ *and* $J\mathbf{g}^\lambda(\mathbf{t})$ *have opposite signs, then the two orientations are opposite.*

PROOF. By (7.33) with $\mathbf{h}_j = \mathbf{g}_j(\mathbf{t})$, the λth component of $\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t})$ is $J\mathbf{g}^\lambda(\mathbf{t})$. Therefore $\alpha_0^\lambda(\mathbf{x}) = [\mathscr{J}\mathbf{g}(\mathbf{t})]^{-1}J\mathbf{g}^\lambda(\mathbf{t})$. Since $\alpha_0(\mathbf{x}) = \pm\mathbf{o}(\mathbf{x})$, $\alpha_0(\mathbf{x}) = \mathbf{o}(\mathbf{x})$ if and only if $\alpha_0^\lambda(\mathbf{x}) = o^\lambda(\mathbf{x})$. □

EXAMPLE 3. Let H be the hemisphere in Example 3, Section 8.3, oriented so that $o^{12}(\mathbf{x}) > 0$ for every $\mathbf{x} \in H$. If (ϕ, θ) are spherical coordinates and $\mathbf{g}$ is as

in that example, then

$$J\mathbf{g}^{12}(\phi, \theta) = \frac{\partial(\mathbf{g}^1, \mathbf{g}^2)}{\partial(\phi, \theta)} = \sin\phi \cos\phi > 0.$$

Therefore, the orientation induced by $\mathbf{g}$ agrees with the given orientation $\mathbf{o}$. In this example the vector $\mathbf{n}(\mathbf{x}) = {}^*\mathbf{o}(\mathbf{x})$ is normal to H, and its third component $n^3(\mathbf{x}) = o^{12}(\mathbf{x})$. We have oriented H so that the normal "points upward."

Problems

1. Let $M \subset E^2$ be a 1-manifold determined by a $C^{(1)}$ function Φ, namely, $M = \{(x, y) : \Phi(x, y) = 0, \text{ grad } \Phi(x, y) \neq \mathbf{0}\}$. Let $\mathbf{v}(x, y) = |\text{grad } \Phi(x, y)|^{-1}(\Phi_2(x, y)\mathbf{e}_1 - \Phi_1(x, y)\mathbf{e}_2)$.
   (a) Show that $\mathbf{v}$ is an orientation for M.
   (b) Show that the only orientations for M are $\mathbf{v}$ and $-\mathbf{v}$, if M is connected.
   (c) Find all possible orientations for M if $\Phi(x, y) = x^2 - y^2 - 1$.

2. Let $S = \mathbf{g}(R)$, where $\mathbf{g}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + \phi(x, y)\mathbf{e}_3$ and $R \subset E^2$ is open. Find the orientation $\mathbf{o}$ on S induced by $\mathbf{g}$ from the positive orientation of E^2. Find the normal $\mathbf{n}(\mathbf{x}) = {}^*\mathbf{o}(\mathbf{x})$.

3. Let $M = \{(x, y, z) : y^2 = x^2 + z^2 + 1, y < 0\}$, and $\mathbf{o}$ the orientation for M such that $o^{31}(x, y, z) < 0$. Let (r, θ) be polar coordinates in the (x, z) plane, and $\mathbf{g}(r, \theta) = (r\cos\theta)\mathbf{e}_1 - (r^2 + 1)^{1/2}\mathbf{e}_2 + (r\sin\theta)\mathbf{e}_3$. Use Proposition 8.8 to show that the induced orientation agrees with $\mathbf{o}$.

4. Suppose that M is the r-manifold determined by Φ, in the sense that M satisfies (4.26), Section 4.7. Show that M is orientable.

8.7 Integrals of r-forms

Let us first define the integral of a differential form ω of degree r over a portion A of an r-manifold M. The integral depends on the orientation assigned to M, and changes sign if the orientation is reversed.

Let M be an r-manifold with orientation $\mathbf{o}$, and ω an r-form continuous on M. For each $\mathbf{x} \in M$ consider $\omega(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x})$, the scalar product of the r-covector $\omega(\mathbf{x})$ and the r-vector $\mathbf{o}(\mathbf{x})$. Since ω and $\mathbf{o}$ are continuous functions, $\omega \cdot \mathbf{o}$ is a continuous real valued function. Let A be an r-measurable subset of M.

Definition. The integral of ω over A with the orientation $\mathbf{o}$ is

$$\int_{A^\mathbf{o}} \omega = \int_A \omega(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x}) dV_r(\mathbf{x}), \tag{8.14}$$

provided $\omega \cdot \mathbf{o}$ is integrable over A.

In particular, if M is compact, then $\omega \cdot \mathbf{o}$ is continuous, bounded, and hence integrable over any r-measurable subset of M. The integral has the

following elementary properties:

(1) $\int_{A^o}(\omega^1 + \omega^2) = \int_{A^o} \omega^1 + \int_{A^o} \omega^2$.
(2) $\int_{A^o} c\omega = c \int_{A^o} \omega$, for any scalar c.
(3) $\int_{A^-o} \omega = -\int_{A^o} \omega$.
(4) If $|\omega(\mathbf{x})| \leq C$ for every $\mathbf{x} \in A$, then $|\int_{A^o} \omega| \leq CV_r(A)$.
(5) $\int_{A^o} \omega = \int_{A_1^o} \omega + \int_{A_2^o} \omega$ if $A = A_1 \cup A_2$ and $A_1 \cap A_2$ is empty.

These follow at once from corresponding elementary properties of the right-hand side of (8.14) (see Problem 11, Section 8.3).

For instance, in (3),

$$\int_A \omega(\mathbf{x}) \cdot [-\mathbf{o}(\mathbf{x})] dV_r(\mathbf{x}) = -\int_A \omega(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x}) dV_r(\mathbf{x}).$$

Since $|\mathbf{o}(\mathbf{x})| = 1$, $|\omega(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x})| \leq |\omega(\mathbf{x})|$. Then in (4),

$$\left| \int_A \omega(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x}) dV_r(\mathbf{x}) \right| \leq \int_A |\omega(\mathbf{x})| dV_r(\mathbf{x}) \leq CV_r(A).$$

The case $r = n$

Let A^+ denote A with the positive orientation $\mathbf{e}_{1\ldots n}$ of E^n. Let $\omega = f\,dx^1 \wedge \cdots \wedge dx^n$ be an n-form. Then

$$\omega(\mathbf{x}) \cdot \mathbf{e}_{1\ldots n} = \omega_{1\ldots n}(\mathbf{x}) = f(\mathbf{x}),$$

and (8.14) becomes

$$\int_{A^+} f\,dx^1 \wedge \cdots \wedge dx^n = \int_A f\,dV_n. \tag{8.15}$$

The left-hand side of (8.15) changes sign if either the orientation of A is reversed or two differentials dx^i and dx^j are interchanged. For instance, if $n = 2$ then

$$\int_{A^+} f\,dx \wedge dy = \int_A f\,dV_2,$$

$$\int_{A^-} f\,dx \wedge dy = \int_{A^+} f\,dy \wedge dx = -\int_A f\,dV_2.$$

Let us next show how to rewrite $\int_{A^o} \omega$ if $r < n$ and A is an r-measurable subset of some coordinate patch S. Let $\mathbf{g}$ be regular from an open set $\Delta \subset E^r$ onto S. We recall from Section 7.7 that $\omega^\sharp$ is the corresponding r-form on Δ, obtained formally by substituting everywhere $\mathbf{g}(\mathbf{t})$ for $\mathbf{x}$ and dg^i for dx^i.

Proposition 8.9. *If* $\mathbf{o}$ *is the orientation on* S *induced by* $\mathbf{g}$ *from the positive orientation on* Δ, *then*

$$\int_{A^o} \omega = \int_{B^+} \omega^\sharp, \qquad A = \mathbf{g}(B), \tag{8.16}$$

provided either integral exists.

PROOF. Given $\mathbf{t}$, let $\mathbf{x} = \mathbf{g}(\mathbf{t})$ and $\mathbf{L} = D\mathbf{g}(\mathbf{t})$. Recall that $\mathbf{g}_j(\mathbf{t}) = \mathbf{L}(\boldsymbol{\varepsilon}_j)$, where $\{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_r\}$ denotes the standard basis for E^r. By (7.39),

$$\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t}) = \mathbf{L}_r(\boldsymbol{\varepsilon}_1 \wedge \cdots \wedge \boldsymbol{\varepsilon}_r) = \mathbf{L}_r(\boldsymbol{\varepsilon}_{1\ldots r}).$$

By (7.49), $\boldsymbol{\omega}^\#(\mathbf{t}) = \mathbf{L}_r^*[\boldsymbol{\omega}(\mathbf{x})]$. Since $\boldsymbol{\varepsilon}_{1\ldots r}$ is the positive orientation for E^r,

$$(*) \qquad \int_{B^+} \boldsymbol{\omega}^\# = \int_B \boldsymbol{\omega}^\#(\mathbf{t}) \cdot \boldsymbol{\varepsilon}_{1\ldots r}\, dV_r(\mathbf{t}).$$

In (7.41), let $\boldsymbol{\beta} = \boldsymbol{\varepsilon}_{1\ldots r}$. Then

$$\boldsymbol{\omega}[\mathbf{g}(\mathbf{t})] \cdot (\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t})) = \boldsymbol{\omega}^\#(\mathbf{t}) \cdot \boldsymbol{\varepsilon}_{1\ldots r}.$$

On the other hand, let $f = \boldsymbol{\omega} \cdot \mathbf{o}$ in (8.4). Then

$$(**) \qquad \int_{A^\mathbf{o}} \boldsymbol{\omega} = \int_A \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x}) dV_r(\mathbf{x}) = \int_B \boldsymbol{\omega}[\mathbf{g}(\mathbf{t})] \cdot \mathbf{o}[\mathbf{g}(\mathbf{t})] \mathscr{J}\mathbf{g}(\mathbf{t}) dV_r(\mathbf{t}).$$

Since $\mathbf{o}$ is the induced orientation, we have $\mathbf{o}(\mathbf{x}) = \boldsymbol{\alpha}_0(\mathbf{x})$ in (8.13). Therefore

$$\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t}) = \mathscr{J}\mathbf{g}(\mathbf{t})\mathbf{o}[\mathbf{g}(\mathbf{t})].$$

Then (8.16) follows from (*) and (**). □

EXAMPLE 1. Let $r = 1, \Delta \subset E^1$. Then $\boldsymbol{\omega}^\#(t) = \boldsymbol{\omega}[\mathbf{g}(t)] \cdot \mathbf{g}'(t)$. If B is an interval, then $\int_{B^+} \boldsymbol{\omega}^\#$ is the line integral of $\boldsymbol{\omega}$ along the curve in E^n represented parametrically on B by $\mathbf{g}$.

EXAMPLE 2. Consider the hyperboloid $M = \{(x, y, z) : x^2 = y^2 + z^2 + 1\}$. M is a 2-manifold. Let $A = \{(x, y, z) \in M : 1 \leq x < \sqrt{2}\}$, and $\mathbf{o}$ the orientation such that $o^{23}(x, y, z) > 0$. It is convenient to coordinate A using polar coordinates (r, θ) in the (y, z) plane. Thus, let $\mathbf{g}(r, \theta) = (r^2 + 1)^{1/2}\mathbf{e}_1 + (r \cos\theta)\mathbf{e}_2 + (r \sin\theta)\mathbf{e}_3$, and $B = \{(r, \theta) : 0 < r < 1, 0 < \theta < 2\pi\}$. Now

$$\begin{aligned}
(dx)^\# &= d[(r^2 + 1)^{1/2}] = r(r^2 + 1)^{-1/2}\, dr \\
(dy)^\# &= d(r \cos\theta) = \cos\theta\, dr - r \sin\theta\, d\theta \\
(dz)^\# &= d(r \sin\theta) = \sin\theta\, dr + r \cos\theta\, d\theta \\
(dx \wedge dy)^\# &= (dx)^\# \wedge (dy)^\# = -r^2(r^2 + 1)^{-1/2} \sin\theta\, dr \wedge d\theta \\
(dz \wedge dx)^\# &= (dz)^\# \wedge (dx)^\# = -r^2(r^2 + 1)^{-1/2} \cos\theta\, dr \wedge d\theta \\
(dy \wedge dz)^\# &= (dy)^\# \wedge (dz)^\# = r\, dr \wedge d\theta.
\end{aligned}$$

In the process of computing $(dy \wedge dz)^\#$ we have computed the Jacobian

$$J\mathbf{g}^{23}(r, \theta) = \frac{\partial(g^2, g^3)}{\partial(r, \theta)} = r.$$

Compare with Example 1, Section 7.7. Since $J\mathbf{g}^{23}$ and o^{23} are both positive, Proposition 8.8 implies that $\mathbf{o}$ agrees with the orientation induced by $\mathbf{g}$. If $\boldsymbol{\omega} = P\, dx \wedge dy + Q\, dz \wedge dx + R\, dy \wedge dz$, then

$$\boldsymbol{\omega}^\# = P \circ \mathbf{g}(dx \wedge dy)^\# + Q \circ \mathbf{g}(dz \wedge dx)^\# + R \circ \mathbf{g}(dy \wedge dz)^\#.$$

We may use Proposition 8.9 here. Note that $\mathbf{g}(B)$ differs from A by a portion of the curve on M corresponding to $\theta = 0$ or 2π. The curve contributes nothing to integrals over A. For instance, let $\omega = xz\,dx \wedge dy$. Then $\omega^\sharp = -r^3 \sin^2\theta\, dr \wedge d\theta$. By (8.16),

$$\int_{A^\circ} \omega = \int_{B^+} (-r^3 \sin^2\theta) dr \wedge d\theta = -\int_B r^3 \sin^2\theta\, dV_2(r,\theta)$$
$$= -\int_0^{2\pi} \sin^2\theta\, d\theta \int_0^1 r^3\, dr = -\frac{\pi}{4}.$$

Example 2 suggests how Proposition 8.8 and 8.9 can be generally applied. By Formula (7.23) it suffices to consider $\omega = f\,dx^{i_1} \wedge \cdots \wedge dx^{i_r}$, since any r-form is the sum of terms of this type. Then

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_r})^\sharp = dg^{i_1} \wedge \cdots \wedge dg^{i_r} = J\mathbf{g}^\lambda(\mathbf{t}) dt^1 \wedge \cdots \wedge dt^r.$$

If $o^\lambda(\mathbf{x})$ and $J\mathbf{g}^\lambda(\mathbf{t})$ have the same sign, then

$$\int_{A^\circ} f\,dx^{i_1} \wedge \cdots \wedge dx^{i_r} = \int_B f \circ \mathbf{g}\, J\mathbf{g}^\lambda\, dV_r. \tag{8.17}$$

A second version of the divergence theorem

Let $D \subset E^n$ be a regular domain. The exterior unit normal $\mathbf{v}$ provides an orientation $\mathbf{o}$ for fr D, called the *positive orientation*. For each $\mathbf{x} \in$ fr D, $\mathbf{o}(\mathbf{x})$ is the $(n-1)$-vector such that $\mathbf{v}(\mathbf{x}) = {}^*\mathbf{o}(\mathbf{x})$. Let us write ∂D^+, instead of (fr $D)^\circ$, for fr D with this orientation. (The symbol ∂ is widely used to denote a boundary.)

Divergence theorem (Second version). *Let D be a regular domain and ω an $(n-1)$-form of class $C^{(1)}$ on* cl D. *Then*

$$\int_{\partial D^+} \omega = \int_{D^+} d\omega. \tag{8.18}$$

PROOF. Consider the 1-form $\zeta = {}^*\omega$. By Formulas (7.56) and (7.59),

$$\omega(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x}) = {}^*\omega(\mathbf{x}) \cdot {}^*\mathbf{o}(\mathbf{x}) = \zeta(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}),$$
$$d\omega = \operatorname{div} \zeta\, dx^1 \wedge \cdots \wedge dx^n.$$

Therefore

$$\int_{\partial D^+} \omega = \int_{\mathrm{fr}\, D} \omega \cdot \mathbf{o}\, dV_{n-1} = \int_{\mathrm{fr}\, D} \zeta \cdot \mathbf{v}\, dV_{n-1},$$

$$\int_{D^+} d\omega = \int_{D^+} \operatorname{div} \zeta\, dx^1 \wedge \cdots \wedge dx^n = \int_D \operatorname{div} \zeta\, dV_n.$$

Then (8.18) follows from (8.7). □

One can characterize the orientation $\mathbf{o}(\mathbf{x})$ in terms of frames. Let $(\mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$ be a frame for the tangent space to the $(n-1)$-manifold fr D at $\mathbf{x}$. This frame determines the orientation $\mathbf{o}(\mathbf{x})$ if $(\mathbf{v}(\mathbf{x}), \mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$ is a positively oriented frame for E^n; otherwise it determines the orientation $-\mathbf{o}(\mathbf{x})$.

The case $n = 2$

Suppose that fr $D = C_1 \cup \cdots \cup C_m$, where each C_k is the trace of a simple closed curve γ_k, and $C_1, \ldots, C_m$ are disjoint. The orientation is chosen by selecting the unit tangent vector $\mathbf{v}(x, y)$ so that $(\mathbf{v}(x, y), \mathbf{v}(x, y))$ is a positively oriented orthonormal frame for E^2. Intuitively speaking, this means that as the boundary is traversed, D is always on the left. Then

$$\int_{\partial D^+} \omega = \int_{\gamma_1} \omega + \cdots + \int_{\gamma_m} \omega.$$

If we write $\omega = M\,dx + N\,dy$, then (8.18) becomes

$$\sum_{k=1}^{m} \int_{\gamma_k} M\,dx + N\,dy = \int_{D^+} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \wedge dy. \tag{8.19}$$

This is known as *Green's theorem* (see Figure 8.13).

