UNDERGRADUATE TEXTS IN MATHEMATICS

This book is intended as an undergraduate text on real analysis and includes all the standard material such as sequences, infinite series, continuity, differentiation, and integration, together with worked examples and exercises. By unifying and simplifying all the various notions of limit, the author has successfully presented a unique and novel approach to the subject matter that has not previously appeared in book form.

The author defines what is meant by a limit just once, and all of the subsequent limiting processes are viewed as special cases of this one definition. In this way the subject matter attains a unity and coherence that is missing in the traditional approach, and students will be able to fully appreciate and understand the common source of the topics they are studying. These topics are presented as "variations on a theme" rather than essentially different ideas, and this leads to a clearer global view of the subject.

The book is divided into three sections. Part I contains preliminary material on sets, and on real and complex numbers. Part II starts with the definition of a limit and its basic properties, and continues with three basic results: the Intermediate Value Theorem, the Mean Value Inequality, and the Cauchy Criterion—all of which are proved by bisection arguments. The last chapter in this section contains a detailed discussion of infinite series, including a treatment of unordered sums. Part III comprises the standard material in analysis.

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LIMITS
A New Approach to Real Analysis

Alan F. Beardon

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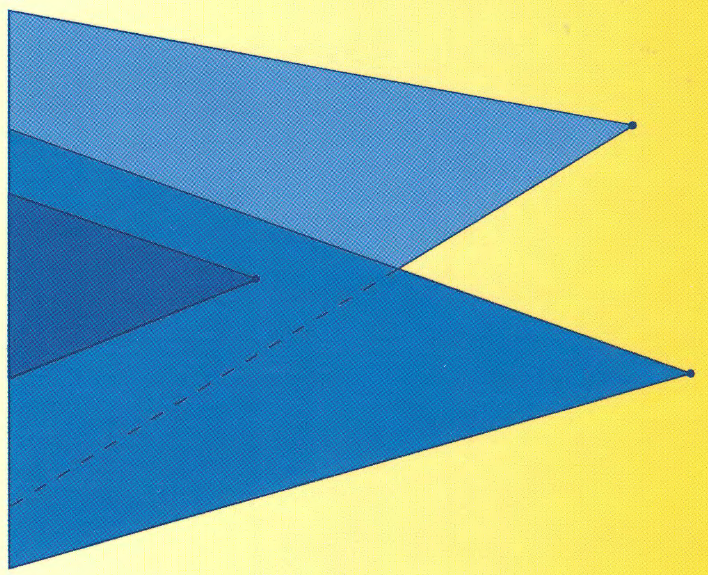

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Alan F. Beardon

Limits

A New Approach to Real Analysis

Springer

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Preface

Broadly speaking, analysis is the study of limiting processes such as summing infinite series and differentiating and integrating functions, and in any of these processes there are two issues to consider; first, there is the question of whether or not the limit exists, and second, assuming that it does, there is the problem of finding its numerical value. By convention, analysis is the study of limiting processes in which the issue of existence is raised and tackled in a forthright manner. In fact, the problem of existence overshadows that of finding the value; for example, while it might be important to know that every polynomial of odd degree has a zero (this is a statement of existence), it is not always necessary to know what this zero is (indeed, if it is irrational, we may never know what its true value is).

Despite the fact that this book has much in common with other texts on analysis, its approach to the subject differs widely from any other text known to the author. In other texts, each limiting process is discussed, in detail and at length before the next process. There are several disadvantages in this approach. First, there is the need for a different definition for each concept, even though the student will ultimately realise that these different definitions have much in common. Next, there is the repetition of remarkably similar results and proofs (perhaps in the hope that by the third or fourth time they will seem easier than before). Thirdly, a rigorous development of 'school mathematics' (the exponential and trigonometric functions, for instance—and this must surely be one of the initial aims in teaching analysis) requires a combination of results taken from different topics in analysis, and this means that in the conventional approach all this has to be left until late in the development. Finally, and perhaps most significantly, in the traditional approach many students finish the course feeling sure that all the ideas they have met have a common thread but are unable to give substance to this feeling.

In this text, we shall define what is meant by a limit just once, and *all of the subsequent limiting processes will be seen as special cases of this one definition*. Accordingly, the subject matter attains a unity and coherence that is missing in the traditional approach. As a by-product of this, we can talk of differentiation, infinite series, continuity, and so on, as early as we wish and, in particular, when we are discussing a careful treatment of school mathematics. Lest those who know the subject should be worried about the level of difficulty, let it be said now that the general definition of a limit is no more complicated than the definition of an equivalence relation (which is standard fare in almost all first-year university courses in mathematics).

The plan of the book is as follows. Part I comprises two chapters; these include some preliminary material on sets, and on real and complex numbers. Many readers will be able to omit most of the material in these chapters. Part II starts with the definition of a limit and its basic properties (in Chapter 3). Chapter 4 contains three basic results (the Intermediate Value Theorem, the Mean Value inequality, and the Cauchy Criterion, all of which are proved by bisection arguments. Chapter 5 contains a detailed discussion of infinite series, including a treatment of unordered sums (which fits well into the general notion of a limit). Part II ends with Chapter 6, which contains a rigorous account of the exponential and trigonometric functions up to and including the periodicity of the exponential function of a complex variable. One of the benefits of this approach is the availability of this material at an early stage. Parts I and II would suffice for a shorter course on the main ideas in analysis at this level, and there can hardly be a better motivation for the analysis than 'closing' the gaps left in earlier treatments. Part III comprises the standard material in analysis, and even this progresses smoothly, since much follows easily from the earlier basic ideas.

The ideas in this text have their roots in discussions of the limiting processes (nets and filters) in topology around 1940 or so. These ideas are given here in a form that (in the view of the author) is suitable for teaching a first course in analysis. The material covered is the standard material that would be found in any first 'serious' course in analysis. Examples occur throughout the text, and there are routine exercises at the end of each section.

Cambridge, England, United Kingdom Alan F. Beardon

Contents

PART I

Foundations

Chapters 1 and 2 contain a brief summary of the basic ideas on sets, functions, and real and complex numbers. Many readers will be able to omit these two chapters, apart perhaps from Section 2.3, in which we discuss the Least Upper Bound Axiom for real numbers.

1 CHAPTER Sets and Functions

Abstract
This chapter contains a brief and informal discussion of sets and functions. Real numbers will be discussed in Chapter 2, and they are used here only to illustrate ideas.

1.1 Sets

We begin with a brief, informal introduction to the elementary ideas in set theory, and we shall ignore completely the substantial difficulties associated with a rigorous discussion. As far as we are concerned a *set* is simply a collection of objects, and to define a set we must describe which objects are in the collection and which objects are not. The objects in a set are the *elements*, or *members*, of that set, and membership is such a basic idea that there is a universal notation for it: $x \in A$ means that x *is a member of the set* A, and $x \notin A$ means that x *is not a member of* A. To avoid tedious repetition, each of the phrases *is a member of*, *is an element of*, *belongs to*, *lies in*, or even just *is in*, are used interchangeably. Likewise, the words *class* and *family* are sometimes used in place of *set*.

If the elements of a set are known explicitly, the set is referred to by enclosing the list of the objects in braces { }; for example, the set containing exactly the objects a, b, and c is $\{a, b, c\}$. If the set is given by some defining condition, it is designated by braces enclosing a variable and (separated by a colon) the condition of membership. For example, $\{x \in \mathbb{R} : x > 0\}$ is the set of objects x in the set $\mathbb{R}$ of real numbers that satisfy $x > 0$. To check that this notation is clear, we confirm that $0 \notin \{1, 2, 3\}$ and $-1 \in \{x \in \mathbb{R} : x \leq 0\}$.

If every element of a set A is also an element of a set B, we say that A is a *subset of* B, or that A is *contained in* B, and write $A \subset B$, or that B *contains* A, and write $B \supset A$. Obviously, $A \subset B$ if and only if for every x, if $x \in A$ then $x \in B$. Two sets A and B are *equal* when they have the same members, and to show that $A = B$ one must show both that $A \subset B$, and that $B \subset A$. We say that A is a *proper subset* of B if $A \subset B$ and $A \neq B$. Note that $A \subset A$, and that $\{x, y\} = \{y, x\}$. Also, $\{1, 2, 2\} = \{1, 2\}$, for any member of one of these sets is also a member of the other; informally, any repetition of an element is redundant.

There is a set with *no* elements; for example, $\{x \in \mathbb{R} : x \neq x\}$. Any set with no elements is automatically a subset of every set and it follows from this that if X and Y are sets without any elements, then $X \subset Y$ and $Y \subset X$, so that $X = Y$. Thus there is exactly one set with no elements; it is called the *empty set* and is denoted by $\varnothing$. A set A is *nonempty* if $A \neq \varnothing$, that is, if A has at least one element.

If A and B are any sets, we define their *union* $A \cup B$ by the condition that $x \in A \cup B$ if and only if $x \in A$ or $x \in B$. Note that in mathematics we always use 'or' in the inclusive sense; that is, 'P or Q' means P, or Q, *or both*. We also define the *intersection* $A \cap B$ of A and B to be the set of objects that lie in both A and B; thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$. Clearly, $A \cup B = B \cup A$ and $A \cap B = B \cap A$. The two sets A and B are *disjoint* if $A \cap B = \varnothing$, and *meet* if $A \cap B \neq \varnothing$. Note that while *disjoint* is standard terminology, *meet* is not.

These ideas generalise to any number of sets. First, given sets A, B, and C, say, the union $A \cup B \cup C$ is the set of objects that are in at least one of these three sets, and their intersection $A \cap B \cap C$ consists of those elements that are in all of them. More generally, suppose that $A_1, A_2, A_3, \ldots$ are sets. Then the sets

$$\bigcup_{n=1}^{\infty} A_n, \quad \bigcap_{n=1}^{\infty} A_n$$

denote the set of objects that lie in A_n for at least one n, and the set of objects that lie in A_n for every n, respectively.

The set of elements in A but not in B is called the *complement of* B *in* A and is denoted by $A - B$, or by $A \backslash B$. Formally, $x \in A - B$ if and only if $x \in A$ and $x \notin B$. Of course, $A - A = \varnothing$, and $A - B = \varnothing$ if and only if $A \subset B$. Note also that for any pair of sets A and B, $A \cup B$ is the union of the three mutually disjoint sets $A - B$, $A \cap B$, and $B - A$.

Exercises

1. Show that $\{x \in \mathbb{R} : -5 < x^2 < 4\} = \{x \in \mathbb{R} : -2 < x < 2\}$.
2. Prove that $A = B$ if and only if $A - B = \varnothing$ and $B - A = \varnothing$.
3. Show that $A - B = A - (A \cap B)$. Show that $A - (A - B) \subset B$, and give an example in which $A - (A - B) \neq B$. Find, and verify, a necessary and sufficient condition on A and B for $A - (A - B) = B$ to be true.
4. Prove each of the following:

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C),$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C), \quad (A \cup B) \cap C = (A \cap C) \cup (B \cap C),$$
$$C - (A \cup B) = (C - A) \cap (C - B), \quad C - (A \cap B) = (C - A) \cup (C - B).$$

5. The *symmetric difference* of two sets A and B is the set $A \Delta B$ of objects in *exactly* one of the sets A and B. Show that

$$A \Delta B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

[See Exercise 2.1.7 for more on this topic.]

6. Show that if a set A contains exactly n elements, then the set of all subsets of A has exactly 2^n elements.
7. Let $A_1, A_2, \ldots$ be any sets. Show that

$$x \in \bigcup_{n=1}^{\infty} \left(\bigcap_{m=n}^{\infty} A_m \right)$$

if and only if $x \in A_k$ for all but a finite number of positive integers k.

Describe what it means for x to belong to

$$x \in \bigcap_{n=1}^{\infty} \left(\bigcup_{m=n}^{\infty} A_m \right).$$

1.2 Ordered pairs

The reader will be familiar with the representation of a point in the plane by an ordered pair (x, y) of real numbers x and y. The numbers x and y are the *first* and *second* coordinates, respectively, of (x, y), and as the reader knows, (x, y) is not the same as (y, x). This description of an ordered pair requires the notion of order (through the words 'first' and 'second'), and it is of theoretical interest to construct a definition of an ordered pair that makes no prior assumptions about, or reference to, order. We begin with a simple result about sets.

Theorem 1.2.1.
Let $X = \{\{a\},\{a,b\}\}$ and $Y = \{\{c\},\{c,d\}\}$. Then $X = Y$ if and only if $a = c$ and $b = d$.

Proof
If $a = c$ and $b = d$, then $X = Y$. Suppose now that $X = Y$. As $\{c\} \in Y$, we see that $\{c\} \in X$, so that $\{c\} = \{a\}$ or $\{c\} = \{a, b\}$. As $\{a\}$ and $\{a, b\}$ both contain a, $\{c\}$ also contains a, so that $a = c$. As $d \in \{c, d\}$, d belongs to one of the elements of X, so that $d = a$ or $d = b$. Likewise, $b = c$ or $b = d$. As $a = c$, these imply that $b = d$, and the proof is complete. ∎

The crucial property of an ordered pair that distinguishes it from a set with two elements is that we can identify its 'first' and 'second' coordinates, so that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. Theorem 1.2.1 provides a purely set-theoretic definition of a set with this property, for if we define

$$(a, b) = \big\{ \{a\}, \{a, b\} \big\}, \tag{1.2.1}$$

then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. The important point here is that (1.2.1) defines an *ordered pair* (a, b) without any reference to the words 'first' and 'second'. Given any two sets X and Y, the set of all ordered pairs (x, y), where $x \in X$ and $y \in Y$, is denoted by $X \times Y$ and is called the *Cartesian product* of X and Y.