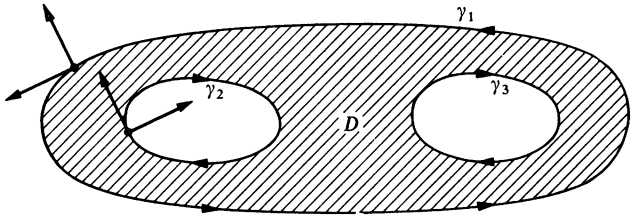


Figure 8.13

EXAMPLE 3. Let $\omega = \frac{1}{2}(x\,dy - y\,dx)$. Then $d\omega = dx \wedge dy$ and $V_2(D) = \int_{D^+} dx \wedge dy$. Hence the area of D can be written as an integral over the boundary:

$$V_2(D) = \frac{1}{2} \int_{\partial D^+} x\,dy - y\,dx.$$

PROBLEMS

1. Let $\Delta \subset E^n$ have the positive orientation and let $\mathbf{g}$ be a regular flat transformation from Δ into E^n.
   (a) Show that $\mathbf{g}$ induces the positive orientation on $\mathbf{g}(\Delta)$ if and only if $J\mathbf{g}(\mathbf{t}) > 0$ for every $\mathbf{t} \in \Delta$.
   (b) Show that (8.16) becomes

$$\int_{A^+} f\,dx^1 \wedge \cdots \wedge dx^n = \int_{B^+} f \circ \mathbf{g}\, J\mathbf{g}\,dt^1 \wedge \cdots \wedge dt^n,$$

   provided $J\mathbf{g}(\mathbf{t}) > 0$ for every $\mathbf{t} \in \Delta$.

2. Let $A = \{(x, y, z) : y = x^2 + z^2, y \leq 4\}$, oriented so that $o^{31}(\mathbf{x}) > 0$. Evaluate:
(a) $\int_{A^o} z\, dx \wedge dy$. (b) $\int_{A^o} \exp y\, dz \wedge dx$.
[*Hint*: Use polar coordinates in the (x, z)-plane.]

3. Let A be the triangle in E^3 with vertices $\mathbf{e}_1, -\mathbf{e}_2, 2\mathbf{e}_3$.
(a) Show that $\mathbf{o} = \frac{1}{3}(2\mathbf{e}_{23} - 2\mathbf{e}_{31} + \mathbf{e}_{12})$ is an orientation for the plane containing A.
(b) Evaluate $\int_{A^o} x\, dy \wedge dz$. [*Hint*: Take $\mathbf{g}$ affine such that $\mathbf{g}(\Sigma) = A$, where Σ is the standard 2-simplex.]

4. Let $A = \{(x, y, z) : x^2 + y^2 = z^2, x > 0, 0 < z < 1\}$, oriented so that $o^{12}(\mathbf{x}) < 0$. Evaluate

$$\int_{A^o} z^2\, dy \wedge dz.$$

5. Let $n = 4$ and $M = \{\mathbf{x} : (x^1)^2 + (x^2)^2 = 1, (x^3)^2 + (x^4)^2 = 1\}$. Let $\mathbf{g}(s, t) = (\cos s)\mathbf{e}_1 + (\sin s)\mathbf{e}_2 + (\cos t)\mathbf{e}_3 + (\sin t)\mathbf{e}_4$, $0 \leq s, t \leq 2\pi$.
(a) Find the orientation $\mathbf{o}$ induced by $\mathbf{g}$ from the positive orientation of E^2.
(b) Evaluate

$$\int_{M^o} dx^3 \wedge dx^4 + x^1 x^3\, dx^2 \wedge dx^4.$$

6. Let $A = \mathbf{g}(B)$, where $\mathbf{g}$ is as in Problem 2, Section 8.6, and $\mathbf{o}$ the orientation induced by $\mathbf{g}$. Using (8.16), show that

$$\int_{A^o} P(x, y, z) dx \wedge dy = \int_B P(x, y, \phi(x, y)) dV_2(x, y).$$

7. Let $D \subset E^2$ be a regular domain. Show that:
(a) $V_2(D) = -\int_{\partial D^+} y\, dx$.
(b) $\int_D (x^2 + y^2) dV_2 = \frac{1}{3}\int_{\partial D^+} x^3\, dy - y^3\, dx$.

8. Evaluate $\int_{\partial\Sigma^+} y^2\, dx \wedge dz$, where Σ is the standard 3-simplex.

9. Let D be the disk $x^2 + y^2 < 1$ and $\omega = (x\, dy - y\, dx)/(x^2 + y^2)$. Then $\int_{\partial D^+} \omega = 2\pi$ while $\int_{D^+} d\omega = 0$. Why does this not contradict Green's theorem?

10. Let $n = 4$ and $D = \{\mathbf{x} : (x^1)^2 + (x^2)^2 + (x^3)^2 < (x^4)^2, 0 < x^4 < 1\}$. Evaluate:
(a) $\int_{\partial D^+} (x^2 + x^4) dx^1 \wedge dx^2 \wedge dx^3$.
(b) $\int_{\partial D^+} |\mathbf{x}|^2\, dx^1 \wedge dx^2 \wedge dx^3$.

11. Suppose that $D = \{(x, y) : f(x) < y < g(x), a < x < b\} = \{(x, y) : \phi(y) < x < \psi(y), c < y < d\}$. Show directly from the fundamental theorem of calculus and properties of line integrals that

$$\int_{\partial D^+} N\, dy = \int_D \frac{\partial N}{\partial x} dV_2,$$

$$\int_{\partial D^+} M\, dx = -\int_D \frac{\partial M}{\partial y} dV_2.$$

Adding, we get the Green's theorem for regular domains of this special type (see Figure 8.14).

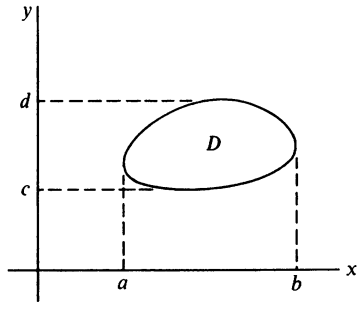


Figure 8.14

12. The *winding number* of a closed curve γ in E^2 about a point (x_0, y_0) not in the trace of γ is

$$w(x_0, y_0) = \frac{1}{2\pi} \int_\gamma \frac{(x - x_0)dy - (y - y_0)dx}{(x - x_0)^2 + (y - y_0)^2}.$$

Let γ be the positively oriented boundary of a regular domain D.

(a) Show that $w(x_0, y_0) = 1$ if $(x_0, y_0) \in D$. [*Hint*: Apply Green's theorem to

$$D_\varepsilon = \{(x, y) \in D : (x - x_0)^2 + (y - y_0)^2 \geq \varepsilon\},$$

where $\varepsilon < \text{dist}[(x_0, y_0), \text{fr } D]$. Note that $m = 2$ in Formula (8.19).]

(b) Show that $w(x_0, y_0) = 0$ if $(x_0, y_0) \notin \text{cl } D$.

8.8 Stokes's formula

The divergence theorem is a special case of a result which is nowadays called *Stokes's formula*. Let ω be an $(r - 1)$-form. Stokes's formula equates the integral of $d\omega$ over a portion A of an oriented r-manifold M and the integral of ω over the (suitably oriented) boundary of A.

Let us begin with the following particular case and afterward generalize. Let $B \subset E^r$ be a regular domain, and let $A = \mathbf{g}(B)$ where $\mathbf{g}$ is a regular transformation of class $C^{(2)}$ from some open set containing cl B into M. Let $\mathbf{o}$ be the orientation induced on A from the positive orientation of E^r. The set $K = \mathbf{g}(\text{fr } B)$ is the boundary of A relative to M.

Let us quote two results whose proofs are somewhat technical and are deferred to Section 8.9. When $r = 2$, these results can also be proved directly (Problem 7). The set K is an $(r - 1)$-manifold by Theorem 8.3. We orient K in a manner consistent with the orientation $\mathbf{o}$ on A, as follows. Let $\boldsymbol{\alpha}$ be the positive orientation on fr B, and let $\boldsymbol{\beta}_1(\mathbf{x}) = \mathbf{L}_{r-1}[\boldsymbol{\alpha}(\mathbf{t})]$, where $\mathbf{x} = \mathbf{g}(\mathbf{t})$, $\mathbf{L} = D\mathbf{g}(\mathbf{t})$, and $\mathbf{L}_{r-1}$ is the induced linear transformation of $(r - 1)$-vectors (Section 7.6). The desired orientation on K is $\boldsymbol{\beta} = |\boldsymbol{\beta}_1|^{-1}\boldsymbol{\beta}_1$. Let $\partial A^{\mathbf{o}}$ denote K with this orientation.

Let ω be an $(r - 1)$-form of class $C^{(1)}$ on cl A. By Proposition 8.12,

$$\int_{\partial A^{\mathbf{o}}} \omega = \int_{\partial B^+} \omega^\sharp.$$

On the other hand, by (8.16),

$$\int_{A^\circ} d\omega = \int_{B^+} (d\omega)^\sharp = \int_{B^+} d\omega^\sharp.$$

By (8.18), the right-hand sides are equal. Therefore we have

Stokes's formula:

$$\int_{\partial A^\circ} \omega = \int_{A^\circ} d\omega. \tag{8.20}$$

The case $r = 2, n = 3$

Here $\omega = P\,dx + Q\,dy + R\,dz$ is a 1-form and $d\omega$ is a 2-form. In this case, (8.20) equates the integral of $d\omega$ over an oriented surface A in E^3 with the integral over the boundary of A oriented consistently with the orientation on A. Figure 8.15 illustrates a case when ∂A° is a single simple closed curve γ. The set A is the image under $\mathbf{g}$ of B, and the orientation $\mathbf{o}$ on A is that induced by $\mathbf{g}$ from the positive orientation on B. The curve $\tilde{\gamma}$ bounding B is taken onto γ by $\mathbf{g}$, with corresponding orientations.

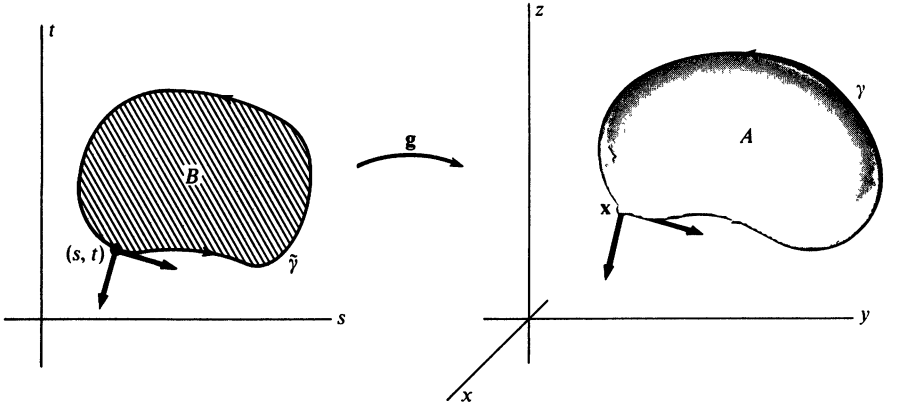


Figure 8.15

EXAMPLE 1. Let A be the portion of the hyperboloid $x^2 = y^2 + z^2 + 1$ in Example 2, Section 8.7. Then γ is a circle in the plane $x = \sqrt{2}$, represented by $\mathbf{f}(\theta) = \mathbf{g}(1, \theta) = \sqrt{2}\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2 + (\sin\theta)\mathbf{e}_3$, $0 \le \theta \le 2\pi$. Let $\eta = y\,dy \wedge dz$. Then $\eta = d\omega$, where $\omega = \frac{1}{2}y^2\,dz$. By Stokes's formula,

$$\int_{A^\circ} y\,dy \wedge dz = \frac{1}{2}\int_\gamma y^2\,dz = \frac{1}{2}\int_0^{2\pi} \cos^2\theta\,d(\sin\theta) = 0.$$

At the last step we used $\cos^2\theta = 1 - \sin^2\theta$. In this example, one also has $\int_\gamma \omega = \int_{\partial B^+} \omega^\sharp$, where ∂B^+ is the perimeter of the rectangle $B = (0, 1) \times (0, 2\pi)$

in the (r, θ) plane traversed counterclockwise. We evaluated $\int_\gamma \omega$ by integrating along the segment λ where $r = 1$, which corresponds to γ. The reader should verify that the rest of ∂B^+ corresponds to an arc on M traversed once in each direction. Since a line integral of ω changes sign when direction is reversed, the contribution from the rest of ∂B^+ is 0 (Problem 3).

When $r = 2$, $n = 3$, Stokes's formula can be rewritten in vector-analysis notation. The 1-form $*d\omega$ is called curl ω, and the vector $\mathbf{n}(\mathbf{x}) = *\mathbf{o}(\mathbf{x})$ is a unit normal tb A. Since $d\omega(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x}) = \text{curl}\, \omega(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$, Formula (8.20) becomes

$$\int_{\partial A^\circ} \omega = \int_A \text{curl}\, \omega(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) dV_2(\mathbf{x}). \tag{8.21}$$

The name *Stokes's formula* was traditionally applied to (8.21) and not its generalization (8.20).

The normal $\mathbf{n}(\mathbf{x})$ varies continuously on A. At a boundary point $\mathbf{x}$ of A, $\mathbf{n}(\mathbf{x})$ can be visualized in the following way. Let $\mathbf{x} = \mathbf{g}(s, t)$, where $(s, t) \in \text{fr}\, B$. Let $\mathbf{v}$ be the exterior normal and $\mathbf{v}$ the positively oriented unit tangent vector to $\tilde{\gamma}$ at (s, t). The vector $\mathbf{h} = D\mathbf{g}(s, t)(\mathbf{v})$ is a tangent vector to γ at $\mathbf{x}$. If $\mathbf{h}_0 = D\mathbf{g}(s, t)(\mathbf{v})$, then $(\mathbf{h}_0, \mathbf{h})$ is a frame for the tangent space to M at $\mathbf{x}$ and has the required orientation $\mathbf{o}(\mathbf{x})$. Hence $(\mathbf{n}(\mathbf{x}), \mathbf{h}_0, \mathbf{h})$ is a positively oriented frame for E^3 (see Figure 8.15).

In particular, let A lie in a plane II, oriented by a unit vector $\mathbf{n}_0$ normal to II. Then $\mathbf{n}(\mathbf{x}) = \mathbf{n}_0$ for every $\mathbf{x} \in A$. Let $\mathbf{x}_0$ be a point of the domain of ω. If A contains $\mathbf{x}_0$ and A has small diameter, then the right-hand side of (8.21) is approximately curl $\omega(\mathbf{x}_0) \cdot \mathbf{n}_0\, V_2(A)$. More precisely,

$$\text{curl}\, \omega(\mathbf{x}_0) \cdot \mathbf{n}_0 = \lim_{\text{diam}\, A \to 0} \frac{1}{V_2(A)} \int_\gamma \omega. \tag{8.22}$$

This is proved using a lemma similar to that for the proof of the corresponding formula (8.8) for the divergence. Note that curl $\omega = \mathbf{0}$ if and only if ω is closed ($d\omega = \mathbf{0}$).

**Note on fluid flow.* In Section 8.5 we defined the circulation c of a fluid along a curve γ lying in the region D_0 of flow. We saw that $c = \int_\gamma \zeta$, where ζ is a 1-form with the same components as the velocity vector. By Stokes's formula, $c = \int_{A^\circ} d\zeta$ if $\gamma = \partial A^\circ$ and $A \subset D_0$. We call the flow irrotational if $d\zeta = \mathbf{0}$. Thus irrotational flows are those for which the circulation is 0 along any simple closed curve γ that bounds such a surface A.

Some generalizations. Let M be an orientable manifold of class $C^{(2)}$. We stated Stokes's formula above in case cl A is contained in some coordinate patch. By using partitions of unity, this restriction can be removed.

Proposition 8.10. *Let M be compact and* $\mathbf{o}$ *an orientation for M. Then*

$$\int_{M^\circ} d\omega = 0$$

for every $(r-1)$-form ω of class $C^{(1)}$ on M.

PROOF. Let $\{\pi_1, \ldots, \pi_m\}$ be a partition of unity for M. Let $\mathbf{g}^{(k)}$ be a regular transformation of class $C^{(2)}$ from an open set $\Delta_k \subset E^r$ onto a coordinate patch S_k containing the support of π_k. Then

$$\int_{M^\circ} d(\pi_k \omega) = \pm \int_{\Delta_k^+} [d(\pi_k \omega)]^\# = \pm \int_{\Delta_k^+} d(\pi_k \omega)^\# = 0,$$

by the divergence theorem and the fact that $(\pi_k \omega)^\# = 0$ near $\operatorname{fr} \Delta_k$ since $\pi_k(\mathbf{x}) = 0$ outside a compact subset of S_k. Since $\sum_k \pi_k = 1$ and $\sum_k d\pi_k = d(\sum_k \pi_k) = 0$,

$$0 = \sum_k \int_{M^\circ} d(\pi_k \omega) = \int_{M^\circ} \left(\sum_k \pi_k\right) d\omega + \int_{M^\circ} \left(\sum_k d\pi_k\right) \omega = \int_{M^\circ} d\omega. \qquad \square$$

Since M has empty boundary relative to itself, one would expect to obtain 0 on the left-hand side of (8.20) when $A = M$. Proposition 8.10 states that this is correct.

Now let M be any orientable r-manifold of class $C^{(2)}$. Let us call a relatively open set $A \subset M$ a *regular domain on M* if:

(1) cl A is a compact subset of M;
(2) the boundary K of A relative to M is an $(r-1)$-manifold of class $C^{(2)}$;
(3) A is the interior, relative to M, of cl A.

Condition (3) rules out oddities of the sort in Example 1, Section 8.4, in which A is on both sides of K. Problem 8 gives a condition equivalent to (3), stated in terms of coordinate systems on M.

Let $\mathbf{o}$ be an orientation for M. If S is a coordinate patch that intersects K, then $K \cap S$ can be oriented consistently with $\mathbf{o}$, as explained earlier in the present section. It can be shown that these orientations agree in overlapping coordinate patches. This defines an orientation $\boldsymbol{\beta}$ on K. Let K with orientation $\boldsymbol{\beta}$ be denoted by ∂A°.

Theorem 8.1. *Let A be a regular domain on M, and let ω be an $(r-1)$-form of class $C^{(1)}$ on* cl A. *Then*

$$\int_{\partial A^\circ} \omega = \int_{A^\circ} d\omega. \tag{8.20}$$

This theorem can be proved using the divergence theorem and a partition of unity in much the same way as for Proposition 8.10. We do not give the details.