Naturally, we should go on to discuss ordered triples (to represent points in Euclidean space), ordered quadruples, and so on, and the simplest way to do this is by induction. Given sets $X_1, X_2, \ldots$, we have already defined what we mean by $X_1 \times X_2$. We now define what is meant by $X_1 \times X_2 \times \cdots \times X_n$ by the inductive definition

$$X_1 \times X_2 \times \cdots \times X_n = \big(X_1 \times X_2 \times \cdots \times X_{n-1}\big) \times X_n.$$

Exercises

1. Show that $(x, y) = (y, x)$ if and only if $x = y$.
2. Show that $A \times B = \varnothing$ if and only if $A = \varnothing$ or $B = \varnothing$.
3. Show that $(A - B) \times C = (A \times C) - (B \times C)$.
4. Show that

$$A \times \left(\bigcap_{n=1}^{\infty} B_n \right) = \bigcap_{n=1}^{\infty} (A \times B_n).$$

State and prove a corresponding result for unions.

5. Show that

$$\left(\bigcap_{n=1}^{\infty} A_n\right) \times \left(\bigcap_{n=1}^{\infty} B_n\right) = \bigcap_{n=1}^{\infty} (A_n \times B_n).$$

Show also that the corresponding statement for unions is false.

1.3 Functions

The reader will be familiar with the idea of a function as a 'rule' that takes an object x to another object $f(x)$. We wish to place the definition of a function on a firm foundation and to dispense with the undefined term 'rule'. The *graph* of a function f is the set of all ordered pairs $(x, f(x))$, so that *two functions are identical if and only if their graphs are identical.* This observation is the key to our formal definition, for it implies that a function and its graph are just two different representations of the same object. As we have defined what we mean by an ordered pair, we can now define a function by its graph without reference to the word 'rule'. Of course, not every set of ordered pairs is the graph of a function, and the crucial requirement for such a set to be a function is that if (x, y) and (x, y') are in the set, then $y = y'$ (that is, each y is uniquely determined by x). These comments motivate the following formal definition of a function.

Definition 1.3.1.
A *function* f is a set of ordered pairs that has the property that if (x, y) and (x, y') are in f, then $y = y'$.

As an illustration, the function f given by the formula $f(x) = x^2$, where $x \in \mathbb{R}$, is now represented by the set $\{(x, x^2) : x \in \mathbb{R}\}$. We shall not persist with this cumbersome notation, and we shall continue to use $f(x) = x^2$ or $x \mapsto x^2$ for this function. However, the point is that our definition now has a firm foundation to which we can return if we need to. If f is a function, and if (x, y) is one of the ordered pairs in f, then y is uniquely determined by f and x, so that we can replace it by the symbol $f(x)$. In this way f is a set of ordered pairs of the form $\big(x, f(x)\big)$ and so is formally identified with its graph. The notation $f(x)$ is not obligatory; algebraists often use xf (with functions 'on the right' and no brackets), and analysts frequently use f_n when the variable n is an integer. We shall sometimes use fx in an expression in which the inclusion of too many brackets is likely to distract the reader.

The *domain* of a function f is the set of x for which some pair (x, y) is in f, and the *image* of f is the set of y for which some pair (x, y) is in f. If f has domain X and image Y, it is automatically a subset of the

Cartesian product set $X \times Y$, and we record this by referring to the function $f : X \to Y$. While this notation is convenient and concise, it is too restrictive, because although we usually know the domain of a function f, only rarely do we know its image. More often, we only know a set that *contains* the image of f, and to be more flexible we agree that in future, the notation $f : X \to Y$ shall mean that X is the domain of f, and that Y contains the image of f.

Given any subset E of the domain of f, we write

$$f(E) = \{f(x) : x \in E\};$$

then the image of $f : X \to Y$ is $f(X)$, and $f(X) \subset Y$. If $f(X) = Y$, we say that $f : X \to Y$ is *surjective*, and that f maps X *onto* Y; otherwise, we say that f maps X *into* Y. Note the important implication that if $f : X \to Y$ is surjective, then every y in Y is of the form $f(x)$ for some x in X.

Let f be a function. As f is a set of ordered pairs, we can always create a new set, which we denote by f^{-1}, of ordered pairs by reversing the order in each pair in f; thus

$$f^{-1} = \{(y, x) : (x, y) \in f\}.$$

We emphasize that the *inverse* f^{-1} of f *always exists as a set of ordered pairs*, but *it need not be a function*. Naturally, it is of interest to determine when f^{-1} is a function. A function f is *injective*, or is an *injection* (or, in older terminology, is *one-to-one*) if, for all pairs (x, y) and (x', y') in f, $y = y'$ implies $x = x'$. An equivalent condition (that is perhaps easier to grasp) is that $x \neq x'$ implies $y \neq y'$ (so that distinct points map to distinct points). The condition for f^{-1} to be a function is that f be injective.

Theorem 1.3.2.
Let f be any function. Then f^{-1} is a function if and only if f is injective, and then f^{-1} is also injective.

Proof
The statements that f^{-1} is a function and that f is injective are both equivalent to the statement that whenever (x, y) and (x', y') are in f, $y = y'$ implies that $x = x'$. Thus f^{-1} is a function if and only if f is injective. Suppose now that f^{-1} is a function. As $(f^{-1})^{-1}$ is a function, namely f, by what we have just proved, f^{-1} is injective. ∎

In order to understand how the inverse function f^{-1} 'reverses' the action of f, we need to discuss the composition of two functions. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions. Then for each x in X there is a unique y in Y such that $(x, y) \in f$, and for this y there is a unique z in Z such that $(y, z) \in g$ (in more familiar language, $y = f(x)$ and $z = g(y)$). The *composition* gf is the set of ordered pairs (x, z) formed in this way, and it is an easy exercise to show that gf is a function from X to Z. Reverting to the usual notation for functions, $(gf)(x) = g(f(x))$, so that $gf(x)$ is obtained by applying f first, and then g. It is now immediate that if f is an injective function with domain X and image Y, then

$$f^{-1}f = \{(x,x) : x \in X\}, \quad ff^{-1} = \{(y,y) : y \in Y\},$$

or, more informally, $f^{-1}f$ is the identity map on X, and ff^{-1} is the identity map on Y. Note that $ff^{-1} \neq f^{-1}f$ unless $X = Y$.

Functions that are both injective and surjective are sufficiently important to warrant their own name.

Definition 1.3.3.
The function $f : X \to Y$ is a *bijection from X to Y* if it is both injective and surjective.

It is immediate that if $f : X \to Y$ is a bijection, then $f^{-1} : Y \to X$ is a bijection. Also, the composition of bijections is a bijection.

Theorem 1.3.4.
If $f:X \to Y$ and $g:Y \to Z$ are bijections, then both $gf:X \to Z$ and $f^{-1}g^{-1}:Z \to X$ are bijections.

Exercises

1. Use the definition of a function to show that the composition gf, where

$$gf = \{(x,z) : \text{for some } y,\ (x,y) \in f \text{ and } (y,z) \in g\},$$

of the two functions $f : X \to Y$ and $g : Y \to Z$ is a function.

2. Prove Theorem 1.3.4.

3. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{E}$ the set of even positive integers. Show that f, defined by $f(n) = n/2$, is a bijection from $\mathbb{E}$ to $\mathbb{N}$. [This shows that there can be a bijection between a set and a proper subset of itself.]

4. Let $\mathbb{N}$ be the set of positive integers. Show that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m,n) = 2^m 3^n$ is injective. Which property of the integers is being used in your proof?

5. Let $f : X \to Y$ be any function. Show
   (i) that $f : X \to Y$ is injective if and only if there is a function $g : Y \to X$ such that for every x in X, $gf(x) = x$, and
   (ii) that $f : X \to Y$ is surjective if and only if there is a function $h : Y \to X$ such that for every y in Y, $fh(y) = y$.

2 CHAPTER Real and Complex Numbers

Abstract

The set of real numbers, whose existence is taken for granted, is described as an ordered field that satisfies the Least Upper Bound Axiom. Complex numbers are introduced, and their basic algebraic properties are discussed. The argument of a complex number is, however, left until later.

2.1 Algebraic properties of real numbers

Real numbers are the the numbers that we have been familiar with since a very early age, and they include the *natural numbers* $1, 2, \ldots$, the *integers* $\ldots, -1, 0, 1, \ldots$, the *rational numbers*, or *fractions* (for example, $1/2$ and $2/3$), the *irrational numbers* (for example, $\sqrt{2}$), and other 'important' numbers like π and e. We shall try to strike a balance between assuming that properties of the real numbers are self-evident (for they are not) and giving a formal treatment of the foundations of real numbers. Briefly, we shall describe their defining properties but take their existence for granted.

The most familiar properties of the real numbers are the algebraic properties of addition (denoted by $+$) and multiplication (denoted by $\times$), and these are best described in the context of a group. A *binary operation* $\circ$ on a set G is a function that takes an ordered pair (x, y) of elements of G

to a third element $x \circ y$ of G. A set G with a binary operation $\circ$ is a *group* if

(i) for all x, y, and z in G, $(x \circ y) \circ z = x \circ (y \circ z)$;
(ii) there is some (necessarily unique) e in G such that for all x in G, $x \circ e = x = e \circ x$;
(iii) for each x in G, there is some x' in G such that $x \circ x' = e = x' \circ x$. We emphasize that the operation $\circ$ acts on the *ordered* pair (x, y), and that we are *not* assuming that $x \circ y = y \circ x$. If, in addition, we have
(iv) for all x and y in G, $x \circ y = y \circ x$, then G is said to be an *abelian*, or *commutative*, group.

For brevity, we now write xy for $x \circ y$. It is easy to see that there can be only one element with the property attributed to e in (ii), for suppose that $\hat{e}$ also has this property. As e satisfies (ii), $\hat{e}e = \hat{e}$, and as $\hat{e}$ satisfies (ii), $\hat{e}e = e$. Thus $\hat{e} = e$. We call e the *identity* (element) of G. Strictly speaking, this argument should have been given before we stated (iii), which refers to e. It is equally easy to see that for each x in G, there is only one element x' as described in (iii); this is *inverse* of x. As $ee = e$, the uniqueness tells us that $e' = e$.

The real numbers are an abelian group with respect to $+$, and the nonzero real numbers are an abelian group with respect to $\times$. Moreover, there is the link

$$a(b + c) = ab + ac$$

between addition and multiplication. Any algebraic structure with these properties is known as a field. More precisely, a *field* is a set F with operations $+$ and $\times$ such that

(F1) F is an abelian group with respect to $+$, with identity 0;
(F2) $F^* = \{x \in F : x \neq 0\}$ is an abelian group with respect to $\times$;
(F3) for all x, y, and z in F, $x \times (y + z) = (x \times y) + (x \times z)$.

For brevity, we shall write xy for $x \times y$, x^2 for xx, and so on. The identity elements for $+$ and $\times$ are denoted by 0 and 1, respectively, and, because $1 \in F^*$, $0 \neq 1$. The inverse of x with respect to $+$ is denoted by $-x$, and subtraction is given by $x - y = x + (-y)$. The inverse of a nonzero y with respect to $\times$ is denoted by y^{-1}, and division by y is given by $x(y^{-1})$. The familiar notation x/y is also used for $x(y^{-1})$.

Many familiar facts about real numbers are valid for a general field; for example, we have the following result.

Theorem 2.1.1.
For all x and y in any field F,

(i) $x0 = 0 = 0x$;
(ii) $xy = 0$ *if and only if* $x = 0$ *or* $y = 0$;

(iii) $(-1)x = -x$.

Proof
For all x, $x = x1 = x(1+0) = x1 + x0 = x + x0$. If we now add $-x$ to both sides of this equation (and use the axioms for a field), we obtain

$$0 = x + (-x) = x + (-x) + x0 = 0 + x0 = x0.$$

Of course, $0x = x0 = 0$.

Suppose now that $xy = 0$ and $y \neq 0$. Then y^{-1} exists and by (i),

$$0 = 0y^{-1} = (xy)y^{-1} = x(yy^{-1}) = x1 = x.$$

We leave the proof of (iii) to the reader. ■

The set $\mathbb{R}$ of real numbers is a field, as are the set of rational numbers, the set of numbers of the form $a + b\sqrt{2}$ with a and b rational, and the integers modulo 3. In addition to its algebraic properties, $\mathbb{R}$ supports an order $>$ (which gives inequalities between numbers), and we discuss this in the next section.

Exercises

In Questions 1–4, all elements are assumed to lie in some given field F.

1. Prove Theorem 2.1.1(iii).
2. Show that $x^2 = y^2$ if and only if $x = y$ or $x = -y$.
3. Show that $-(-x) = x$. Deduce that $x^2 = (-x)^2$, and hence that $(-1)^2 = 1$.
4. Prove that $-1 \neq 0$, and that $(-1)^{-1} = -1$. Show also that $x = x^{-1}$ if and only if $x = 1$ or $x = -1$.
5. Prove that $\{0, 1, 2\}$ with arithmetic modulo 3 is a field in which $2^{-1} = 2$.
6. Show that the set of numbers of the form $a + b\sqrt{2}$, where a and b are rational, is a field. [You should assume here that $\sqrt{2}$ exists and is not rational.]
7. Let X be a nonempty set. Show that for each subset A of X, $A \Delta \varnothing = A$ and $A \Delta A = \varnothing$. [See Exercise 1.1.5.]

   Deduce that the class of all subsets of X with the binary operation $A \Delta B$ is a commutative group with identity $\varnothing$.

   Suppose that $X = \{1, 2, \ldots, 9\}$, $A = \{1, 2, 3, 4, 5\}$, and $C = \{5, 6, 7, 8, 9\}$. By considering $A^{-1} \Delta C$, find all sets B such that $A \Delta B = C$.

2.2 Order

We can discuss order either in terms of the relation > or (equivalently) in terms of the set of positive numbers, and we choose the latter. A field F is an *ordered field* if there is a subset F^+ of *positive elements* of F such that

(O1) if x and y are in F^+, then so are $x+y$ and xy;
(O2) for each x in F, exactly one of $x \in F^+$, $x = 0$, $-x \in F^+$ holds.

We say that x is *negative* if $-x \in F^+$. The *Trichotomy Law* (O2) implies that every x is either positive or negative or zero, and that it is only one of these.