We have assumed that M is of class $C^{(2)}$, but Theorem 8.1 is still valid for manifolds of class $C^{(1)}$. Moreover, the relative boundary K may be piecewise of class $C^{(1)}$ in the sense explained at the end of Section 8.4. For instance, if M is an r-plane and A an r-simplex contained in M, then the boundary of A relative to M is piecewise of class $C^{(1)}$.

Problems

1. Let $\omega = yz\,dx + x\,dy + dz$. Let γ be the unit circle in the xy-plane, oriented in the counterclockwise direction. Calculate $\int_\gamma \omega$ and $\int_{A^\circ} d\omega$ and verify that they are equal, where the orientation $\mathbf{o}$ is chosen so that $\partial A^\circ = \gamma$ and:
(a) A is the disk $x^2 + y^2 < 1$ in the xy-plane.
(b) $A = \{(x, y, 1 - x^2 - y^2) : x^2 + y^2 < 1\}$.

2. Let $\omega = z \exp(-y)dx + z\,dy + y\,dz$. Evaluate $\int_{A^\circ} d\omega$ when A is:
(a) The ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ oriented by the exterior normal.
(b) The square with vertices $\mathbf{0}$, $\mathbf{e}_1 + \mathbf{e}_2$, $\sqrt{2}\mathbf{e}_3$, $\mathbf{e}_1 + \mathbf{e}_2 + \sqrt{2}\mathbf{e}_3$, oriented so that $o^{23}(\mathbf{x}) > 0$.
(c) The paraboloid $y = x^2 + z^2$ oriented so that $o^{31}(\mathbf{x}) > 0$.

3. In Example 1, show that $\int_{\partial B^+} \omega^\sharp = \int_\lambda \omega^\sharp$.

4. Let $M = \operatorname{fr} D$, where D is a regular domain in E^n. Show that $\int_M (*d\omega) \cdot \mathbf{v}\, dV_{n-1} = \iota$ if ω is an $(n-2)$-form of class $C^{(1)}$ on M.

5. Prove the following:

$$d\omega(\mathbf{x}_0) \cdot \alpha_0 = \lim_{\operatorname{diam} A \to 0} [V_r(A)]^{-1} \int_{\partial A^{\mathbf{x}_0}} \omega,$$

where $\mathbf{x}_0 \in A$ and A lies in an r-plane Π oriented by α_0.

6. Let α be the r-vector of an r-simplex S_0 and $\beta_0, \beta_1, \ldots, \beta_r$ the $(r-1)$-vectors of its oriented faces (Problem 12, Section 7.5). Show that

$$d\omega(\mathbf{x}_0) \cdot \alpha = \sum_{i=0}^{r} (-1)^i \omega(\mathbf{x}_0) \cdot \beta_i.$$

[*Hint*: Consider simplexes S similar to S_0 and containing $\mathbf{x}_0$. Apply Problem 5 with $A = S$.]

7. Let $r = 2$, and A, B, K, ω as in the discussion preceding (8.20).
(a) Show that K is a 1-manifold.
(b) Show that $\int_{\partial A^\circ} \omega = \int_{\partial B^+} \omega^\sharp$.

8. Show that condition (3) in the definition of regular domain A is equivalent to the following property. Let S be any coordinate patch on M such that $S \cap K$ is not empty. Let $B = \mathbf{g}^{-1}(A)$, where $\mathbf{g}$ is regular from an open set Δ onto S. Then for any $\mathbf{t}_0 \in (\operatorname{fr} B) \cap \Delta$ there exist a neighborhood U of $\mathbf{t}_0$ and Φ of class $C^{(1)}$ on U such that $\operatorname{grad} \Phi(\mathbf{t}) \neq \mathbf{0}$ for every $\mathbf{t} \in U$ and $(\operatorname{fr} B) \cap U = \{\mathbf{t} \in U : \Phi(\mathbf{t}) = 0\}$, $B \cap U = \{\mathbf{t} \in U : \Phi(\mathbf{t}) < 0\}$. (cf. Section 8.4.)

8.9 Regular transformations of submanifolds

Throughout the present chapter we consider regular transformations from $\Delta \subset E^r$ into an r-manifold M, but there remains the following question. Given a transformation $\mathbf{g}$ from Δ into E^n, under what conditions is $\mathbf{g}(\Delta)$ an r-manifold if we no longer assume that $\mathbf{g}(\Delta)$ is contained in an r-manifold M?

Proposition 8.11. *Let* $\mathbf{g}$ *be of class* $C^{(1)}$ *from an open set* $\Delta \subset E^r$ *into* E^n. *If* $D\mathbf{g}(\mathbf{t}_0)$ *has rank* r, *then there exists a neighborhood* Ω *of* $\mathbf{t}_0$ *such that* $\mathbf{g}(\Omega)$ *is an* r*-manifold.*

PROOF. Since $D\mathbf{g}(\mathbf{t}_0)$ has rank r, there exists $\lambda = (i_1, \ldots, i_r)$ such that $J\mathbf{g}^\lambda(\mathbf{t}_0) \neq 0$, where $\mathbf{g}^\lambda = (g^{i_1}, \ldots, g^{i_r})$. By the inverse function theorem, $\mathbf{t}_0$ has a neighborhood Ω such that $\mathbf{g}^\lambda|\Omega$ is a regular flat transformation and $R_0 = \mathbf{g}^\lambda(\Omega)$ is open. Then $\mathbf{g}|\Omega = \mathbf{G} \circ (\mathbf{g}^\lambda|\Omega)$, with $\mathbf{G}$ as in Figure 8.3 (Section 8.1), and $\mathbf{g}(\Omega) = \mathbf{G}(R_0)$ is an r-manifold. □

Definition. A transformation $\mathbf{g}$ from an open set Δ into E^n is *open* if $\mathbf{g}(\Delta_1)$ is a relatively open subset of $\mathbf{g}(\Delta)$ for any open set $\Delta_1 \subset \Delta$.

Theorem 8.2. *Let* $\mathbf{g}$ *satisfy conditions* (1), (2), *and* (3) *in the definition of regular transformation* (*Section* 8.1). *Moreover, let* $\mathbf{g}$ *be open. Then* $\mathbf{g}(\Delta)$ *is an* r*-manifold.*

PROOF. By Proposition 8.11, each $\mathbf{t}_0 \in \Delta$ has a neighborhood Ω such that $\mathbf{g}(\Omega)$ is an r-manifold. Since $\mathbf{g}$ is an open transformation, $\mathbf{g}(\Omega)$ is a relatively open subset of $\mathbf{g}(\Delta)$. Hence $\mathbf{g}(\Delta)$ is an r-manifold. □

Corollary. *If* $\mathbf{g}$ *satisfies* (1), (2), *and* (3) *and* $\mathbf{g}^{-1}$ *is continuous, then* $\mathbf{g}(\Delta)$ *is an* r*-manifold.*

PROOF. Let $\Delta_1 \subset \Delta$ be open. Then $\mathbf{g}(\Delta_1) = (\mathbf{g}^{-1})^{-1}(\Delta_1)$ is open relative to $\mathbf{g}(\Delta)$ by Theorem 2.6. Therefore, $\mathbf{g}$ is an open transformation. □

Problem 3, Section 8.1, shows that $\mathbf{g}(\Delta)$ need not be an r-manifold if merely (1), (2), and (3) are assumed.

In this section we give conditions under which the image of a manifold N under a transformation $\mathbf{g}$ is a manifold, and under which the inverse image of a manifold is a manifold. In particular, we are concerned with manifolds of lower dimension s, where $1 \leq s < r$ and the domain of $\mathbf{g}$ is an open subset of E^r. Many of the results have already been quoted in earlier sections, especially Section 8.8, when $s = r - 1$.

In showing that sets are manifolds, we use the following observation, which is an immediate consequence of the definition of manifold in Section

4.7. A set $P \subset E^n$ is an s-manifold if any $\mathbf{x}_0 \in P$ is contained in a set Q open relative to P, such that Q is an s-manifold.

Theorem 8.3. *Let* $\mathbf{g}$ *be regular from an open set* $\Delta \subset E^r$ *into an* r*-manifold* M. *Let* $N \subset \Delta$ *and* $P = \mathbf{g}(N)$. *Let* $1 \leq s < r$. *Then*

(a) P *is an* s*-manifold if and only if* N *is an* s*-manifold.*
(b) *If* N *is an* s*-manifold and* $\mathbf{x} = \mathbf{g}(\mathbf{t})$, $\mathbf{t} \in N$, *the tangent space* $T_P(\mathbf{x})$ *is the image under* $D\mathbf{g}(\mathbf{t})$ *of* $T_N(\mathbf{t})$.

PROOF. We divide the proof into three parts.

(i) Suppose that N is an s-manifold. Let $\mathbf{x}_0 \in P$, $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$ and let Σ be a coordinate patch on N containing $\mathbf{t}_0$. There exists an open set $\Gamma \subset E^s$ and a regular transformation $\boldsymbol{\psi}$ from Γ onto Σ. Let $\mathbf{f} = \mathbf{g} \circ \boldsymbol{\psi}$. Since

$$D\mathbf{f}(\boldsymbol{\tau}) = D\mathbf{g}(\mathbf{t}) \circ D\boldsymbol{\psi}(\boldsymbol{\tau}), \quad \text{if } \mathbf{t} = \boldsymbol{\psi}(\boldsymbol{\tau}),$$

and $D\mathbf{g}(\mathbf{t})$, $D\boldsymbol{\psi}(\boldsymbol{\tau})$ have maximum rank, $D\mathbf{f}(\boldsymbol{\tau})$ also has maximum rank s. By Propositions 8.3 and 8.4, $\mathbf{g}(\Delta)$ is a relatively open subset of M, and $\mathbf{g}^{-1}$ is continuous. Similarly, Σ is a relatively open subset of N, and $\boldsymbol{\psi}^{-1}$ is continuous. Let $P_1 = \mathbf{g}(\Sigma)$. Then P_1 is open relative to $P = \mathbf{g}(N)$, since $\mathbf{g}$ is a homeomorphism. Moreover, $\mathbf{f}^{-1} = \boldsymbol{\psi}^{-1} \circ (\mathbf{g}^{-1} | P_1)$ is continuous. By the corollary above, $\mathbf{f}(\Gamma)$ is an s-manifold. But $P_1 = \mathbf{f}(\Gamma)$. Since any $\mathbf{x}_0 \in P$ lies in such a relatively open set P_1, P is an s-manifold.

(ii) Continuing the notation in (i), let $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_s$ be the partial derivatives of $\boldsymbol{\psi}$. Given $\mathbf{t} = \boldsymbol{\psi}(\boldsymbol{\tau})$, let

$$\mathbf{k}_l = \boldsymbol{\psi}_l(\boldsymbol{\tau}), \qquad \mathbf{h}_l = D\mathbf{g}(\mathbf{t})(\mathbf{k}_l), \qquad l = 1, \ldots, s.$$

Since $\mathbf{f} = \mathbf{g} \circ \boldsymbol{\psi}$, the composite function theorem implies that $\mathbf{h}_l = \mathbf{f}_l(\boldsymbol{\tau})$. By Proposition 8.1, the vectors $\mathbf{k}_1, \ldots, \mathbf{k}_s$ form a basis for $T_N(\mathbf{t})$ and $\mathbf{h}_1, \ldots, \mathbf{h}_s$ a basis for $T_P(\mathbf{x})$, if $\mathbf{x} = \mathbf{g}(\mathbf{t})$. This proves (b).

(iii) It remains to show that N is an s-manifold if P is an s-manifold. Let $\mathbf{t}_0 \in N$, $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$. Choose $\lambda = (i_1, \ldots, i_r)$ and a coordinate patch S on M as in the proof of Proposition 8.3. Let Q be a coordinate patch on P containing $\mathbf{x}_0$, with $Q \subset S$, and $\boldsymbol{\chi}$ a regular transformation from an open set $B \subset E^s$ onto Q. Let $\boldsymbol{\chi}^\lambda = \mathbf{X}^\lambda \circ \boldsymbol{\chi}$; in other words, $\boldsymbol{\chi}^\lambda = (\chi^{i_1}, \ldots, \chi^{i_r})$. Let $T = \mathbf{X}^\lambda(Q) = \boldsymbol{\chi}^\lambda(B)$. Now $\boldsymbol{\chi}^\lambda$ is $C^{(1)}$ and univalent. The differential $D\boldsymbol{\chi}^\lambda(\boldsymbol{\tau})$ has maximum rank s (Problem 1). Since $\boldsymbol{\chi}^{-1}$ and $\mathbf{G} = (\mathbf{X}^\lambda | S)^{-1}$ are continuous, $(\boldsymbol{\chi}^\lambda)^{-1}$ is continuous. By the corollary, T is an s-manifold. Let Ω be a neighborhood of $\mathbf{t}_0$ such that $\mathbf{g}^\lambda | \Omega$ is regular, from Ω onto an open set $R_1 = \mathbf{g}^\lambda(\Omega)$; see the proof of Proposition 8.3. Let $C = \{\mathbf{x} : \mathbf{x}^\lambda \in R_1\}$. Then C is open; by choosing Ω small enough, we may assume $P \cap C \cap S = Q \cap C$. The set $N_1 = \Omega \cap N \cap \mathbf{g}^{-1}(C \cap S)$ is open relative to N. Moreover, $N_1 = (\mathbf{g}^\lambda | \Omega)^{-1}(T \cap R_1)$. Since $(\mathbf{g}^\lambda | \Omega)^{-1}$ is regular and $T \cap R_1$ is an s-manifold, the corollary implies that N_1 is an s-manifold. Since any $\mathbf{t}_0 \in N$ lies in such a set N_1, N is an s-manifold. □

If α is an orientation for N, then $\mathbf{g}$ induces an orientation β on P as follows. Let $\mathbf{x} = \mathbf{g}(\mathbf{t})$, $\mathbf{t} \in N$, and let $\beta_1(\mathbf{x}) = \mathbf{L}_s[\alpha(\mathbf{t})]$, where $\mathbf{L}_s$ is the linear transformation of s-vectors induced by $\mathbf{L} = D\mathbf{g}(\mathbf{t})$. We take $\beta(\mathbf{x}) = |\beta_1(\mathbf{x})|^{-1}\beta_1(\mathbf{x})$.

Proposition 8.12. *Let* $\mathbf{g}$, N, *and* P *be as in Theorem* 8.3. *Let* α *be an orientation for the* s*-manifold* N *and* β *the induced orientation on* P. *Let* Y *be an* s*-measurable subset of* P, *and* ω *an* s*-form. Then*

$$\int_{Y^\beta} \omega = \int_{Z^\alpha} \omega^\#, \qquad Y = \mathbf{g}(Z), \tag{8.23}$$

provided either integral exists.

PROOF. First let us consider the case when $Z \subset \Sigma$, with Σ a coordinate patch on N. Then $\Sigma = \psi(\Gamma)$, with Γ an open subset of E^s and ψ regular from Γ onto Σ. Let $\mathbf{f} = \mathbf{g} \circ \psi$, as in the proof of Theorem 8.3. Then $\mathbf{f}$ is regular from Γ onto $\mathbf{g}(\Sigma)$. Let us suppose that α is the orientation on Σ induced by ψ from the positive orientation on E^s. Then β is the orientation on $\mathbf{g}(\Sigma)$ induced by $\mathbf{f}$ from the positive orientation on E^s. Let us write $\omega_\mathbf{g}^\#$ to denote the dependence on $\mathbf{g}$ of $\omega^\#$. Then $\omega_\mathbf{f}^\# = (\omega_\mathbf{g}^\#)_\psi^\#$. Let $Z = \psi(W)$. By Proposition 8.9,

$$\int_{Y^\beta} \omega = \int_{W^+} \omega_\mathbf{f}^\# = \int_{W^+} (\omega_\mathbf{g}^\#)_\psi^\# = \int_{Z^\alpha} \omega_\mathbf{g}^\#.$$

If α is induced by Σ from the negative orientation, then we replace W^+ by W^-. This proves (8.23) if Z is contained in a coordinate patch.

The general case can be reduced to this one by introducing a partition of unity. One requires that each function π_k of the partition of unity has support in an open set U_k such that $\mathbf{g}^{-1}(U_k \cap P)$ is contained in some coordinate patch on N. □

PROBLEMS

1. Let $\chi^\lambda = \mathbf{X}^\lambda \circ \chi$ as in Part (iii) of the proof of Theorem 8.3. Show that $D\chi^\lambda(\tau)$ has rank s for any $\tau \in B$. *Hint.* $[D\mathbf{G}(\mathbf{x}^\lambda)]^{-1} = D\mathbf{X}^\lambda(\mathbf{x})|\,T_s(\mathbf{x})$.

2. Let Q and M be r-manifolds, with $Q \subset E^m$, $M \subset E^n$. Let $\mathbf{g}$ be a transformation of class $C^{(1)}$ on an open set containing Q, with $\mathbf{g}(Q) \subset M$. Call $\mathbf{g}|Q$ regular if $\mathbf{g}|Q$ is univalent and $D\mathbf{g}(\mathbf{t})|\,T_Q(\mathbf{t})$ has rank r for each $\mathbf{t} \in Q$. Let $N \subset Q$ and $P = \mathbf{g}(N)$. Prove that statements (a) and (b) in Theorem 8.3 remain true provided $\mathbf{g}|Q$ is regular.

8.10 Closed and exact differential forms

Any exact differential form $\omega = d\eta$ is closed, provided η is of class $C^{(2)}$. This is a consequence of the formula in Section 7.4 $d(d\eta) = \mathbf{0}$. Whether, conversely, every closed form ω is exact depends on the topological nature of the domain D of ω. In this section we give two sufficient conditions that every closed r-form with domain D be exact. The first is that D be simply connected and

applies when $r = 1$. The second is that D be star-shaped and applies for any degree r.

The results in this section do not depend on the rest of Chapter 8, except for the use of Green's theorem in the easy special case of a rectangle.