Given an ordered field F, we introduce the usual relation $>$, namely that $x > y$ if and only if $x - y \in F^+$. Thus (in agreement with the familiar terminology) x is positive if and only if $x > 0$. The relations $<$, $\leq$, and $\geq$ are now defined in the usual way. Once again, many familiar facts about $\mathbb{R}$ are valid in this more general situation. For example, for every x, y, and z, if $x > y$ then $x + z > y + z$. Indeed, $(x+z)-(y+z) = x - y$ and this is in F^+. The corresponding rule for multiplication has two cases: if $x > y$ and $z > 0$ then $xz > yz$, whereas if $x > y$ and $z < 0$ then $yz > xz$. Now take any x in F. If $x \in F^+$, then $x^2 \in F^+$. If $-x \in F^+$, then $(-x)^2 \in F^+$, and as $x^2 = (-x)^2$ (see Exercise 2.1.3), again $x^2 \in F^+$. If $x = 0$ then $x^2 = 0$. We have now shown that for every x, $x^2 \geq 0$. In particular, as $1^2 = 1$ and $1 \neq 0$, we see that $1 > 0$, and hence $1 \in F^+$. From this we have $1 + 1 \in F^+$, and so on, and because (from (O2)) $0 \notin F^+$, we see now that, for example, $1 + 1 + 1 \neq 0$. Because of this, an ordered field cannot be a finite set.

Exercises

1. Prove that in any ordered field, if $x > y$ and $z > 0$ then $xz > yz$, and if $x > y$ and $z < 0$ then $yz > xz$.
2. Show that $\{0, 1, 2\}$ with arithmetic modulo 3 is not an ordered field.
3. Let F be an ordered field, and define the function $x \mapsto |x|$ (of F into itself) by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Prove that for all x and y, $|x - y| = |y - x|$ and $|x + y| \leq |x| + |y|$. Deduce that

$$|x - y| \leq |x - z| + |z - y|.$$

This provides a 'distance' on F without any prior geometric assumptions.

2.3 Upper and lower bounds

The arithmetic and order properties of a field are not enough by themselves to establish many properties of $\mathbb{R}$ that we have accepted without question in the past. It is not true, for example, that any ordered field contains a solution to the equation $x^2 = 2$; indeed, the set $\mathbb{Q}$ of rational numbers is an ordered field, yet as is well known, there is no rational solution of $x^2 = 2$. We are forced to conclude that $\mathbb{R}$ must have some additional property that $\mathbb{Q}$ fails to have, and that leads to the existence of $\sqrt{2}$ (and much more). We turn now to this additional property.

A nonempty subset E of an ordered field F is

(i) *bounded below* if there is some a such that for all x in E, $x \geq a$;
(ii) *bounded above* if there is some b such that for all x in E, $x \leq b$;
(iii) *bounded* if it is bounded both above and below.

Any a satisfying (i) is a *lower bound* of E, and any b satisfying (ii) is an *upper bound* of E. Note that upper and lower bounds of E *need not be in E*; for example, 0 is a lower bound of the set of positive numbers but is not itself positive.

If b is an upper bound of E, and if $b \leq c$, then c is also an upper bound of E, and this suggests that we should seek the smallest possible upper bound of E. Accordingly, we say that u the *least upper bound*, or the *supremum*, of E if u is an upper bound of E and if $u \leq b$ for every upper bound b of E. There is a linguistic ambiguity here that can be a source of confusion. We are *not* defining the least upper bound in the sense that we are applying the adjective 'least' to 'upper bound', for we do not know that there is a smallest object of this type (by analogy, we cannot talk of the 'least' positive number). What we are doing is to define the entire (inseparable) phrase 'least upper bound' (which might be better written as least-upper-bound). Of course, when the least-upper-bound exists, it is indeed the smallest possible upper bound, and this linguistic ambiguity disappears.

While it is clear that any set E can have *at most one* supremum (for if u and u' are two such, then $u \leq u'$ and $u' \leq u$, so that $u = u'$), it is not clear that it actually has a supremum. In fact, it is not possible to derive the existence of a supremum from the arithmetic and ordering properties of the field, and in view of this, we introduce the following axiom.

The Least Upper Bound Axiom.

The set of $\mathbb{R}$ of real numbers is an ordered field in which every nonempty subset that is bounded above has a supremum.

With this we have reached our objective of characterising $\mathbb{R}$, for this defines $\mathbb{R}$ in the sense that any other set that has these properties is merely a relabelling of $\mathbb{R}$. We omit both a formal statement and a proof of this.

There is an analogous discussion of lower bounds, and as $x < y$ if and only if $-y < -x$, this can be extracted (without further assumptions) from the theory of upper bounds. The number ℓ is the *infimum*, or the *greatest lower bound*, of a nonempty set E if ℓ is a lower bound of E and if $b \leq \ell$ for every lower bound b of E. For emphasis, we state and prove the *theorem* that corresponds to the Least Upper Bound Axiom.

Theorem 2.3.1.
Let E be a subset of $\mathbb{R}$ that is nonempty and bounded below. Then E has a greatest lower bound.

Proof
Let $E' = \{x \in \mathbb{R} : -x \in E\}$. Take x in E; then $-x \in E'$, so that $E' \neq \varnothing$. Let b be a lower bound of E. If $y \in E'$ then $-y \in E$, so that $-y \geq b$, and hence $y \leq -b$. We deduce that the least upper bound, say k, of E' exists, and it is easy to see that $-k$ is the greatest lower bound of E. We leave the reader to complete the details of this proof. ■

In order to appreciate both the necessity and the power of the Least Upper Bound Axiom, we use it to prove the following two results.

Theorem 2.3.2.
The set $\mathbb{N}$ of natural numbers is not bounded above.

Theorem 2.3.3.
Each positive number a has a unique positive square root $\sqrt{a}$.

Proof of Theorem 2.3.2.
Suppose that $\mathbb{N}$ is bounded above. Then $\mathbb{N}$ has a least upper bound, say N. As $N - 1 < N$, $N - 1$ is not an upper bound of $\mathbb{N}$, so that there is some m in $\mathbb{N}$ with $m > N - 1$. This, however, implies that $m + 1 \in \mathbb{N}$ and $m > N$, and this is a contradiction. ■

Observe that Theorem 2.3.2 yields the following fact.

Corollary 2.3.4.
There is no smallest positive number.

Indeed, if $x > 0$, then $1/x$ is not an upper bound of $\mathbb{N}$, so there is a positive integer n with $0 < 1/n < x$. This argument shows that Theorem 2.3.2 is equivalent to the assertion that $1/n \to 0$ as $n \to \infty$, a result that is truly fundamental in any discussion of limits.

Proof of Theorem 2.3.3.

The set $E = \{x \in \mathbb{R} : x \geq 0,\ x^2 < a\}$ contains 0 and is bounded above by $1 + a$. Thus E has a least upper bound, say k, where $k \geq 0$. Suppose now that $k^2 < a$. Then for for each positive integer n,

$$\left(k + \frac{1}{n}\right)^2 = k^2 + \frac{2k}{n} + \frac{1}{n^2} \leq k^2 + \frac{2k+1}{n} < a$$

when $n \geq (2k+1)/(a - k^2)$. As such n exist (by Theorem 2.3.2), we have found a number $k + 1/n$ that is in E and that is greater than k. As this cannot be so, we conclude that $k^2 \geq a$. A similar argument shows that $k^2 \leq a$, and hence that $k^2 = a$; we omit the details. The uniqueness is clear. ■

Next, we state the Principle of Induction (see the Appendix for more details) and the Binomial Theorem (which can be proved by induction).

The Principle of Mathematical Induction.

Suppose that a property $P(n)$ of an integer n is true when $n = 1$, and that for every positive integer n, if $P(n)$ is true then $P(n+1)$ is true. Then $P(n)$ is true for every positive integer n.

The Binomial Theorem.

For any real numbers a and b, and any positive integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

where $0! = 1$, and the binomial coefficients are given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Exercises

1. Theorem 2.3.3 guarantees that $\sqrt{2}$ exists. Prove that it is not rational.
2. Given a nonempty set E of real numbers, let $E' = \{x^2 : x \in E\}$. Show that E' is bounded above if and only if E is bounded.
3. Suppose that the sets A and B are nonempty and bounded above. Write down an expression for the supremum of $A \cup B$ in the terms of the suprema of A and B, and justify your answer.
4. Prove, by induction, that for n in $\mathbb{N}$,

$$\sqrt{\frac{5}{4(4n+1)}} \leq \frac{1.3.5\ldots(2n-1)}{2.4.6.\ldots(2n)} \leq \sqrt{\frac{3}{4(2n+1)}}.$$

5. Show (i) by induction, and (ii) by using the Least Upper Bound Axiom, that any set of n real numbers contains a largest element. Show also that it contains a smallest element.

6. Use induction to prove Bernoulli's inequality that: if $x > -1$, then $(1+x)^n \geq 1+nx$. Where do you need to know that $x > -1$? Give an example in which $1+nx > (1+x)^n$.

2.4 Complex numbers

The Cartesian coordinate system on the Euclidean plane associates to each point a unique ordered pair of coordinates. Instead of referring to a point by its coordinates (x, y), we can refer to it as a *complex number* $x+iy$. There is nothing mysterious about this, and $x+iy$ is simply another way of writing (x, y). The set of complex numbers is denoted by $\mathbb{C}$.

The addition and multiplication of complex numbers are defined by

$$(x+iy)+(x'+iy') = (x+x')+i(y+y'),$$
$$(x+iy)(u+iv) = (xu-yv)+i(xv+yu),$$

and these lead to the following result.

Theorem 2.4.1.
The set $\mathbb{C}$ of complex numbers is a field.

Proof
The proof involves much tedious checking, and we only mention the key facts. The identity element for addition is $0+i0$, and the additive inverse of $x+iy$ is $(-x)+i(-y)$. In the context of a field, $x+iy$ is nonzero if it is not the additive identity $0+i0$, and this is so if x and y are not both zero, for $x+iy = 0+i0$ if and only if $(x, y) = (0, 0)$. The multiplicative identity is $1+i0$, and if $x+iy \neq 0+i0$, the multiplicative inverse of $x+iy$ is

$$\left(\frac{x}{x^2+y^2}\right)+i\left(\frac{-y}{x^2+y^2}\right).$$

The verification of the axioms for a field is straightforward, and we omit the details. ∎

Note that $\mathbb{C}$ cannot be made into an ordered field. Indeed, we saw in §2.2 that if a is in an ordered field F, then $a^2 \geq 0$, so that there is no solution in F of $x^2 = -1$. However, this equation does have a solution in $\mathbb{C}$, for

$$(0+i1)^2 = -1+i0 = -(1+i0), \tag{2.4.1}$$

so that $\mathbb{C}$ cannot be made into an ordered field (in any way). This means, and this is worth emphasizing here, that *we cannot write down inequalities between complex numbers*. Depite this lack of order, there are immense advantages in working in $\mathbb{C}$. For example, in an ordered field only non-negative numbers can have square roots, whereas in $\mathbb{C}$, every number has a square root (see Exercise 2.4.1).

The arithmetic of complex numbers allows us to regard each complex number, say $x + iy$, as an expression in the algebraic symbols x, i, and y, and to add and multiply complex numbers as though these symbols were real numbers, but with the additional condition that $i^2 = -1$. Moreover, as

$$(a + i0) + (x + iy) = (a + x) + iy,$$

$$(a + i0)(x + iy) = (ax) + i(ay),$$

it is not necessary to adhere strictly to the notation $x + iy$. In future, we shall write x in place of $x + i0$, iy instead of $0 + iy$, and $x + i$ instead of $x + 1i$, and this will not introduce any inconsistencies into our algebra. As a consequence of this, we can regard the set $\mathbb{R}$ of real numbers as a subset of $\mathbb{C}$ (this is the same as regarding $\mathbb{R}$ as the first coordinate axis in the plane $\mathbb{C}$). The existence of $\sqrt{-1}$ (which caused much debate in the distant past) is not a problem for us, for (2.4.1) can now be written as $i^2 = -1$.

It is not necessary always to use the format $x+iy$ for a complex number, and any single symbol will suffice. By far the most common symbol used is z. We shall frequently write $z = x + iy$, where x and y are real, and in future (and unless we state otherwise), we always assume that in any expression such as this the variables x and y are real. We call x and y the *real part*, and the *imaginary part*, of z, respectively, and we write $x = \text{Re}[z]$, and $y = \text{Im}[z]$. The *conjugate* $\bar{z}$ of z is defined by $\bar{z} = x - iy$, so that in the plane, $\bar{z}$ is the mirror image of z in the real axis. It is easy to see that

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}, \quad \overline{z + w} = \bar{z} + \bar{w}, \quad \overline{zw} = \bar{z}\,\bar{w}.$$

Before moving on, we remark that there is a Binomial Theorem for complex numbers, namely that if z and w are complex numbers, then

$$(z + w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}.$$

The proof, as in the real case, is by induction.

As a complex number is just a representation of a point in the Euclidean plane, we already have a (geometric) concept of distance between complex numbers. Let $z = x + iy$ and $w = u + iv$. Then the *distance* $|z - w|$ between z and w is the distance between the points (x, y) and (u, v) in the plane, so that we now take, *as a definition,*

$$|z - w| = \sqrt{(x-u)^2 + (y-v)^2},$$

where the nonnegative square root is taken. In particular,

$$|z| = \sqrt{x^2 + y^2},$$

and this is the *modulus* of the complex number z. The important formula

$$|z|\,|w| = |zw|$$

follows directly from the identity

$$(x^2 + y^2)(u^2 + v^2) = (xu - yv)^2 + (xv + yu)^2. \qquad (2.4.2)$$

Also, and these are trivial, we have

$$z\bar{z} = |z|^2, \quad |\bar{z}| = |z|, \quad z^{-1} = \frac{\bar{z}}{|z|^2},$$

where in the last formula we are assuming that $z \neq 0$.

The properties of the distance given in the next result are used constantly, and these must, of course, be derived from the algebraic definition of distance and not taken as 'self-evident' geometric facts.

Theorem 2.4.2.

For all complex numbers z_1, z_2, and z_3,

(i) $|z_1 - z_2| \geq 0$, *with equality if and only if* $z_1 = z_2$;
(ii) $|z_1 - z_2| = |z_2 - z_1|$;
(iii) $|z_1 - z_3| \leq |z_1 - z_2| + |z_2 - z_3|$.