Homotopies

Let $\mathbf{f}$ and $\mathbf{g}$ be transformations of class $C^{(2)}$ from a set $B \subset E^m$ into a set $A \subset E^n$. We are interested in whether it is possible to smoothly interpolate in A between $\mathbf{f}$ and $\mathbf{g}$. If this is possible then $\mathbf{f}$ and $\mathbf{g}$ are called *homotopic in A*. To state this more precisely, let us consider the subset $[0, 1] \times B$ of E^{m+1}.

Definition. If there is a transformation $\mathbf{H}$ of class $C^{(2)}$ on $[0, 1] \times B$ such that $\mathbf{H}(s, \mathbf{t}) \in A$ for every $(s, \mathbf{t}) \in [0, 1] \times B$ and $\mathbf{H}(0, \mathbf{t}) = \mathbf{f}(\mathbf{t})$, $\mathbf{H}(1, \mathbf{t}) = \mathbf{g}(\mathbf{t})$ for every $\mathbf{t} \in B$, then $\mathbf{f}$ and $\mathbf{g}$ are *homotopic in A*.

In the usual definition of homotopy in topology, $\mathbf{H}$ is required to be merely continuous. What we call homotopy is then called a *homotopy of class* $C^{(2)}$.

EXAMPLE 1. Let A be convex. Then we may take

$$\mathbf{H}(s, t) = s\mathbf{g}(\mathbf{t}) + (1 - s)\mathbf{f}(\mathbf{t}).$$

Therefore, any two transformations $\mathbf{f}$ and $\mathbf{g}$ of class $C^{(2)}$ with values in a convex set A are homotopic in A. In particular, this is true when $A = E^n$.

To define simple connectedness one may take B to be a circle. However, instead of a circle it is more convenient to let B be an interval $[a, b]$ with the endpoints identified. Let $\mathbf{f}$ and $\mathbf{g}$ be transformations from $[a, b]$ into A such that $\mathbf{f}(a) = \mathbf{f}(b)$ and $\mathbf{g}(a) = \mathbf{g}(b)$. Then $\mathbf{f}$ and $\mathbf{g}$ are *strictly homotopic in A* if the homotopy $\mathbf{H}$ in the definition above can be chosen so that $\mathbf{H}(s, a) = \mathbf{H}(s, b)$ for every $s \in [0, 1]$.

If $\partial\mathbf{H}/\partial t \neq 0$, then for each s the transformation $\mathbf{H}(s, \ )$ represents on $[a, b]$ a closed curve γ_s of class $C^{(2)}$ in the sense of Section 6.2. Intuitively, one may regard a homotopy as a smooth interpolation by the curves γ_s between the curve γ_0 represented by $\mathbf{f}$ and the curve γ_1 represented by $\mathbf{g}$. However, for technical reasons it is disadvantageous to include the conditions $\partial\mathbf{H}/\partial t \neq 0$ in the definition of homotopy.

Definition. If $\mathbf{g}$ is strictly homotopic in A to a constant transformation $\mathbf{f}$, then $\mathbf{g}$ is *null homotopic* in A.

If $\mathbf{f}(t) = \mathbf{x}_0$ for every $t \in [a, b]$, then one should think intuitively that γ_s shrinks to the point $\mathbf{x}_0$ as $s \to 0^+$. When A is an open subset of E^2 this is possible roughly speaking provided γ_s does not loop around any holes which may be present in A. In Figure 8.16, A has two holes and the curves γ_s in the figure are not null homotopic in A.

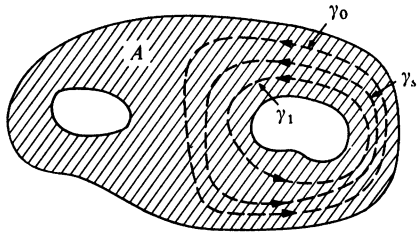


Figure 8.16

Let D be an open set, and ω a 1-form with domain D. Let us set

$$\langle \mathbf{g}, \omega \rangle = \int_a^b \omega[\mathbf{g}(t)] \cdot \mathbf{g}'(t) dt.$$

In case $\mathbf{g}'(t) \neq \mathbf{0}$, $\langle \mathbf{g}, \omega \rangle$ is just another notation for the line integral of ω along the curve represented by $\mathbf{g}$.

Proposition 8.13. *Let ω be closed. If $\mathbf{f}$ and $\mathbf{g}$ are strictly homotopic in D, then* $\langle \mathbf{f}, \omega \rangle = \langle \mathbf{g}, \omega \rangle$.

PROOF. Let $\omega^\#$ be the 1-form on the rectangle $R = [0, 1] \times [a, b]$ induced by the transformation $\mathbf{H}$. Since $d\omega = \mathbf{0}$, $d\omega^\# = (d\omega)^\# = \mathbf{0}$. By Green's theorem,

$$\int_{\partial R^+} \omega^\# = \int_{R^+} d\omega^\# = 0.$$

The integral over ∂R^+ is the sum of the integrals over the four segments $\lambda_1, \ldots, \lambda_4$ indicated in Figure 8.17. Now

$$\omega^\# = \sum_{i=1}^n \omega_i \circ \mathbf{H}(dx^i)^\# = \sum_{i=1}^n \omega_i \circ \mathbf{H}\left(\frac{\partial H^i}{\partial s} ds + \frac{\partial H^i}{\partial t} dt\right),$$

$$\int_{\lambda_2} \omega^\# = \sum_{i=1}^n \int_a^b \omega_i \circ \mathbf{H} \frac{\partial H^i}{\partial t} dt,$$

$\mathbf{H}$ and $\partial H^i/\partial t$ being evaluated at $(1, t)$. Since $\mathbf{H}(1, t) = \mathbf{g}(t)$, the right-hand side is just $\langle \mathbf{g}, \omega \rangle$. Similarly, since $\mathbf{H}(0, t) = \mathbf{f}(t)$,

$$\int_{\lambda_4} \omega^\# = -\langle \mathbf{f}, \omega \rangle.$$

Since $\mathbf{H}(s, a) = \mathbf{H}(s, b)$,

$$\int_{\lambda_1} \omega^\# = -\int_{\lambda_3} \omega^\#. \qquad \square$$

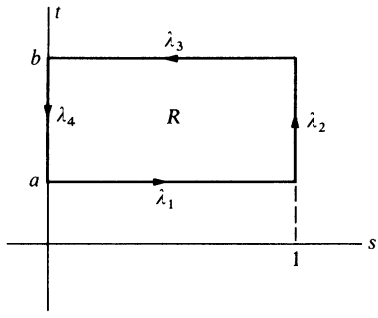


Figure 8.17

EXAMPLE 2. Let $n = 2$ and let D be the plane with $(0, 0)$ removed. Let $\omega = (x\,dy - y\,dx)/(x^2 + y^2)$. Formally, $\omega = d\Theta$, where $\Theta(x, y)$ is the angle from the positive x-axis to (x, y), $0 < \Theta(x, y) < 2\pi$. However, Θ is defined only in the plane with a slit removed even though ω is defined and of class $C^{(\infty)}$ in D. For each integer $m \neq 0$ let $\mathbf{g}_m(t) = (\cos mt)\mathbf{e}_1 + (\sin mt)\mathbf{e}_2$, $0 \leq t \leq 2\pi$. Then $\langle \mathbf{g}_m, \omega \rangle = 2m\pi$, which shows that $\mathbf{g}_m$ and $\mathbf{g}_l$ are not strictly homotopic in D when $m \neq l$. The transformation $\mathbf{g}_m$ represents the unit circle traversed $|m|$ times, counterclockwise if $m > 0$ and clockwise if $m < 0$.

Proposition 8.13 has the following corollaries.

Corollary 1. *If* $\mathbf{g}$ *is null homotopic in* D, *then* $\langle \mathbf{g}, \omega \rangle = 0$.

PROOF. If $\mathbf{f}$ is constant, then $\langle \mathbf{f}, \omega \rangle = 0$. □

Definition. An open set D is *simply connected* if every transformation $\mathbf{g}$ of class $C^{(2)}$ from an interval $[a, b]$ into D, satisfying $\mathbf{g}(a) = \mathbf{g}(b)$, is null homotopic in D.

Roughly speaking, D is simply connected if every closed curve in D can be shrunk in D to a point. When $D \subset E^2$ this amounts to saying that D "has no holes." Removal of a single point, as in Example 2, must be counted as introducing a hole.

If $D = \{\mathbf{x} \in E^3 : |\mathbf{x}| > 1\}$, then D is simply connected, yet D has a "hole."

Theorem 8.4. *If* D *is a simply connected open subset of* E^n, *then every closed* 1*-form with domain* D *is exact.*

PROOF. By Theorem 6.1, it suffices to show that $\int_\gamma \omega = 0$ for every piecewise smooth closed curve γ lying in D. Let $\mathbf{g}$ be a representation of such a curve γ on $[0, 1]$, such that $\mathbf{g}$ is piecewise of class $C^{(1)}$. There is a sequence $\mathbf{g}_1, \mathbf{g}_2, \ldots$ of transformations of class $C^{(\infty)}$ on $[0, 1]$ such that: (1) $\mathbf{g}_m(0) = \mathbf{g}_m(1)$ for $m = 1, 2, \ldots$; (2) $\mathbf{g}_m(t) \to \mathbf{g}(t)$ for every $t \in [0, 1]$, and $\mathbf{g}'_m(t) \to \mathbf{g}'(t)$ except at the (finitely many) points of discontinuity of $\mathbf{g}'$, as $m \to \infty$; (3) $|\mathbf{g}_m(t)|$ and $|\mathbf{g}'_m(t)|$

are bounded by some number C. Such a sequence can be found by a standard smoothing technique (Problem 4). By Lebesgue's dominated convergence theorem, $\langle \mathbf{g}_m, \boldsymbol{\omega} \rangle \to \langle \mathbf{g}, \boldsymbol{\omega} \rangle$ as $m \to \infty$. By Problem 7, Section 5.11, $\mathbf{g}_m$ tends to $\mathbf{g}$ uniformly on $[0, 1]$. Since $\mathbf{g}([0, 1])$ is a compact subset of the open set D, there exists m_1 such that $\mathbf{g}_m([0, 1]) \subset D$ for all $m \geq m_1$. By Corollary 1, $\langle \mathbf{g}_m, \boldsymbol{\omega} \rangle = 0$ for each $m \geq m_1$. Therefore $\langle \mathbf{g}, \boldsymbol{\omega} \rangle = \int_\gamma \boldsymbol{\omega} = 0$. □

Let us turn to the question of finding a condition on D that ensures that any closed form of arbitrary degree r is exact. For this purpose, let B be an open subset of E^m. Let us introduce an operation which changes any r-form $\boldsymbol{\eta}$ of class $C^{(1)}$ on $[0, 1] \times B$ into an $(r-1)$-form of class $C^{(1)}$ on B. The latter form is denoted by $\int_0^1 \boldsymbol{\eta}$. If $r = 1$, then $\boldsymbol{\eta} = f\,ds + \boldsymbol{\eta}^1$, where $\boldsymbol{\eta}^1$ involves the differentials $dt^1, \ldots, dt^m$. In this case $\int_0^1 \boldsymbol{\eta} = \int_0^1 f(s, \ )ds$, which is of class $C^{(1)}$ on B. Next, if $\boldsymbol{\eta} = ds \wedge \boldsymbol{\theta} = \sum_{[\mu]} \theta_\mu\, ds \wedge dt^{j_1} \wedge \cdots \wedge dt^{j_{r-1}}$, then we set

$$\int_0^1 \boldsymbol{\eta} = \sum_{[\mu]} \left(\int_0^1 \theta_\mu\, ds \right) dt^{j_1} \wedge \cdots \wedge dt^{j_{r-1}}, \tag{8.24}$$

Finally, any r-form $\boldsymbol{\eta}$ on $[0, 1] \times B$ can be written $\boldsymbol{\eta} = ds \wedge \boldsymbol{\theta} + \boldsymbol{\eta}^1$, where $\boldsymbol{\eta}^1$ involves only the differentials $dt^1, \ldots, dt^m$:

$$\boldsymbol{\eta}^1 = \sum_{[\lambda]} \eta_\lambda^1\, dt^{i_1} \wedge \cdots \wedge dt^{i_r}.$$

We set $\int_0^1 \boldsymbol{\eta} = \int_0^1 ds \wedge \boldsymbol{\theta}$. Using the rules for exterior differentiation, we find that

$$d\boldsymbol{\eta} = d(ds \wedge \boldsymbol{\theta}) + d\boldsymbol{\eta}^1 = -ds \wedge d'\boldsymbol{\theta} + ds \wedge \frac{\partial \boldsymbol{\eta}^1}{\partial s} + d'\boldsymbol{\eta}^1,$$

where d' denotes the differential with respect to $\mathbf{t}$ of a form on $[0, 1] \times B$ and the components of the r-form $\partial \boldsymbol{\eta}^1 / \partial s$ are the partial derivatives $\partial \eta_\lambda^1 / \partial s$. Therefore

$$\begin{aligned} \int_0^1 d\boldsymbol{\eta} &= -\int_0^1 ds \wedge d'\boldsymbol{\theta} + \int_0^1 ds \wedge \frac{\partial \boldsymbol{\eta}^1}{\partial s} \\ &= -\int_0^1 ds \wedge d'\boldsymbol{\theta} + \boldsymbol{\eta}^1(1) - \boldsymbol{\eta}^1(0), \end{aligned} \tag{*}$$

where $\boldsymbol{\eta}^1(s)$ is the r-form on B with coefficients $\eta_\lambda^1(s, \ )$. Differentiating under the integral sign,

$$\frac{\partial}{\partial t^j} \int_0^1 f\, ds = \int_0^1 \frac{\partial f}{\partial t^j} ds, \qquad j = 1, \ldots, m,$$

provided f is of class $C^{(1)}$. Hence

$$d \int_0^1 f\, ds = \sum_{j=1}^m \left(\frac{\partial}{\partial t^j} \int_0^1 f\, ds \right) dt^j = \int_0^1 ds \wedge d'f.$$

Applying this in (8.24), if η is of class $C^{(1)}$ we get

$$(**) \qquad \begin{aligned} d\int_0^1 \eta &= \sum_{[\mu]} \int_0^1 ds \wedge d'\theta_\mu \wedge dt^{j_1} \wedge \cdots \wedge dt^{j_{r-1}} \\ &= \int_0^1 ds \wedge \left(\sum_{[\mu]} d'\theta_\mu \wedge dt^{j_1} \wedge \cdots \wedge dt^{j_{r-1}} \right) = \int_0^1 ds \wedge d'\boldsymbol{\theta}. \end{aligned}$$

From (*) and (**) we get

$$\int_0^1 d\eta + d\int_0^1 \eta = \eta^1(1) - \eta^1(0). \tag{8.25}$$

Now let ω be an r-form of class $C^{(1)}$ on A. Let $\mathbf{H}$ be a homotopy between transformations $\mathbf{f}$ and $\mathbf{g}$, and let $\omega_{\mathbf{f}}^{\#}$, $\omega_{\mathbf{g}}^{\#}$, $\omega_{\mathbf{H}}^{\#}$ denote the r-forms induced respectively by $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{H}$. Let $\eta = \omega_{\mathbf{H}}^{\#}$. Then $\eta^1(1) = \omega_{\mathbf{g}}^{\#}$ and $\eta^1(0) = \omega_{\mathbf{f}}^{\#}$. Therefore

$$\int_0^1 d\omega_{\mathbf{H}}^{\#} + d\int_0^1 \omega_{\mathbf{H}}^{\#} = \omega_{\mathbf{g}}^{\#} - \omega_{\mathbf{f}}^{\#}. \tag{8.26}$$

With this formula we can readily deduce a result about closed forms, called *Poincaré's lemma.*

Definition. A set A is *star-shaped* if there is a point $\mathbf{x}_0 \in A$ such that for every $\mathbf{x} \in A$ the line segment joining $\mathbf{x}_0$ and $\mathbf{x}$ is contained in A (Figure 8.18).

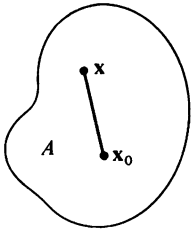


Figure 8.18

Poincaré's lemma. *Let D be a star-shaped open set and let $1 \leq r \leq n$. Then every closed r-form with domain D is exact.*

PROOF. Let $\mathbf{x}_0$ be a point with respect to which D is star-shaped. Let $\mathbf{f}(\mathbf{x}) = \mathbf{x}_0$, $\mathbf{g}(\mathbf{x}) = \mathbf{x}$, $\mathbf{H}(s, \mathbf{x}) = \mathbf{x}_0 + s(\mathbf{x} - \mathbf{x}_0)$, $B = D$. (This homotopy merely shrinks D radially to the point $\mathbf{x}_0$.) Then $\omega_{\mathbf{g}}^{\#} = \omega$; and since $r > 0$ and $df^i = \mathbf{0}$, $\omega_{\mathbf{f}}^{\#} = \mathbf{0}$. Since ω is closed, $d\omega_{\mathbf{H}}^{\#} = (d\omega)_{\mathbf{H}}^{\#} = \mathbf{0}$. Let $\zeta = \int_0^1 \omega_{\mathbf{H}}^{\#}$. Then by (8.26), $d\zeta = \omega$. □

Note. Poincaré's lemma gives only a sufficient condition on D that every closed form be exact. A necessary and sufficient condition can be obtained from DeRham's theorem [21, Chapter IV; 24, Chapter IV].

Let us state without proof the following version of the theorem. Let $\mathscr{Z}^r(D)$ denote the set of all closed r-forms of class $C^{(\infty)}$ on D. If ω and ζ are

closed, then $\omega + \zeta$ is closed and $c\omega$ is closed for any scalar c. Thus $\mathscr{Z}^r(D)$ is a vector space over E^1. Similarly, let $\mathscr{E}^r(D)$ denote the vector space consisting of all exact r-forms of the type $\omega = d\zeta$ where ζ is of class $C^{(\infty)}$ on D. Then $\mathscr{E}^r(D) \subset \mathscr{Z}^r(D)$. According to DeRham's theorem, the quotient vector space $\mathscr{H}^r(D) = \mathscr{Z}^r(D)/\mathscr{E}^r(D)$ is isomorphic to the r-dimensional cohomology group of D with real coefficients. (The homology and cohomology groups of a space are defined in algebraic topology. They contain a great deal of topological information about the space.) In particular, every closed r-form is exact if and only if $\mathscr{H}^r(D) = 0$.