Proof

First, (i) and (ii) are trivial. Next, (iii) is equivalent to

$$|z + w| \leq |z| + |w|, \qquad (2.4.3)$$

where $z = z_1 - z_2$ and $w = z_2 - z_3$, so it suffices to prove this inequality. Let $z = x + iy$ and $w = u + iv$. Then, using (2.4.2) with y replaced by $-y$,

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= |z|^2 + 2(xu + yv) + |w|^2 \\ &\leq |z|^2 + 2|z|\,|w| + |w|^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

We now take the nonnegative square root of each side of this inequality and obtain (2.4.3). The inequality (iii) in Theorem 2.4.2 is known as the *Triangle Inequality,* because when interpreted geometrically, it implies that the length of a side of the triangle (with vertices z_1, z_2, and z_3) is not more than the sum of the lengths of the other two sides. ∎

We have seen that the Triangle Inequality is equivalent to the inequality (2.4.3), and we end with a useful corollary of this. For all z and w,

$$|z| \leq |z-w| + |w|, \quad |w| \leq |w-z| + |z|,$$

so that

$$\Big| |z| - |w| \Big| \leq |z-w|. \tag{2.4.4}$$

After replacing w by $-w$ we also obtain

$$\Big| |z| - |w| \Big| \leq |z+w|. \tag{2.4.5}$$

We end with a comment about the use of polar coordinates for complex numbers. In elementary treatments of complex numbers we often find the expressions

$$z = re^{i\theta}, \quad z = r(\cos\theta + i\sin\theta),$$

where $r = |z|$. We have already defined $|z|$, but there is a problem with θ. What, for example, is an 'angle', and how do we measure 'angles' without calculus? The solution to these difficulties is to defer any discussion of the argument of z until after we have placed the theory of the trigonometric functions cos and sin on a firm analytic foundation (which we do in Chapter 6). Then the argument of z can be defined in terms of the inverse trigonometric functions $\sin^{-1}$ and $\cos^{-1}$, still without depending on the notion of an angle. As convenient as the argument of a complex number is, *it is not at this point justifiable or necessary.*

Exercises

1. Suppose that $w = u + iv$, where $v \neq 0$. Show that

$$\left(\sqrt{\frac{|w|+u}{2}} + i\frac{v}{|v|}\sqrt{\frac{|w|-u}{2}}\right)^2 = w.$$

2. Show that for any complex numbers z and w,

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2.$$

   Interpret this in terms of the quadilateral in the plane with vertices 0, z, w, and $z + w$.

3. Show (by induction) that $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$.

4. Show that

$$|z_1 + z_2 + \cdots + z_n| \geq |z_1| - \Big(|z_2| + \cdots + |z_n|\Big).$$

Show also that for any r in $\{1, 2, \dots, n\}$,

$$|z_1 + z_2 + \cdots + z_n| \geq \Big(|z_1 + \cdots + z_r|\Big) - \Big(|z_{r+1}| + \cdots + |z_n|\Big),$$

but that it is not always true that for any r,

$$|z_1 + z_2 + \cdots + z_n| \geq \Big(|z_1| + \cdots + |z_r|\Big) - \Big(|z_{r+1}| + \cdots + |z_n|\Big).$$

5. Prove by induction that if z is any complex number with $z \neq 1$, then

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Find a formula for $1 + 2z + 3z^2 + \cdots + (n-1)z^n$ and prove this by induction.

6. Suppose that $z_1 + z_2 + z_3 = z_1 z_2 z_3$. Show that z_1, z_2, and z_3 cannot all lie above the real axis.

2.5 Notation

We end this section by reminding the reader of the following notation that will be in force throughout this book:

$\mathbb{N}$ denotes the set $\{1, 2, 3, \dots\}$ of natural numbers;
$\mathbb{Z}$ denotes the set of integers;
$\mathbb{Q}$ denotes the set of rational numbers;
$\mathbb{R}$ denotes the set of real numbers; and
$\mathbb{C}$ denotes the set of complex numbers.

Of course, $\mathbb{N}$ is also the set of positive integers.

Finally, for any nonempty subset E of $\mathbb{R}$, the supremum, or the least upper bound, of E (when it exists) is denoted by $\sup E$ or $\operatorname{lub} E$. Likewise, when it exists, the infimum, or greatest lower bound, of E is denoted by $\inf E$ or $\operatorname{glb} E$.

PART II

Limits

Chapter 3 contains an introduction to limits through one single definition and many illustrative examples. These examples include sequences, series, continuous functions, derivatives and integrals. Chapter 4 contains some elementary basic theorems about limits that are used in Chapter 5 to study infinite series. In Chapter 6 we place the theory of the exponential and trigonometric functions on a firm analytic foundation and then use this to develop the argument of a complex number.

3 CHAPTER Limits

Abstract
In this chapter we define what is meant by a limit. We give many examples and prove some elementary theorems.

3.1 Introduction

Many results in analysis can be proved by using sequences. Informally, a *sequence* is an infinite list of numbers; for example, $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$. The notation $\ldots$ is meant to imply that the omitted terms are exactly what you would expect them to be, but as an appeal to each reader's personal expectation is unsatisfactory, it is desirable (and some would say essential) to write down the formula for the nth term of the sequence. For the sequence given above, the formula is $n/(n+1)$. At this point we have accepted, perhaps without even realising it, that a sequence is nothing more that a function f defined on the set $\{1, 2, 3, \ldots\}$. This set carries its natural order, namely $1 < 2 < 3 < \cdots$, and the terms in the sequence are arranged in this order, namely $f(1), f(2), f(3), \ldots$. *It is precisely this order* (inherited from $\mathbb{N}$) *that distinguishes the sequence* $f(1), f(2), f(3), \ldots$ *from the set* $\{f(n) : n \in \mathbb{N}\}$. For example, if $f(n) = 1$ for all n, the sequence is $1, 1, 1, \ldots$ and this is not the same as the set $\{f(n) : n \in \mathbb{N}\}$, for this set is simply $\{1\}$.

In addition to having a intuitive idea of what a sequence is, most readers will have an intuitive notion of convergence and will accept that roughly speaking, the sequence $f(1), f(2), f(3), \ldots$ *converges* to the value a if $f(n)$ gets arbitrarily close to a as n increases. Notice that the phrase 'as n increases' *only makes sense because the domain of the sequence is ordered.* The essential difference between a sequence and a general function is that the domain of a function is a set, whereas the domain of a sequence is an *ordered* set.

Analysis is the study of all limiting processes, and we propose now to take a broader view and to study limits by allowing ourselves the freedom to use *functions defined on any ordered set.* This mild generalisation allows us to give a single general definition of a limit (that encompasses all the examples one normally meets in analysis) and thereby brings to the discussion a greater economy, unity, and clarity than is otherwise possible. We now embark upon a discussion of limits. First, we shall discuss which types of ordered, or *directed*, sets are appropriate; then we shall define what is meant by a limit and finally, we prove some elementary results about the general limiting process.