PROBLEMS

1. Let D be the solid torus obtained by rotating the circular disk $(y - a)^2 + z^2 < b^2$, $0 < b < a$, about the z-axis. Let γ be the circular path traversed by the center of the disk. Show that $\int_\gamma (x\,dy - y\,dx)/(x^2 + y^2) \neq 0$. Hence by Corollary 1, γ is not null homotopic in D.

2. Let S be the sphere $x^2 + y^2 + z^2 = a^2$, oriented by the unit exterior normal. Let

$$\omega = \rho^{-3}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy),$$
$$\rho^2 = x^2 + y^2 + z^2,$$

the domain of ω being $E^3 - \{\mathbf{0}\}$. Show that:
(a) ω is closed.
(b) $\int_{S^\circ} \omega = 4\pi$. [*Hint*: Find $*\omega(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})$, where $\mathbf{v}(\mathbf{x})$ is the exterior normal.]
(c) $E^3 - \{\mathbf{0}\}$ is simply connected.

3. Let $\tilde{D}$ be star-shaped and let $D = \mathbf{g}(\tilde{D})$, where $\mathbf{g}$ is a regular flat transformation of class $C^{(2)}$. Show that every closed form with domain D is exact.

4. For $m = 1, 2, \ldots$ let h_m be a function of class $C^{(\infty)}$ on E^1 such that $h_m \geq 0$, $\int_{-\infty}^{\infty} h_m\,dx = 1$, $h_m(x) = 0$ whenever $|x| \geq 1/m$. [For instance, we may take $h_m(x) = mh(mx)$, where h is as in Section 3.4.] Let ψ be a piecewise continuous function on E^1 which is periodic of period 1. Let $\psi_m(x) = \int_{-\infty}^{\infty} \psi(y)h_m(x - y)dy = \int_{-\infty}^{\infty} \psi(x + z)h_m(z)dz$. Show that:
(a) If $|\psi(x)| \leq C$ for every x, then $|\psi_m(\mathbf{x})| \leq C$ for every x and $m = 1, 2, \ldots$.
(b) ψ_m is of class $C^{(\infty)}$ and of period 1, $m = 1, 2, \ldots$.
(c) If $\int_0^1 \psi\,dx = 0$, then $\int_0^1 \psi_m\,dx = 0$, $m = 1, 2, \ldots$.
(d) At each point x_0 of continuity of ψ, $\psi_m(x_0) \to \psi(x_0)$ as $m \to \infty$. [*Hint*: $\psi_m(x_0) - \psi(x_0) = \int_{-1/m}^{1/m} [\psi(x_0 + z) - \psi(x_0)]h_m(z)dz$.]

[*Note*: In the proof of Theorem 8.4, let ψ be a periodic extension of $g^{i'}$, and let $g_m^i(t) = g^i(0) + \int_0^t \psi_m(x)dx$, $i = 1, \ldots, n$.]

8.11 Motion of a particle

In this section we consider the motion of a particle E^3, subject to a force $\mathbf{F}$ of the type (8.28). It is shown that energy is conserved, and that angular momentum is also conserved if (8.30) holds. Kepler's laws of motion of the planets about the sun are derived from these two conservation principles.

Let $\mathbf{x} = (x, y, z)$ denote a possible position of particle. We denote time by t, and by $\mathbf{x}(t)$, the position of the particle at time t. [Note: In the notation of earlier sections the position would be denoted by $\mathbf{g}(t)$. It should be clear from the context whether $\mathbf{x}$ stands for a point in E^3 or a function of t.] Following traditional notation in mechanics, time derivatives are denoted by $\cdot$. Thus $\dot{\mathbf{x}}(t) = (d\mathbf{x}/dt)$. This is the *velocity vector*, $\mathbf{v} = \dot{\mathbf{x}}$.

Similarly, $\dot{\mathbf{v}} = \ddot{\mathbf{x}}$ is the *acceleration vector*. Let m denote the mass of the particle. Then Newton's law states that

$$m\ddot{\mathbf{x}} = \mathbf{F}, \tag{8.27}$$

where $\mathbf{F}$ is the force vector. A general model for the motion of particles assumes that $\mathbf{F} = \mathbf{F}(t, \mathbf{x}, \mathbf{v})$ depends on time, position, and velocity. However, we consider here only the case when $\mathbf{F} = \mathbf{F}(\mathbf{x})$. Moreover, we assume that

$$\mathbf{F}(\mathbf{x}) = -\operatorname{grad} U(\mathbf{x}), \tag{8.28}$$

where U is a function of class $C^{(2)}$ on an open set $D \subset E^3$. D is the set of possible positions $\mathbf{x}$ for the particle. $U(\mathbf{x})$ is called the *potential energy* of position $\mathbf{x}$. The *kinetic energy* is $\frac{1}{2}m|\mathbf{v}|^2$. The *total energy* of a particle in position $\mathbf{x}$ with velocity $\mathbf{v}$ is

$$E = \tfrac{1}{2}m|\mathbf{v}|^2 + U(\mathbf{x}). \tag{8.29}$$

Since $\dot{\mathbf{x}} = \mathbf{v}$, $\ddot{\mathbf{x}} = \dot{\mathbf{v}}$, we have

$$\begin{aligned} \dot{E} &= m\mathbf{v} \cdot \dot{\mathbf{v}} + \operatorname{grad} U \cdot \mathbf{v} \\ &= \mathbf{v} \cdot (m\dot{\mathbf{v}} - \mathbf{F}) = 0. \end{aligned}$$

Thus $E[\mathbf{x}(t)]$ is constant. In other words, *total energy is conserved during motion of the particle*. This property is not sufficient to determine the motion. To proceed further, one may seek other quantities that are conserved.

Let us now suppose that potential energy depends only on the distance r from the origin $\mathbf{0}$:

$$U(\mathbf{x}) = u(r), \qquad r = |\mathbf{x}|. \tag{8.30}$$

The vector $\mathbf{p} = m\mathbf{v}$ is called the *momentum* of the particle. The basic equation (8.27) can be rewritten as $\dot{\mathbf{p}} = \mathbf{F}$. The 2-vector $\mathbf{x} \wedge \mathbf{p}$ is called the *angular momentum* of the particle (about $\mathbf{0}$). Using the product rule, one gets

$$\frac{d}{dt}(\mathbf{x} \wedge \mathbf{p}) = \mathbf{x} \wedge \mathbf{F}(\mathbf{x}) \tag{8.31}$$

(see Problem 1). The product $\mathbf{x} \wedge \mathbf{F}(\mathbf{x})$ is called the *torque*. When (8.30) holds,

$$\mathbf{x} \wedge \mathbf{F}(\mathbf{x}) = \mathbf{x} \wedge \left(-\frac{u'(r)}{r}\mathbf{x}\right) = -\frac{u'(r)}{r}\mathbf{x} \wedge \mathbf{x} = \mathbf{0}.$$

Thus, *angular momentum is constant during motion of the particle*. This result has the following two consequences.

(1) *The particle moves in a plane P.*

In fact, if $\alpha = \mathbf{x} \wedge \mathbf{p}$, then P is the 2-dimensional vector subspace of E^3 whose 2-space is α (Section 7.5).

Let us make a rotation in E^3 such that P is the (x, y) plane E^2. From now on, we suppose that the particle moves in E^2. In terms of polar coordinates (r, θ),

$$\mathbf{x} = r[(\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2]$$

$$\mathbf{p} = m\dot{r}[(\cos\theta)\mathbf{e}_1 + (\sin\theta)\mathbf{e}_2] + mr[(-\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2]\dot{\theta}.$$

$$\mathbf{x} \wedge \mathbf{p} = mr^2\dot{\theta}\mathbf{e}_{12}.$$

Since angular momentum is conserved, $mr^2\dot{\theta} = A$, for the constant $A = |\mathbf{x} \wedge \mathbf{p}|$. We assume $A > 0$, and therefore $\dot{\theta} > 0$. The case $A < 0$ is entirely similar; see Problem 2 for the exceptional case $A = 0$. Since $\dot{\theta} > 0$, θ is an increasing function of t.

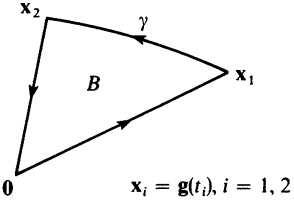


Figure 8.19

Consider two times t_1, t_2, with $0 < t_2 - t_1$ and $\theta(t_2) - \theta(t_1) \leq 2\pi$. Let γ denote the curve along which the particle moves for $t_1 \leq t \leq t_2$, and B the region shown in Figure 8.19. Then using Green's theorem

$$\frac{1}{2}\int_{t_1}^{t_2} r^2\dot{\theta}\,dt = \frac{1}{2}\int_{\gamma} r^2\,d\theta = \frac{1}{2}\int_{\partial B^+} r^2\,d\theta = \int_{B^+} r\,dr \wedge d\theta.$$

The right-hand side is the area $V_2(B)$. Since $r^2\dot{\theta} = A/m$,

$$\frac{A(t_2 - t_1)}{2m} = V_2(B). \tag{8.32}$$

This expression depends only on the difference $t_2 - t_1$. Thus we have:

(2) *The particle sweeps out area at a constant rate.*

This is Kepler's second law.

We have not yet used the fact that total energy E is constant during motion of the particle. For motion in E^2, $|\mathbf{v}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$. Since $\dot{\theta} = A(mr^2)^{-1}$, we have from (8.29) and (8.30),

$$E = \frac{m\dot{r}^2}{2} + \frac{A^2}{2mr^2} + u(r).$$

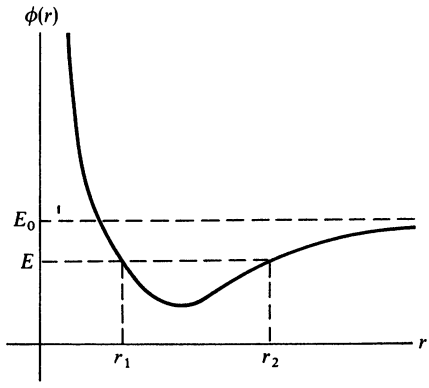


Figure 8.20

This is a first-order differential equation for $r(t)$. Let us suppose that $\phi(r) = A^2(2mr^2)^{-1} + u(r)$ has the shape indicated in Figure 8.20.

Now $(m/2)\dot{r}^2 = E - \phi(r)$, and hence $\phi(r) \leq E$. Consider the case $E < E_0$, and r_1, r_2 the two solutions of $\phi(r) = E$ (Figure 8.20). Then $r_1 \leq r(t) \leq r_2$. We seek those solutions for which $\dot{r}(t) = 0$ only for isolated values of t. By separation of variables, one can express $t = t(r)$ as a function of r on each time interval where $r(t)$ is monotone. In fact, on such an interval

$$t(r) - t(r_0) = \pm\left(\frac{m}{2}\right)^{1/2} \int_{r_0}^{r} \frac{dy}{(E - \phi(y))^{1/2}},$$

Moreover, for any r_0 in this interval, $r(t)$ is a periodic function with period $T/2$, where

$$T = 4\left(\frac{m}{2}\right)^{1/2} \int_{r_1}^{r_2} \frac{dy}{(E - \phi(y))^{1/2}}. \tag{8.33}$$

The integral converges since $\phi'(r) \neq 0$ for $r = r_1, r_2$. On each time interval where $r(t)$ is monotone, we can write $\theta(t) = \Theta[r(t)]$. Using the facts that $\dot{\theta} = \Theta'[r(t)]\dot{r}$ and $\dot{\theta} = A(mr^2)^{-1}$,

$$\Theta(r) = \pm\left(\frac{1}{2m}\right)^{1/2} \int_{r_0}^{r} \frac{A\,dy}{y^2(E - \phi(y))^{1/2}} + \Theta(r_0). \tag{8.34}$$

We leave the details of this argument to the reader (Problem 3). When $E < E_0$, the particle moves periodically in orbits of period T. If $E \geq E_0$, the particle escapes toward infinity as $t \to \infty$ (Problem 4).

Let us consider the important particular case $u(r) = -kr^{-1}$, where k is a positive constant. This corresponds to the inverse square law of gravitational attraction. In this case,

$$\phi(r) = \frac{A^2}{2mr^2} - \frac{k}{r}$$

and $E_0 = 0$. Let us take $r_0 = r_1$. By a rotation of axes, we may suppose that $\Theta(r_1) = 0$. The integral in (8.34) can be evaluated to give

$$\Theta(r) = \arccos \frac{(A/r) - (mk/A)}{(2mE + (m^2k^2/A^2))^{1/2}}, \qquad -\frac{\pi}{2} < \Theta(r) < \frac{\pi}{2}.$$

If we take

$$c = \frac{A^2}{mk}, \qquad e = \left(1 + \frac{2EA^2}{mk^2}\right)^{1/2},$$

then the particle moves on the curve

$$\frac{c}{r} = 1 + e\cos\theta. \tag{8.35}$$

This is the equation of a conic section (Kepler's first law). For $E < 0$, $e < 1$. In this case the curve is an ellipse. For $E = 0$ and $E > 0$, the curve is a parabola and hyperbola, respectively. When $E < 0$, the minor and major semiaxes of the ellipse are

$$r_1 = \frac{c}{(1 - e^2)^{1/2}} = \frac{A}{(2m|E|)^{1/2}}$$

$$r_2 = \frac{c}{1 - e^2} = \frac{k}{2|E|}.$$

By Equation (8.32), with B the interior of the ellipse and $t_2 - t_1 = T$,

$$\frac{AT}{2m} = \pi r_1 r_2 = \pi r_2 \frac{A}{(2m|E|)^{1/2}}.$$

Therefore

$$T = 2\pi(r_2)^{3/2}\left(\frac{m}{k}\right)^{1/2}.$$

This is Kepler's third law, stating that the period is proportional to the linear dimension to the power $\frac{3}{2}$.

Problems

1. (a) Let $\mathbf{f}$ and $\mathbf{g}$ be functions of class $C^{(1)}$ on $[a, b]$. Show that

$$\frac{d}{dt}(\mathbf{f} \wedge \mathbf{g}) = \frac{d\mathbf{f}}{dt} \wedge \mathbf{g} + \mathbf{f} \wedge \frac{d\mathbf{g}}{dt}.$$

(b) Derive (8.31).

2. Suppose that (8.28) and (8.30) hold and that angular momentum is $\mathbf{0}$.
(a) Show that the particle moves on a straight line through $\mathbf{0}$.
(b) Let $u(r) = r + r^{-1}$. Show that the motion is periodic for any energy level $E > 2$.

3. Let $\phi(r)$ be as in Figure 8.20, $A > 0$, and T as in (8.33). Show that:
   (a) The distance $r(t)$ from $\mathbf{0}$ oscillates between r_1 and r_2, taking time $T/4$ to go from r_1 to r_2 or vice versa.
   (b) The angle $\theta(t)$ is periodic with period T.

4. Let $E \geq E_0$ in Figure 8.20, and let r_1 be the unique r such that $\phi(r) = E$. Show that:
   (a) If $\dot{r} > 0$ initially, then $r(t)$ is increasing and $r(t) \to \infty$ as $t \to \infty$.
   (b) If $\dot{r} < 0$ initially, then the particle first decreases to r_1 and afterward increases to ∞ as in (a).

8.12 Motion of several particles

An important part of classical theoretical mechanics concerns the motion of N particles, $N \geq 2$, subject to forces depending on the positions of all of the particles. The motion may be subject to certain physical restrictions. Thus, let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ denote the positions of the particles, with $\mathbf{x}_j \in E^3$ for each $j = 1, \ldots, N$. Let us suppose that $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is restricted to belong to a manifold $M \subset E^{3N}$. Let n be the dimension of M. The equations of motion have the form

$$m_j \ddot{\mathbf{x}}_j = \mathbf{F}_j(\mathbf{x}), \qquad j = 1, \ldots, N, \tag{8.36}$$

where $m_1, \ldots, m_N$ are the masses of the particles. The force functions $\mathbf{F}_1, \ldots, \mathbf{F}_N$ cannot be arbitrary, since the particles must move in M.

Let $S \subset M$ be a coordinate patch. For simplicity we consider only motions during which $\mathbf{x}$ remains in S. We find an analog of the potential energy function U in Section 8.11, and rewrite equations (8.36) in terms of a coordinate system for S. We denote the coordinates of $\mathbf{x}$ in this system by $\mathbf{q} = (q^1, \ldots, q^n)$. Thus $\mathbf{x} = \mathbf{G}(\mathbf{q})$, where $\mathbf{G}$ is a regular transformation from an open set $\Delta \subset E^n$ into M. At time t, the coordinates of the position $\mathbf{x}(t)$ are $\mathbf{q}(t) = \mathbf{G}^{-1}[\mathbf{x}(t)]$. The position of the jth particle is $\mathbf{x}_j(t) = \mathbf{G}_j[\mathbf{q}(t)]$, where $\mathbf{G} = (\mathbf{G}_1, \ldots, \mathbf{G}_N)$. By the chain rule the velocity vectors satisfy

$$\dot{\mathbf{x}}_j = \sum_{k=1}^{n} \frac{\partial \mathbf{G}_j}{\partial q_k} \dot{q}^k, \qquad j = 1, \ldots, N.$$

Consider the following function T, defined on $\Delta \times E^n$:

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_{k,l=1}^{n} a_{kl}(\mathbf{q}) \dot{q}^k \dot{q}^l,$$

where

$$a_{kl}(\mathbf{q}) = \sum_{j=1}^{N} m_j \frac{\partial \mathbf{G}_j}{\partial q^k} \frac{\partial \mathbf{G}_j}{\partial q^l}.$$

Then

$$T[\mathbf{q}(t), \dot{\mathbf{q}}(t)] = \sum_{j=1}^{N} \frac{m_j}{2} |\dot{\mathbf{x}}_j|^2 \tag{8.37}$$

is the kinetic energy at time t. The matrix $(a_{kl}(\mathbf{q}))$ is symmetric and positive definite for each $\mathbf{q}$. For $k = 1, \ldots, n$,

$$\frac{\partial T}{\partial \dot{q}^k} = \sum_{l=1}^{n} a_{kl}\dot{q}^l = \sum_{j=1}^{N} m_j \dot{\mathbf{x}}_j \cdot \frac{\partial \mathbf{G}_j}{\partial q^k} \tag{8.38}$$

$$\frac{\partial T}{\partial q^k} = \frac{1}{2} \sum_{p,l=1}^{n} \frac{\partial a_{pl}}{\partial q^k} \dot{q}^p \dot{q}^l \tag{8.39}$$

By a short calculation (Problem 3), the equations of motion (8.36) imply (in the coordinates $\mathbf{q}$)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^k}\right) = \frac{\partial T}{\partial q^k} + \mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial q^k}, \qquad k = 1, \ldots, n, \tag{8.40}$$

where $\mathbf{F} = (\mathbf{F}_1, \ldots, \mathbf{F}_N)$ is the function with values in E^{3N} describing all of the forces.

Let us write $\mathbf{F} = \mathbf{F}^a + \mathbf{F}^c$, where $\mathbf{F}^a(\mathbf{x})$ and $\mathbf{F}^c(\mathbf{x})$ are the components of $\mathbf{F}(\mathbf{x})$ tangent to M and normal to M at $\mathbf{x}$, respectively. $\mathbf{F}^a(\mathbf{x})$ is called the *applied force* at $\mathbf{x}$, and $\mathbf{F}^c(\mathbf{x})$ the *constraining force* at $\mathbf{x}$. By Proposition 8.1, $\partial \mathbf{G}/\partial q^k$ is a tangent vector to M at $\mathbf{x} = \mathbf{G}(\mathbf{q})$. Since $\mathbf{F}^c(\mathbf{x})$ is a normal vector to M at $\mathbf{x}$, $\mathbf{F}^c \cdot (\partial \mathbf{G}/\partial q^k) = 0$. Thus, constraining forces contribute nothing to the last term of (8.40). Generalizing (8.28), let us assume that there is a potential energy function U, satisfying

$$\mathbf{F}_j^a(\mathbf{x}) = -\operatorname{grad}_j U(\mathbf{x}), \tag{8.41}$$

where grad_j denotes gradient in the variables $\mathbf{x}_j$. Let $V(\mathbf{q}) = U[\mathbf{G}(\mathbf{q})]$, and let $L = T - V$. Then Equations (8.40) become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k}\right) = \frac{\partial L}{\partial q^k}, \qquad k = 1, \ldots, n. \tag{8.42}$$

These are called *Lagrange's equations* for the motion of the particles. The total energy $E = T + V$ is constant during the motion (Problem 2).

Lagrange's equations furnish a starting point for further development of theory. Hamilton's equations furnish another useful formulation (Problem 5) [10; 17, Chapter 13].

Problems

1. A single particle moves on a 2-dimensional manifold $M \subset E^3$, subject only to a constraining force ($\mathbf{F}^a = \mathbf{0}$). Show that the principal normal vector at $\mathbf{x}(t)$ (Section 6.2, Problem 9) to the curve of motion is also a normal vector to M at $\mathbf{x}(t)$. [*Note:* Such curves are called *geodesics* of M.]

2. Show that $E[\mathbf{q}(t)]$ is constant, where $E = T + V$ is the total energy. [*Hint*: By Euler's formula (Section 3.3, Problem 8),

$$2T = \sum_{k=1}^{n} \frac{\partial T}{\partial \dot{q}^k} \dot{q}^k.\Big]$$

3. Derive (8.40), using (8.38) and (8.39).

4. Let $M = \{\mathbf{x} : \mathbf{x}_j \neq \mathbf{x}_i \text{ for } i, j = 1, \ldots, N, i \neq j\}$. Suppose that $U(\mathbf{x})$ depends only on the mutual distances between particles, namely,

$$U(\mathbf{x}) = \Psi(|\mathbf{x}_2 - \mathbf{x}_1|, |\mathbf{x}_3 - \mathbf{x}_1|, \ldots, |\mathbf{x}_n - \mathbf{x}_{n-1}|).$$

In (8.41), take $\mathbf{F} = \mathbf{F}^a$ (no constraining forces, since M is an open subset of E^{3N}, $n = 3N$). Let $\mathbf{p}_j = \frac{1}{2} m_j \dot{\mathbf{x}}_j$. Show that:
(a) $\sum_{j=1}^{N} \mathbf{p}_j(t)$ is constant.
(b) $\sum_{j=1}^{N} \mathbf{x}_j(t) \wedge \mathbf{p}_j(t)$ is constant.

5. Let $\pi = (\pi_1, \ldots, \pi_n)$ denote a covector, $\pi \in (E^n)^*$. Let

$$H(\mathbf{q}, \pi) = \min\{L(\mathbf{q}, \dot{\mathbf{q}}) + \pi \cdot \dot{\mathbf{q}} : \dot{\mathbf{q}} \in E^n\}. \tag{8.43}$$

Show that:
(a) The minimum (for fixed $\mathbf{q}$) is attained at the unique $\dot{\mathbf{q}}$ such that

$$\frac{\partial L}{\partial \dot{q}^k} = -\pi_k, \qquad k = 1, \ldots, n. \tag{8.44}$$

Let us write $\dot{\mathbf{q}} = \Phi(\mathbf{q}, \pi)$.
(b) $L(\mathbf{q}, \dot{\mathbf{q}}) = \max\{\mathbf{q}, \pi) - \pi \cdot \dot{\mathbf{q}} : \pi \in (E^n)^*\}$.
The maximum is attained at the unique π such that

$$\frac{\partial H}{\partial \pi_k} = \dot{q}^k, \qquad k = 1, \ldots, n. \tag{8.45}$$

Let us write $\pi = \Psi(\mathbf{q}, \dot{\mathbf{q}})$.
(c) For fixed $\mathbf{q}$, the transformation $\Phi(\mathbf{q}, \ )$ from $(E^n)^*$ into E^n is inverse to $\Psi(\mathbf{q}, \ )$.
(d) The Lagrange equations (8.42) are equivalent to the system of $2n$ differential equations

$$\dot{q}^k = \frac{\partial H}{\partial \pi_k}, \qquad \dot{\pi}_k = -\frac{\partial H}{\partial q^k}, \qquad k = 1, \ldots, n, \tag{8.46}$$

if $\pi(t) = \Psi[\mathbf{q}(t), \dot{\mathbf{q}}(t)]$.
(e) $H[\mathbf{q}(t), \pi(t)] = E$, where E is the total energy (Problem 2).

[*Note*: Ψ is called the *canonical transformation* in mechanics, and (8.46) are the equations of motion in *Hamiltonian* form.]

Appendix

A.1 Axioms for a vector space

A *vector space* over the real number field is a nonempty set $\mathscr{V}$ together with two operations called *addition* and *scalar multiplication*. The sum $u + v$ of two elements $u, v \in \mathscr{V}$ is also an element of $\mathscr{V}$ and the scalar multiple cu of $u \in \mathscr{V}$ by the real number c is an element of $\mathscr{V}$. These operations are required to satisfy the following axioms:

(1) Addition is associative and commutative.
(2) There is a zero element θ such that $u + \theta = u$ for every $u \in \mathscr{V}$.
(3) The distributive laws hold:

$$(c + d)u = cu + du, \qquad c(u + v) = cu + cv$$

for every real c, d and $u, v \in \mathscr{V}$.
(4) $(cd)u = c(du)$ for every real c, d, and $u \in \mathscr{V}$.
(5) $0u = \theta$, $1u = u$, for every $u \in \mathscr{V}$.

It is easy to show that E^n satisfies these five axioms. However, a multitude of other important vector spaces besides E^n occur in mathematics.

A subset B of a vector space $\mathscr{V}$ is called a *linearly dependent set* if there exist distinct elements $u_1, \ldots, u_m \in B$ and real numbers $c^1, \ldots, c^m$ not all 0 such that

$$c^1 u_1 + \cdots + c^m u_m = \theta.$$

If B is not linearly dependent, then B is a *linearly independent set*. $\mathscr{V}$ is a *finite dimensional vector space* if some finite subset B of $\mathscr{V}$ spans $\mathscr{V}$, namely, if every element $u \in \mathscr{V}$ is a linear combination $u = c^1 u_1 + \cdots + c^m u_m$ where $u_1, \ldots, u_m \in B$.

A *basis* for $\mathscr{V}$ is a linearly independent set that spans $\mathscr{V}$. If $\mathscr{V}$ is finite dimensional, then every basis B has the same number n of elements [12, p. 43]. The number n is the *dimension* of $\mathscr{V}$. If $n = 0$, then $\mathscr{V}$ has the single element θ. If $n > 0$ and $B = \{u_1, \ldots, u_n\}$ is a basis for $\mathscr{V}$, then every $u \in \mathscr{V}$ can be uniquely written as a linear combination

$$u = c^1 u_1 + \cdots + c^n u_n.$$

A nonempty set $\mathscr{P} \subset \mathscr{V}$ is called a *vector subspace* of $\mathscr{V}$ if: $u, v \in \mathscr{P}$ implies $u + v \in \mathscr{P}$ and $u \in \mathscr{P}$ implies $cu \in \mathscr{P}$ for any real c. In other words, this says that $\mathscr{P}$ when provided with the addition and scalar multiplication in $\mathscr{V}$ is a vector space.

Let $\mathscr{V}$ and $\mathscr{W}$ be vector spaces. Let L be a function with domain $\mathscr{V}$ and values in $\mathscr{W}$. Then L is *linear* if

(a) $L(u + v) = L(u) + L(v)$ for every $u, v \in \mathscr{V}$.
(b) $L(cu) = cL(u)$ for every $u \in \mathscr{V}$ and real c.

Let L and M be linear. The sum $L + M$ is given by

$$(L + M)(u) = L(u) + M(u)$$

for every $u \in \mathscr{V}$. The function $L + M$ has Properties (a) and (b), and thus is linear. If c is a real number, then cL is the linear function given by $(cL)(u) = cL(u)$ for every $u \in \mathscr{V}$.

Let $\mathscr{L}(\mathscr{V}, \mathscr{W})$ denote the set of all linear functions with domain $\mathscr{V}$ and values in $\mathscr{W}$, together with these operations of sum of functions and multiplication of functions by scalars. Then $\mathscr{L}(\mathscr{V}, \mathscr{W})$ satisfies Axioms (1) through (5) for a vector space. The zero element of $\mathscr{L}(\mathscr{V}, \mathscr{W})$ is the function whose value at every $u \in \mathscr{V}$ is the zero element of $\mathscr{W}$.

The dual space of $\mathscr{V}$

Let us now suppose that $\mathscr{W} = E^1$ and set $\mathscr{V}^* = \mathscr{L}(\mathscr{V}, E^1)$. The vector space $\mathscr{V}^*$ is called the *dual space* of $\mathscr{V}$. Let us show that if $\mathscr{V}$ has positive, finite dimension n, then $\mathscr{V}^*$ also has dimension n. Let $B = \{u_1, \ldots, u_n\}$ be a basis for $\mathscr{V}$. Let $L^1, \ldots, L^n$ be the real valued functions such that for each $i = 1, \ldots, n$ and $u = c^1 u_1 + \cdots + c^n u_n$,

$$L^i(c^1 u_1 + \cdots + c^n u_n) = c^i.$$

These functions L^i are linear, and therefore belong to $\mathscr{V}^*$. They are specified by their values at the basis elements:

$$(*) \qquad L^i(u_j) = \delta^i_j, \qquad i, j = 1, \ldots, n,$$

where $\delta^i_j = 1$ if $i = j$ and $\delta^i_j = 0$ if $i \neq j$.

Let us show that $B^* = \{L^1, \ldots, L^n\}$ is a basis for $\mathscr{V}^*$. Suppose that $b_1 L^1 + \cdots + b_n L^n = \theta$, where θ is the zero function. Then, for every $u \in \mathscr{V}$,

$$b_1 L^1(u) + \cdots + b_n L^n(u) = \theta(u) = 0.$$

Taking $u = u_i$ and applying formula (*), $b_i = 0$ for each $i = 1, \ldots, n$. Thus B^* is a linearly independent set. To show that B^* spans $\mathscr{V}^*$, given $L \in \mathscr{V}^*$ let $a_i = L(u_i)$. If $u = c^1u_1 + \cdots + c^nu_n$, then since L is linear $L(\sum c^iu_i) = \sum c^iL(u_i)$. Therefore

$$L(u) = \sum_{i=1}^{n} a_i c^i = \sum_{i=1}^{n} a_i L^i(u).$$

Since this is true for every $u \in \mathscr{V}$,

$$L = a_1L^1 + \cdots + a_nL^n,$$

which shows that B^* spans $\mathscr{V}^*$.

The basis B^* is called *dual* to the basis B.

A function ϕ from a vector space $\mathscr{V}$ into a vector space $\mathscr{W}$ is an *isomorphism* if ϕ is linear and $\phi(u) \neq \phi(v)$ whenever $u \neq v$. If there is such an isomorphism from $\mathscr{V}$ onto $\mathscr{W}$, then $\mathscr{V}$ and $\mathscr{W}$ are *isomorphic* vector spaces. All n-dimensional vector spaces are isomorphic. If $\{u_1, \ldots, u_n\}$ is a basis for $\mathscr{V}$ and $\{w_1, \ldots, w_n\}$ a basis for $\mathscr{W}$, then the linear function ϕ such that $\phi(u_i) = w_i$ for $i = 1, \ldots, n$ is an isomorphism from $\mathscr{V}$ onto $\mathscr{W}$.

In particular, any finite-dimensional vector space $\mathscr{V}$ is isomorphic with its dual $\mathscr{V}^*$. However, this isomorphism is unnatural from several points of view. In this book we maintain the distinction between $\mathscr{V}$ and $\mathscr{V}^*$.

A more natural isomorphism is the following one from a vector space $\mathscr{V}$ into the dual $\mathscr{V}^{**}$ of $\mathscr{V}^*$. For each $u \in \mathscr{V}$ let $\phi(u) = l_u$, where $l_u \in \mathscr{V}^{**}$ is the real valued linear function such that $l_u(L) = L(u)$ for every $L \in \mathscr{V}^*$. This isomorphism is onto $\mathscr{V}^{**}$ if $\mathscr{V}$ is finite dimensional.

A.2 Mean value theorem; Taylor's theorem

In this section and Section A.3 we review some basic theorems from calculus for functions of one variable. In the proof of the Mean Value Theorem we use the fact that a continuous real valued function has a maximum and minimum on any closed interval $[a, b]$ (see Theorem 2.5).

Mean value theorem. *Let f be real valued and continuous on a closed interval $[a, b]$, and let the derivative $f'(x)$ exist for every $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

PROOF. Let $m = [f(b) - f(a)]/(b - a)$ and let $F(x) = f(b) - f(x) - m(b - x)$. Then F is continuous on $[a, b]$ and $F'(x) = -f'(x) + m$ for $x \in (a, b)$. Since $[a, b]$ is compact, F has a maximum and a minimum value on $[a, b]$. If the maximum value is positive, then since $F(a) = F(b) = 0$, the maximum must occur at some $x_1 \in (a, b)$. By elementary calculus $F'(x_1) = 0$ and we may take $c = x_1$. Similarly, if the minimum value is negative we may take $c = x_2$, where $F(x_2)$ is the minimum value. If neither of these possibilities occurs, then $F(x) = 0$ on $[a, b]$ and c is arbitrary. □

The mean value theorem has the following generalization.

Taylor's theorem with remainder. *Let f, together with its derivatives f', f'', $\ldots$, $f^{(q-1)}$, be continuous on a closed interval $[a, b]$ and let the qth-order derivative $f^{(q)}(x)$ exist for every $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots + \frac{f^{(q-1)}(a)}{(q-1)!}(b - a)^{q-1} + R_q,$$

where

$$R_q = \frac{f^{(q)}(c)}{q!}(b - a)^q.$$

PROOF. Let

$$G(x) = f(b) - f(x) - f'(x)(b - x) - \frac{f''(x)}{2!}(b - x)^2 - \cdots - \frac{f^{(q-1)}(x)}{(q-1)!}(b - x)^{q-1} - \frac{K}{q!}(b - x)^q,$$

where the number K is so chosen that $G(a) = 0$. Then $G(b) = 0$ and, using the product rule,

$$G'(x) = \frac{(b - x)^{q-1}}{(q-1)!}[-f^{(q)}(x) + K].$$

Repeating the reasoning in the proof of the mean value theorem, there exists $c \in (a, b)$ such that $G'(c) = 0$. Then $f^{(q)}(c) = K$. □

A.3 Review of Riemann integration

Let f be real valued and continuous on an interval $[a, b]$. Then f has an integral over $[a, b]$, denoted by $\int_a^b f(t)dt$. According to Riemann's definition of the integral, it is the limit of sums:

$$\int_a^b f(t)dt = \lim_{\mu \to 0} \sum_{j=1}^{m} f(s_j)(t_j - t_{j-1})$$

where

$$a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b, \qquad t_{j-1} \le s_j \le t_j,$$

and

$$\mu = \max\{t_1 - t_0, t_2 - t_1, \ldots, t_m - t_{m-1}\}.$$

More generally, the Riemann integral exists for any bounded function with a finite number of discontinuities. It agrees with the integral in Lebesgue's sense, which is defined in Chapter 5 for a much wider class of functions.

Fundamental theorem of calculus. *Let f be continuous on $[a, b]$ and let*

$$F(t) = \int_a^t f(s)ds, \qquad a \le t \le b.$$

Then $F'(t) = f(t)$ for every $t \in [a, b]$.

PROOF. By elementary properties of the integral, if $h > 0$ and $t + h \le b$, then

$$\frac{F(t + h) - F(t)}{h} = \frac{1}{h}\int_t^{t+h} f(s)ds.$$

Since f is continuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that $f(t) - \varepsilon < f(s) < f(t) + \varepsilon$ whenever $|s - t| < \delta$. Then if $h < \delta$,

$$h[f(t) - \varepsilon] < \int_t^{t+h} f(s)ds < h[f(t) + \varepsilon],$$

$$f(t) - \varepsilon < \frac{F(t + h) - F(t)}{h} < f(t) + \varepsilon.$$

Hence

$$f(t) = \lim_{h \to 0^+} \frac{F(t + h) - F(t)}{h}.$$

The right-hand side is the right-hand derivative of F at t. Similarly, $f(t)$ equals the left-hand derivative of f at t. □

In the theorem, $F'(a)$ means the right-hand derivative and $F'(b)$ means the left-hand derivative.

The fundamental theorem says that F is an antiderivative of f. If G is any antiderivative of f, then $G'(t) - F'(t) = 0$ for every $t \in [a, b]$, and by the mean value theorem $G(t) - F(t)$ is constant on $[a, b]$. Thus $G(t) - F(t) = G(a) - F(a) = G(a)$, and upon setting $t = b$ we obtain

$$G(b) - G(a) = \int_a^b f(s)ds.$$

Change of variables in integrals

Let ϕ be any real valued function possessing a continuous derivative on some closed interval $[\alpha, \beta]$ such that

$$\phi'(\tau) \ge 0 \quad \text{for every } \tau \in [\alpha, \beta], \quad \phi(\alpha) = a, \quad \phi(\beta) = b.$$

Then $a \le \phi(\tau) \le b$ for every $\tau \in [\alpha, \beta]$.

If $U = F \circ \phi$, then

$$U'(\tau) = F'[\phi(\tau)]\phi'(\tau) = f[\phi(\tau)]\phi'(\tau)$$

for every $\tau \in [\alpha, \beta]$. Since

$$F(a) = U(\alpha), \qquad F(b) = U(\beta),$$

$$\int_a^b f(t)dt = \int_\alpha^\beta f[\phi(\tau)]\phi'(\tau)d\tau.$$

This is the formula for change of variables in integrals. If $\phi'(\tau) \leq 0$ for every $\tau \in [\alpha, \beta]$, then $\phi(\alpha) \geq \phi(\beta)$. The same formula holds if we agree that

$$\int_a^b = -\int_b^a.$$

A.4 Monotone functions

Let f be real valued with domain $S \subset E^1$.

Definition. If for every $x, y \in S$ such that $x < y$,

$f(x) < f(y)$	then f is *increasing.*
$f(x) \leq f(y)$	then f is *nondecreasing.*
$f(x) > f(y)$	then f is *decreasing.*
$f(x) \geq f(y)$	then f is *nonincreasing.*

If f is either an increasing function or a decreasing function, then f is called *strictly monotone.* If f is either nondecreasing or nonincreasing, then f is *monotone.* If the restriction of f to A is monotone, then f is *monotone on* A.

A function f is univalent if $f(x) \neq f(y)$ whenever $x \neq y$. Clearly, any strictly monotone function is univalent. If S is an interval, then conversely any continuous univalent function must be strictly monotone. This can be proved from the intermediate value theorem.

Let A be an interval and assume that the derivative $f'(x)$ exists for every $x \in A$. It is proved in elementary calculus that:

(a) f is nondecreasing on A if and only if $f'(x) \geq 0$ for every $x \in A$.
(b) If $f'(x) > 0$ except at a finite number of points of A, then f is increasing on A.
(c) f has an inverse f^{-1} if and only if f is strictly monotone on A.
(d) If $f'(x) \neq 0$ for all $x \in A$, then the derivative of the inverse is given by

$$f^{-1'}(t) = \frac{1}{f'(x)}, \qquad \text{if } t = f(x).$$

Among the examples of strictly monotone functions from calculus are the exponential function exp, whose inverse is log. The restriction to $[-\pi/2\ \pi/2]$ of the function sin is strictly monotone. Its inverse is denoted by $\sin^{-1}$.

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Answers to problems

Section 1.1

1 (a) 2, 1.
(b) $\sqrt{2}$, no lower bound.
(c) $\sqrt{2}$, -1.
(d) 0, $-e^{-1}$.
(e) 1, $\frac{1}{2}$.
sup $S \in S$ for (b), (c), (e); inf $S \in S$ for (c), (d).

Section 1.2

1 $4\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3 + 3\mathbf{e}_4$, $-2\mathbf{e}_1 - \mathbf{e}_3 + \mathbf{e}_4$, $\sqrt{30}$, $\sqrt{6}$, $\sqrt{6}$ $\sqrt{12}$, 6.

8 $\mathbf{v}_4 = \pm(\sqrt{2}/10)(4\mathbf{e}_1 + 3\mathbf{e}_2 - 3\mathbf{e}_3 + 4\mathbf{e}_4)$.

9 $\sqrt{m}$, $m = 1, 2, \ldots, n$.

Section 1.3

1 $\{\mathbf{x} : x^1 + x^2 + 4x^3 = 1\}$.

2 (a) $\{\mathbf{x} : 3x^1 - 3x^2 - 3x^3 - x^4 = 0\}$.
(b) $t = \frac{1}{7}$.

3 $-2x^1 + x^2 + x^3 + 2x^4 = -3$.

4 One basis for $\mathscr{V}$ is $\{\mathbf{e}_1 + 2\mathbf{e}_3, \mathbf{e}_2 + 3\mathbf{e}_3\}$.

Section 1.4

2 (a) $\{\mathbf{x} : 0 < |\mathbf{x} - \mathbf{x}_0| < \delta\}$, $\{\mathbf{x} : |\mathbf{x}| = 0 \text{ or } \delta\}$, $\{\mathbf{x} : |\mathbf{x}| \leq \delta\}$.
(b) Empty set, A, A.

(c) A, the union of the half-lines $y = 0$, $y = x + 1$, $x \geq -1$, $\{(x, y) : 0 \leq y \leq x + 1, x \geq -1\}$.
(d) A, the union of the circle $x^2 + y^2 = 1$ and the line segment joining $(0, 0)$ and $(1, 0)$, the closed circular disk $x^2 + y^2 \leq 1$.
(e) Empty set, E^2, E^2.
(f) Empty set, A, A.
(g) Empty set, $A \cup \{0\}$, $A \cup \{0\}$.

3 Open in (c), (d); closed in (b), (f).

Section 1.5

3 $t = \frac{5}{8}$; $t^0 = \frac{1}{2}$, $t^1 = \frac{1}{4}$, $t^2 = \frac{1}{4}$

7 (b) The barycenter is at the intersection of the line segments that join the vertices with the barycenters of the opposite $(r - 1)$-dimensional faces.

Section 2.1

1 (a) $[-1, 1]$, $[0, 1]$, $\left\{x : x = 2m\pi + y, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], m \text{ any integer}\right\}$.

(b) $[-1, 1]$, $\left[0, \frac{\pi}{2}\right]$. Yes.

2 (a) The triangle with vertices $(0, 0)$, $(a, \pm\sqrt{2}a)$.

(b) $\left\{(x, y) : x^2 - xy + y^2 < \frac{a^2}{2}\right\}$.

Section 2.2

1 (a) $\frac{1}{2}$.
(b) No limit.
(c) $\frac{1}{2}$.
(d) $(1, 5) = \mathbf{e}_1 + 5\mathbf{e}_2$.
(e) No limit.
Continuity: In (a), (b), (e) except at $(0, 0)$. In (c) except at 0. If we set $f(0) = \frac{1}{2}$, then f is also continuous at 0. In (d) at every point of E^1.

5 Continuous at any (x, y) with $y \neq 0$, also at $(0, 0)$.

6 (a) 0.
(b) No limit.
(c) No limit.
(d) 1.

Section 2.3

1 (a) 0.
(b) No limit.
(c) 0.
(d) e^2.
(e) 1.

2 (a) $(-\frac{1}{2}, 0)$.
(b) No limit.
(c) $(1, 0)$.

Section 2.4

1 (a) ± 1.
(b) No accumulation points.
(c) $\left\{\left(\cos\frac{2m\pi}{5}, \sin\frac{2m\pi}{5}\right) : m = 0, 1, \ldots, 4\right\}$.
(d) $\{(x, y) : y^2 - x^2 + 1 \leq 0\}$.
(e) $[-1, 1]$.

Section 2.5

1 (a) S is not closed.
(b) S is not bounded.

Section 2.6

3 (a) Both open and closed relative to S.
(b) Closed relative to S.
(c) Neither.

4 (a) Closed relative to S.
(b) Open relative to S.

Section 2.11

1 (a) The square $\{(x, y) : |x| \leq 1, |y| \leq 1\}$.

2 (a) $\|(x, y)\| = [x^2 + xy + 4y^2]^{1/2}$.
(b) 2.

Section 3.1

1 (a) $f_1(x, y) = 1 + \log(xy)$, $f_2(x, y) = x/y$
(b) $f_1(x, y, z) = 6x(x^2 + 2y^2 + z)^2$, $f_2(x, y, z) = 12y(x^2 + 2y^2 + z)^2$, $f_3(x, y, z) = 3(x^2 + 2y^2 + z)^2$
(c) $f_i(\mathbf{x}) = 2x^i$

2 The derivative in direction $(\cos\theta, \sin\theta)$ is $-2(\cos\theta + \sin\theta)$.

3 $\sin 2\theta$, if $\mathbf{v} = (\cos\theta, \sin\theta)$.

5 The derivative is 0 in those directions for which $v^1 + v^2 + v^3 = 0$. There is no derivative in other directions.

Section 3.2

1 (a) $\mathbf{a} = \mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3$.
(b) The plane $x + y + 2z = c$ in E^3. The line through

$$\mathbf{e}_1 + \mathbf{e}_2 + \frac{c-2}{2}\mathbf{e}_3 \quad \text{and} \quad \frac{c}{2}\mathbf{e}_3.$$

SECTION 3.3

1 It has the equation $4x + 5y + z + 4 = 0$.

2 (a) $\sqrt{5}$.
(b) $1/\sqrt{2}e$.
(c) 0.

3 (a) $[2x(x^2 + 2y + 1)^{-1} + \cos(x^2)]\mathbf{e}^1 + 2(x^2 + 2y + 1)^{-1}\mathbf{e}^2$.
(b) 0.09.

4 (a) $\mathbf{x}_0$.
(b) $|\mathbf{x}|^{-1}\mathbf{x}$.
(c) $2(\mathbf{x}_0 \cdot \mathbf{x})\mathbf{x}_0$.

6 (b) $\{(x, y): x = y^2, x \neq 0\}$, $\{(x, y): x = -y^2, x \neq 0\}$.
(c) The union of the sets in 6(b) and the x-axis, with $(0, 0)$ excluded since f is not differentiable there.
(d) $\{(x, y): x = y^2(c^{-1} \pm \sqrt{c^{-2} - 1}), x \neq 0\}$ if $|c| \leq 1$, $c \neq 0$; the x- and y-axes if $c = 0$.

SECTION 3.4

1 $xyz = -z + (y + 1)z - (x - 1)z + (x - 1)(y + 1)z$

2 (a) $f(x, y) = 1 - (x - 1) + R_2(x, y)$;

$$|R_2(x, y)| \leq m^{-3}[(x - 1)^2 + y^2] \qquad \text{where } m = \min\{x, 1\}.$$

7 (c) $f_{12}(0, 0) = -1$, $f_{21}(0, 0) = 1$.

8 $\binom{n + q - 1}{q}$; $\binom{q - 1}{n - 1}$ if $n \leq q$, 0 if $n > q$.

SECTION 3.5

1 (a) Maximum at $\frac{1}{2}\mathbf{e}_1$.
(b) Saddle point at $-\mathbf{e}_1 + 2\mathbf{e}_2$.
(c) Maximum at each point where $xy = \pi/2 + 2m\pi$; minimum at each point where $xy = -\pi/2 + 2m\pi$, m any integer. Saddle point at 0
(d) Saddle points at $\mathbf{0}$ and at $\mathbf{e}_1$.

2 (a) Maximum at $-\mathbf{e}_1 + \mathbf{e}_2$, saddle point at $\frac{1}{3}(-\mathbf{e}_1 + \mathbf{e}_2)$.
(b) Saddle points at $-\mathbf{e}_2$ and at $\mathbf{e}_1 + \mathbf{e}_2$.
(c) Saddle points at $m\pi\mathbf{e}_1$, m any integer.

5 $\mathbf{x} = (1/m)(\mathbf{x}_1 + \cdots + \mathbf{x}_m)$. The minimum value is

$$\frac{1}{m^2} \sum_{j=1}^{m} \left| \sum_{k=1}^{m} (\mathbf{x}_j - \mathbf{x}_k) \right|^2.$$

6 (a) $\frac{1}{4}$, -2.
(b) $\sin \frac{1}{2}$, $-\sin \frac{1}{2}$.

7 (a) Let c be any number such that $\psi'(c) = 0$. Then all points $\mathbf{x}$ on the hyperplane $\{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = c\}$ are critical.
(b) All points on the line $y = x$ are critical.

Section 3.6

1 If $f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$, then $a_0 > 0$, $8a_0a_2 \geq 3a_1^2$.

2 (a) Neither.
(b) Concave, not strictly.
(c) Convex if $p \leq 0$ or $p \geq 1$. Concave if $0 \leq p \leq 1$. Not strictly.
(d) Strictly convex.
(e) Neither. However, f is strictly convex on each half of $\{(x, y) : 2xy < -1\}$.

4 (a) $a = 1$
(b) a^2 satisfies the equation $\cot a^2 = 2a^2$, $a^2 < \pi/2$.

Section 4.1

1 $\begin{pmatrix} 1 & 1 & 5 \\ -2 & 0 & 1 \end{pmatrix}$. The rank is 2. The kernel consists of all scalar multiples of

$$\boldsymbol{\varepsilon}_1 - 11\boldsymbol{\varepsilon}_2 + 2\boldsymbol{\varepsilon}_3.$$

3 (a) The diagonal elements are $c^1, \ldots, c^n$; all other elements are 0.
(b) $(\mathbf{L}^{-1})^i(\mathbf{x}) = (c^i)^{-1}x^i$ provided $c^i \neq 0$ for every i.

Section 4.2

1 (a) Reflection in the line $s = t$.
(b) They are rotations through angle $3\pi/2$, $\pi/2$ respectively.

2 $\mathbf{L}$ is not a rotation.

4 (b) If $\mathbf{g}(\mathbf{t}) = \mathbf{L}(\mathbf{t}) + \mathbf{x}_0$, then $\mathbf{L}$ must be nonsingular.

Section 4.3

1 (a) The parabola $x = c^2 + y^2/4c^2$ if $c \neq 0$; the positive x-axis if $c = 0$.
(b) The lines $t = s(k \pm \sqrt{k^2 - 1})$ if $k = 1/m$, $m^2 < 1$; the line $t = s$ if $m = 1$; the line $t = -s$ if $m = -1$; $\{(0, 0)\}$ if $m^2 > 1$.
(c) $\{(x, y) \in Q : x \leq a^2\}$.

2 $\mathbf{g}(E^2) = \{(x, y) : x \geq 0, y \geq 0\}$.
(a) The parts of the lines $x + y = 2|c|$, $y = x + 2|c|$, $y + 2|c| = x$ in $\mathbf{g}(E^2)$.
(b) The union of the lines $s + t = m(s - t)$ and $s + t = -m(s - t)$.
(c) $\{(x, y) : x^2 + y^2 \leq 2a^2, x \geq 0, y \geq 0\}$.

3 (b) The part of the cone between the plane $z = 0$ and the vertex.
(c) The s-axis; $\{(m, 1) : m \text{ any integer}\}$.

4 (a) $\mathbf{g}(\Delta) = \{(x, x^2) : x \geq \frac{2}{3}\}$.
(b) If $c \geq \frac{2}{3}$, the part of the ellipse $s^2 + st + t^2 = 1/c$ in Δ; if $c < \frac{2}{3}$, the empty set.

5 (a) Except where $s^2 = t^2$; everywhere in E^2; everywhere in Δ.

(b) $\mathbf{g}_1 = \mathbf{e}_1 + \mathbf{e}_2, \mathbf{g}_2 = -\mathbf{e}_1 + \mathbf{e}_2$ in $Q_1 = \{(s, t) : s - t > 0, s + t > 0\}$,
$\mathbf{g}_1 = -\mathbf{e}_1 + \mathbf{e}_2, \mathbf{g}_2 = \mathbf{e}_1 + \mathbf{e}_2$ in $Q_2 = \{(s, t) : s - t < 0, s + t > 0\}$,
with similar expressions in the other two of the four quadrants Q_3, Q_4 into which the lines $s = \pm t$ divide E^2.
$\mathbf{g}_1 = -2\pi t(\sin 2\pi s)\mathbf{e}_1 + 2\pi t(\cos 2\pi s)\mathbf{e}_2$,
$\mathbf{g}_2 = (\cos 2\pi s)\mathbf{e}_1 + (\sin 2\pi s)\mathbf{e}_2 - \mathbf{e}_3$;
$\mathbf{g}_1 = -(s^2 + st + t^2)^{-2}(2s + t)[\mathbf{e}_1 + 2(s^2 + st + t^2)^{-1}\mathbf{e}_2]$,
$\mathbf{g}_2 = -(s^2 + st + t^2)^{-2}(s + 2t)[\mathbf{e}_1 + 2(s^2 + st + t^2)^{-1}\mathbf{e}_2]$,

(c) 2; 2 unless $t = 0$, and 1 if $t = 0$; 1.

(d) 2 in Q_1, Q_3, -2 in Q_2, Q_4; not applicable; 0.

Section 4.4

1 $F_{12} = xf_{12} + xyf_{22} + f_2$, the partial derivatives of f being evaluated at (x, xy).

2 $F_1 = f_1 + f_3 g_1, \quad F_2 = f_2 + f_3 g_2$,
$F_{11} = f_{11} + 2f_{13}g_1 + f_{33}(g_1)^2 + f_3 g_{11}$,
$F_{12} = f_{12} + f_{13}g_2 + f_{32}g_1 + f_{33}g_1 g_2 + f_3 g_{12}$,
$F_{22} = f_{22} + 2f_{23}g_2 + f_{33}(g_2)^2 + f_3 g_{22}$.

3 (a) 162

(b) $-\phi'(\frac{3}{2})\phi'(\frac{1}{2})$.

4 (b) $15\mathbf{E}_1 + 54\mathbf{E}_2, \quad -3\mathbf{E}_1$.

Section 4.5

1 (a) Yes, $\mathbf{g}(E^n) = E^n$, $\mathbf{g}^{-1}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$.

(b) Yes, $\mathbf{g}(E^2) = E^2$. $\mathbf{g}^{-1}(x, y) = \frac{1}{3}[(x + 2y)\boldsymbol{\varepsilon}_1 + (x - y)\boldsymbol{\varepsilon}_2]$.

(c) No, $\mathbf{g}(E^2)$ is the half-plane $4x \geq -9$.

(d) Yes, $\mathbf{g}(\Delta) = \Delta$, $\mathbf{g}$ is not univalent since $\mathbf{g}(-s, -t) = \mathbf{g}(s, t)$.

(e) Yes, $\mathbf{g}(\Delta) = \{(x, y) : 0 < y < \frac{1}{2}\exp(-x)\}$, $\mathbf{g}^{-1}(x, y) = \frac{1}{2}[(a + b)\boldsymbol{\varepsilon}_1 + (a - b)\boldsymbol{\varepsilon}_2]$, where $a = (y^{-1} + 2e^x)^{1/2}$, $b = (y^{-1} - 2e^x)^{1/2}$.

2 $g^{-1}(x) = [(x + 1)^{1/2} - 1]^{1/2}, x > 0$.

3 Its matrix is $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

4 If $0 < \cos c < 1$, the image of the line $t = c$ is the right half of the hyperbola $x^2/\cos^2 c - y^2/\sin^2 c = 1$; and if $-1 < c < 0$ it is the left half. If $\cos c = 0$, it is the y-axis, if $\cos c = 1$ the right half of the x-axis; if $\cos c = -1$, the left half.

5 (b) $(\mathbf{g}|\tilde{\Delta})^{-1}(x, y) = \log R(x, y)\boldsymbol{\varepsilon}_1 + \Theta(x, y)\boldsymbol{\varepsilon}_2$, where $R(x, y) = (x^2 + y^2)^{1/2}$, $\Theta(x, y)$ is the angle from the positive x-axis to (x, y).

(c) $\mathbf{g}(E^2) = E^2 - \{(0, 0)\}$.

Section 4.6

1 $\phi' = -\Phi_1/\Phi_2$. $\phi'' = -[(\Phi_2)^2\Phi_{11} - 2\Phi_1\Phi_2\Phi_{12} + (\Phi_1)^2\Phi_{22}]/(\Phi_2)^3$.

2 $\phi_{11} = -[(\Phi_2)^2\Phi_{11} - 2\Phi_1\Phi_2\Phi_{12} + (\Phi_1)^2\Phi_{22}]/(\Phi_1)^3$.

3 $\phi_1 = -1$, $\psi_1 = 0$, $\phi_2 = 0$, $\psi_2 = 1$ at $(-1, 1)$.

4 (b) Radius $3\sqrt{2}/2$.
(c) Radius $\sqrt{21}$.

Section 4.7

1 $\{(x, y): F(x, y) = c\}$ is an ellipse if $\log c > 2$, is the one point set $\{(0, 0)\}$ if $\log c = 2$, and is empty if $\log c < 2$. Any ellipse is a 1-manifold.

2 (a) The cone is not a 2-manifold.
(b) $2(x - 2) - (y + 1) + 4(z - 1) = 0$.

7 No.

Section 4.8

1 $\frac{3}{5}$.

2 $|c|$ if $c \leq \frac{1}{2}$, $\sqrt{c - \frac{1}{4}}$ if $c \geq \frac{1}{2}$.

3 $\sqrt{14/3}$.

9 (a) $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$.

Section 5.1

2 $V(Y) = 4$, $V(Z) = 12$, $V(Y \cup Z) = 13$, $V(Y \cap Z) = 3$.

3 $V(I_1 \cup I_2) = \frac{19}{2}$, $V(I_1 \cap I_2) = \frac{1}{2}$.

4 $(e - 1)\exp\left(\frac{1}{m}\right)\dfrac{1/m}{\exp(1/m) - 1}$.

Section 5.2

5 $239\pi/240$.

Section 5.3

1 (a) Unbounded; $(-\infty, 0]$.
(b) Bounded; E^2.
(c) Bounded; E^2.
(d) Bounded; $\{(x, y): y^2 \geq x^2, |x| + |y| \leq 1\}$.

3 1.

SECTION 5.5

1 (a) $\frac{9}{2}, (\frac{1}{2}, \frac{8}{5})$.
(b) $1 + \pi/2, (\bar{x}, 0)$ where $\bar{x} = 2/(6 + 3\pi)$.

2 $(1 - 3e^{-2})/4$.

3 $\displaystyle\int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dz \left\{ \int_{\sqrt{x^2+z^2}}^{\sqrt{8-x^2-z^2}} f\, dy + \int_{-\sqrt{8-x^2-z^2}}^{-\sqrt{x^2+z^2}} f\, dy \right\}$.

4 8.

5 $\frac{51}{15}$.

6 (b) $(e - 1)^2$.
(c) $2 - 2\pi$.

SECTION 5.6

1 (a) Exists.
(b) Exists if $p < 1$; divergent if $p \geq 1$.
(c) Exists.
(d) Exists.
(e) Exists if $p < 1, p + q > 1$; divergent otherwise.
(f) Diverges.
(g) Diverges.

5 (a) 0.
(b) π.
(c) 0.

SECTION 5.7

1 $\frac{1}{6}$.

2 7.

SECTION 5.8

1 $\displaystyle\int_2^3 f(x)dx = \int_0^1 f[g(t)](2 - 2t)dt$, provided either integral exists.

2 $2 \log 2$.

3 $\frac{165}{6}$.

SECTION 5.9

1 $2a^5/15$.

2 $\frac{1}{6}$.

3 $\dfrac{1}{\sqrt{2}}$.

4 $\displaystyle\int_0^{\pi/2} d\theta \int_0^{[\cos\theta+\sin\theta)^{-1}} r\,dr \int_0^{g(r\cos\theta,\, r\sin\theta)} f[r\cos\theta, r\sin\theta, z]dz.$

5 $\dfrac{4\pi}{3}(a^2 - b^2)^{3/2}.$

7 $\pi/2.$

10 $\pi^2/4.$

11 (a) $\frac{1}{3}\Gamma(\frac{3}{2})\Gamma(\frac{1}{3})/\Gamma(\frac{11}{6})$.
(b) $\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})/\Gamma(\frac{5}{4})$.

(c) $\dfrac{1}{b}\,\Gamma\!\left(\dfrac{a+1}{b}\right)$.

(d) $\left(\dfrac{-1}{d+1}\right)^{c+1}\Gamma(c+1)$.

(e) $\Gamma(k+1)/\Gamma(n+k+1)$.

(f) $a_1^{-(k+1)/2}\Gamma\!\left(\dfrac{k+1}{2}\right)(a_2\cdots a_n)^{-1/2}\pi^{(n-1)/2}$.

Section 5.11

2 $\displaystyle\int_t f\,dV_2 = \frac{2\pi}{2-p} + \pi t^{1-2/p}\left(\frac{-p}{2-p}\right)$ if $p \neq 2$;

$\displaystyle\int_t f\,dV_2 = \pi(1 + \log t)$ if $p = 2$.

The limit as $t \to \infty$ is $2\pi/(2-p)$ if $0 < p < 2$, and $+\infty$ if $p \geq 2$.

Section 5.12

1 (a) $2\tan^{-1}(1/t)$.

(b) $\dfrac{1 + \exp(\pi t)}{t^2 + 1}$.

3 $\phi(x) = (\sqrt{\pi}/2)\exp(-x^2/4)$.

5 $t^{-1}[2\exp(t^4) - (1 + 1/\log t)\exp(t^2\log^2 t)]$.

Section 6.1

1 $y = 2(x - \sqrt{2})$.

2 $y - 1 = \frac{1}{2}(x-1)$, $z - 1 = \frac{1}{3}(x-1)$.

3 $\sqrt{2}(-\mathbf{e}_1 + \mathbf{e}_2)$.

SECTION 6.2

1 (a) Simple closed curve.
(b) Neither.
(c) Simple arc.

2 (b) $\frac{2}{27}[(4 + 9b)^{3/2} - 8]$.

3 $\mathbf{G}(s) = \mathbf{g}[s/\sqrt{2}], \quad 0 \leq s \leq 2\sqrt{2}\pi$.

4 $-\mathbf{e}_1 - 2\mathbf{e}_2, \mathbf{e}_1 - 2\mathbf{e}_2$, and any scalar multiples of these tangent vectors.

5 (b) No, since $\mathbf{g}'(0) = \mathbf{0}$.

SECTION 6.3

2 $M_2 = N_1, \quad N_3 = O_2, O_1 = M_3$.

3 (a) $f(x, y) = \frac{1}{2}x^2y + c$.
(b) Not exact.
(c) Not exact.
(d) $f(x, y) = x/y - y/x + \phi(x, y)$, where ϕ is constant on each of the four quadrants into which the coordinate axes divide E^2.

SECTION 6.4

1 (a) $\frac{1}{2}ac$.
(b) πab.

2 (a) $\frac{11}{2}$.
(c) $2, \frac{11}{6}$.
(d) $-\frac{5}{6}$.

5 $f(x, y, z) = \frac{1}{2}\phi(\rho^2)$, where $\phi(u) = \int_0^u \psi(v)dv$.

7 $2\sqrt{6}$.

8 (a) $\pi\mathbf{e}_3$.
(b) $\frac{2}{3}\sqrt{2}\pi^3 + 2\sqrt{2}\pi$.

SECTION 6.6

3 (b) $T = cP^{a/(1+a)}$.

SECTION 7.1

1 $\mathbf{e}^{12}, \boldsymbol{\omega} = \omega_{12}\mathbf{e}^{12}; \mathbf{e}^{12}, \mathbf{e}^{13}, \mathbf{e}^{14}, \mathbf{e}^{23}, \mathbf{e}^{24}, \mathbf{e}^{34}, \boldsymbol{\omega} = \omega_{12}\mathbf{e}^{12} + \omega_{13}\mathbf{e}^{13} + \cdots + \omega_{34}\mathbf{e}^{34}$.

3 (a) $-\mathbf{e}^{12} - 3\mathbf{e}^{13} - 3\mathbf{e}^{23}$
(b) $-xy\,dx \wedge dy + x^2y\,dx \wedge dz + (3x + z)dy \wedge dz$.

4 (a) $4xy\,dx \wedge dy$.
(b) 0.
(c) 0.
(d) $(\partial f/\partial x)dx \wedge dy$.

Section 7.2

1 $-1,\ 0,\ 1,\ 0,\ 0.$

2 $-1.$

Section 7.3

1 $\mathbf{e}^1 \wedge \mathbf{e}^{234} = \mathbf{e}^3 \wedge \mathbf{e}^{124} = \mathbf{e}^{1234}, \mathbf{e}^2 \wedge \mathbf{e}^{134} = \mathbf{e}^4 \wedge \mathbf{e}^{123} = -\mathbf{e}^{1234}.$

2 (a) $6\mathbf{e}^{12} + 2\mathbf{e}^{13} - \mathbf{e}^{23}.$
(b) $\mathbf{0}.$
(c) $-3\mathbf{e}^{123}.$
(d) $4\mathbf{e}^{123}.$

3 (a) $-\mathbf{e}^{12345}.$
(b) $\mathbf{e}^{1235} - \mathbf{e}^{1234} - \mathbf{e}^{1345}.$

Section 7.4

1 (a) $2xy \sin(xy^2)dx \wedge dy \wedge dz.$
(b) $3\, dx \wedge dy \wedge dz.$

3 (a) $\frac{1}{n} \sum_{i=1}^{n} (-1)^{i+1} x^i\, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n.$

(b) $\frac{1}{r} \sum_{k=1}^{r} (-1)^{k+1} x^{i_k} \mathbf{E}^{\lambda_k}$, where $\lambda_k = (i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_r).$

5 (a) $\mathbf{0}$ if r is odd; $-2\, d\omega \wedge d\zeta$ if r is even.
(b) $\mathbf{0}.$

Section 7.5

1 (a) $-\mathbf{e}_{2345}.$
(b) $\mathbf{0}.$
(c) $-\mathbf{e}_{146}.$
(d) $2\mathbf{e}_{6'} + \mathbf{e}_{2'} - 2\mathbf{e}_{1'}$, where $i' = (1, \ldots, i-1, i+1, \ldots, 6).$
(e) $2\mathbf{e}_{1\cdot\ 6}.$

2 (a) 2.
(b) 0.
(c) $-1.$
(d) $-3.$

3 Negative; -1

4 Negative.

5 No.

6 $\sqrt{3}/2.$

7 $\frac{1}{2}.$

Section 7.6

1 (a) If $\mathbf{b} = \mathbf{L}^*(\mathbf{a})$, then $b_1 = a_1 - a_2 + 2a_3$, $b_2 = -2a_1 + 3a_3$.
(b) If $\boldsymbol{\beta} = c\boldsymbol{\varepsilon}_{12}$, then $\mathbf{L}_2(\boldsymbol{\beta}) = c\mathbf{v}_1 \wedge \mathbf{v}_2 = c(2\mathbf{e}_{12} + 7\mathbf{e}_{13} - 3\mathbf{e}_{23})$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the column vectors.
(c) If $\boldsymbol{\omega} = \omega_{12}\mathbf{e}^{12} + \omega_{13}\mathbf{e}^{13} + \omega_{23}\mathbf{e}^{23}$, then $\mathbf{L}_2^*(\boldsymbol{\omega}) = (-2\omega_{12} + 7\omega_{13} - 3\omega_{23})\boldsymbol{\varepsilon}^{12}$.
(d) $\mathbf{0}$.
(e) $\sqrt{62}$.

Section 7.7

1 (a) $st(\cos t\, ds - s \sin t\, dt)$.
(b) $-st \cos t \exp t\, ds \wedge dt$.
(c) $\mathbf{0}$
(d) $f(st, s\cos t, \exp t)$.

3 $\boldsymbol{\omega}^\sharp = M \circ \mathbf{g}\, dg^1 + N \circ \mathbf{g}\, dg^2$. $(d\boldsymbol{\omega})^\sharp = \left(\dfrac{\partial N}{\partial y} - \dfrac{\partial M}{\partial x}\right) \circ \mathbf{g}\, \dfrac{\partial(g^1, g^2)}{\partial(s, t)}\, ds \wedge dt$.

4 (a) $f \circ g\, s\, ds \wedge dt \wedge du$.
(b) $s \cos t \sin t\, ds \wedge du + s^2 \cos^2 t\, dt \wedge du$.

Section 7.9

3 $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$.

Section 8.1

1 (a) $4\sqrt{3}$; the triangle with vertices $2\mathbf{e}_3$, $\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$.
(b) $\rho \sin \alpha$; a cone from which points $(x, y, 0)$, $y \geq 0$, are deleted.
(c) $[1 + s^2 + t^2]^{1/2}$; the hyperboloid $x = yz$.

2 A basis for $T_s(\mathbf{x}_0)$ is:
(a) $\{\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3, \mathbf{e}_1 - 3\mathbf{e}_2 + 2\mathbf{e}_3\}$.
(b) $\left\{(\cos\alpha)\mathbf{e}_1 + \dfrac{\sqrt{2}}{2}\sin\alpha(\mathbf{e}_2 + \mathbf{e}_3), \dfrac{\sqrt{2}}{2}\sin\alpha(\mathbf{e}_3 - \mathbf{e}_2)\right\}$.
(c) $\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3\}$.

4 $\dfrac{2\pi}{3}[2^{3/2} - 1]$.

7 (b) $\dfrac{\pi}{18}(13^{3/2} - 1)$.

Section 8.2

1 $\Delta = \{(x, y) : |x| < 1, |x| < y < (3 - x^2)/2\}$,
$\tilde{\Delta} = \{(x, z) : |x| < 1, z > 0, z^2 < 3 - x^2 - 2|x|\}$.
$\mathbf{g}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + (3 - x^2 - 2y)^{1/2}\mathbf{e}_3$,
$\tilde{\mathbf{g}}(x, z) = x\mathbf{e}_1 + \frac{1}{2}(3 - x^2 - z^2)\mathbf{e}_2 + z\mathbf{e}_3$,
$\phi(x, y) = (x, (3 - x^2 - 2y)^{1/2})$.

2 $\mathbf{F}(M) = \{(s, t) : t < \exp s\}$.

SECTION 8.3

1 $\dfrac{\pi a^2 b^2}{2}$.

4 (a) $8\pi/3$.
(b) $\sqrt{3}/6$.

6 (b) $4\pi^2 r_1 r_2$.

SECTION 8.4

1 The integrals equal 0.

2 (a) $-\pi/4$. (b) $\pi/4$.

5 ω is not of class $C^{(1)}$ on cl D.

8 (b) $k = 2 - n$.

SECTION 8.5

2 $\mathbf{v}(x, y) = C_1 r^{-2}(x\mathbf{e}_1 + y\mathbf{e}_2)$.

SECTION 8.6

1 (c) Four possible orientations; on each half of the hyperbola M use either orientation $\mathbf{v}$ or orientation $-\mathbf{v}$, with $\mathbf{v}$ as in part (a).

2 $\mathbf{n}(\mathbf{x}) = [1 + |\text{grad } \phi(x, y)|^2]^{1/2}(-\phi_1(x, y)\mathbf{e}_1 - \phi_2(x, y)\mathbf{e}_2 + \mathbf{e}_3)$.

SECTION 8.7

2 (a) -8π. (b) $\pi(\mathbf{e}^4 - 1)$.

3 $\frac{1}{3}$.

4 $\frac{1}{2}$.

5 (a) $\mathbf{o}(\mathbf{x}) = [(-\sin s)\mathbf{e}_1 + (\cos s)\mathbf{e}_2] \wedge [(-\sin t)\mathbf{e}_3 + (\cos t)\mathbf{e}_1]$.
(b) π^2.

8 $\frac{1}{12}$.

9 ω is not of class $C^{(1)}$ on cl D.

10 (a) $-\dfrac{\pi}{3}$. (b) $-\dfrac{8\pi}{15}$.

SECTION 8.8

1 π.

2 (a) 0.
(b) $\sqrt{2}(1 - e^{-1})$.
(c) 0.

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