3.2 Directed sets

This section is devoted to those orders or, as we shall call them, *directions*, on a set that are fundamental to the limiting process. A direction $\succ$ on a set X is a relation that has much in common with the ordering relation $>$ (greater than) on $\mathbb{R}$, and where for 'greater than' we write $x > y$, for a direction we shall write $x \succ y$. There are, however, two very important differences between $>$ and $\succ$. First, the relation $>$ is defined only on $\mathbb{R}$, whereas a direction $\succ$ can be defined on any set. Next, and this is more subtle, every pair x and y of real numbers are comparable in the sense that either $x > y$ or $x = y$ or $y > x$, but in the case of a direction, *not every pair x and y need be comparable.* Although in many cases of interest they so be, it is not necessary to insist that they so be, and by not doing so we obtain much greater flexibility. We now give a formal definition of a direction on a set.

Definition 3.2.1.
A relation $\succ$ on a nonempty set X is a *direction* on X if $\succ$ has the two properties

(i) for all x, y, and z in X, if $x \succ y$ and $y \succ z$ then $x \succ z$;
(ii) for all x and y in X, there is some z in X with $z \succ x$ and $z \succ y$.

If $\succ$ is a direction on X, we say that $\succ$ *directs* X, and that $(X, \succ)$ is a *directed set.* If $x \succ y$, we say that x *dominates* y. We emphasize again that we are *not* assuming that for all x and y either $x \succ y$, $x = y$, or $y \succ x$. In the case of the usual direction $>$ (greater than) on $\mathbb{R}$, the fact that every two numbers x and y are comparable means that the maximum $\max\{x, y\}$ of the two numbers exists (and is either x or y). In the case of a general direction, if x and y are not comparable we cannot find their maximum, but *the role of the maximum is taken over by the element z whose existence is asserted in Definition 3.2.1(ii).*

The following examples illustrate the rich variety of situations covered by the idea of a direction and in order to gain familiarity with the concept, *we strongly recommend* that in each case, the reader verify that $(X, \succ)$ is a directed set.

(1) Let $X = \mathbb{R}$, and define $x \succ y$ if and only if $x \geq y$.
(2) Let $X = \mathbb{R}$, and define $x \succ y$ if and only if $x < y$.
(3) Let $X = \{1, 2, 3, \ldots\}$, and define $x \succ y$ if and only if $x > y$.
(4) Let $X = \mathbb{R}$, and define $x \succ y$ if and only if $|x| > |y|$.
(5) Let $X = \{x \in \mathbb{R} : x \neq 0\}$, and define $x \succ y$ if and only if $|x| < |y|$. Note that here, $\succ$ is *not* a direction on $\mathbb{R}$, for Definition 3.2.1(ii) would fail when $x = 0$ and $y = 0$. This is an important observation, which has implications for limits, and we shall return to this point later.
(6) Let X be the set of subsets of $\mathbb{R}$ of the form (a, b), where $a < 0 < b$ and

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

For A and B in X, define $A \succ B$ if and only if $A \subset B$. Note that if A and B are in X, then $A \cap B$ is in X and also $A \cap B \succ A$ and $A \cap B \succ B$. In this case, $(-2, 1)$ and $(-1, 2)$ are elements of X that are not comparable.
(7) Let E be any nonempty set, let X be the set of finite subsets of E, and define $A \succ B$ if and only if $A \supset B$.
(8) Let X be the set of real-valued functions on $\mathbb{R}$, and for f and g in X, define $f \succ g$ if and only if for all x, $f(x) \geq g(x)$. Again, not every f and g in X are comparable.

Consider (for the moment, informally) the sequence given by the function $f(n) = [n(n-1)]^{-1}$. As $f(n)$ is not defined when $n = 1$, it is natural to restrict the domain of f to be the set $\{2, 3, 4, \ldots\}$. For exactly the same reason, we shall need to use corresponding subsets of a directed set X, and these will be, of course, the set of elements x that dominate some given element a. We call these sets the *final segments* of X.

Definition 3.2.2.
The *final segment* of $(X, \succ)$ determined by the element w of X is the set $X(w)$ given by $X(w) = \{x \in X : x \succ w\}$.

Just as $\{2, 3, 4, \ldots\}$ inherits an order from $\mathbb{N}$, so a final segment of X inherits an order from X. This is confirmed in our next result.

Theorem 3.2.3.

Suppose that $(X, \succ)$ *is a directed set and* $w \in X$. *Then* $X(w)$ *is nonempty, and* $\succ$ *is a direction on* $X(w)$.

Proof

The proof is easy, but this does not diminish the importance of the result. First, we take $x = w$ and $y = w$ in Definition 3.2.1(ii) to conclude that there is some z in $X(w)$; thus $X(w) \neq \varnothing$. We must now show that $\succ$ is a direction on the set $X(w)$. Because $\succ$ is transitive on X (that is, (i) of Definition 3.2.1 holds), it is also transitive on $X(w)$. Suppose now that x and y are in $X(w)$. Then there is some z in X with $z \succ x$ and $z \succ y$. As $z \succ x$ and $x \succ w$, we have $z \succ w$, so that $z \in X(w)$. Thus $z \in X(w)$, $z \succ x$, and $z \succ y$, and this completes the proof that $\succ$ is a direction on $X(w)$. ■

We now give the basic rules for final segments; these are little more than a restatement of the definition of a direction.

Theorem 3.2.4.

Let $(X, \succ)$ *be a directed set and suppose that* x *and* y *are in* X. *Then*

(i) *if* $x \succ y$ *then* $X(x) \subset X(y)$;
(ii) *there is some* z *in* X *with* $X(z) \subset X(x) \cap X(y)$.

Proof

Suppose first that $x \succ y$. If $z \in X(x)$ then $z \succ x$, so that $z \succ y$. This proves (i). To prove (ii), recall that there is some z in X with $z \succ x$ and $z \succ y$. Thus, by (i), $X(z) \subset X(x) \cap X(y)$. ■

The idea of a direction on a set X can be conveniently, and informally, illustrated in terms of 'shadows'. Given a directed set $(X, \succ)$, the final segment $X(x_0)$ determined by x_0 is $\{x : x \succ x_0\}$, and this is illustrated in Figure 3.2.1. The basic properties (i) and (ii) of Theorem 3.2.4 are then illustrated in Figure 3.2.1 and 3.2.2, respectively.

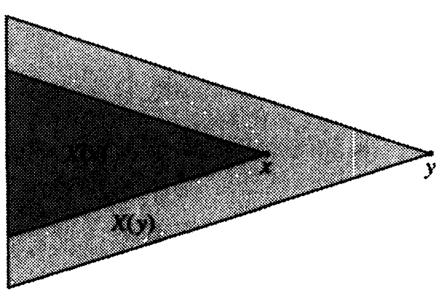

Figure 3.2.1

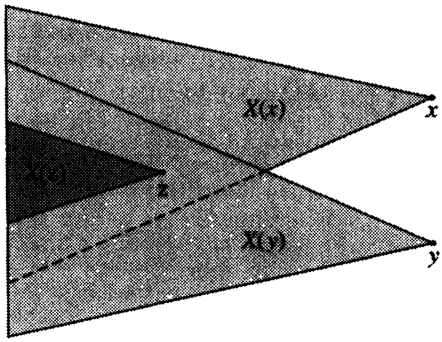

Figure 3.2.2

We end with some useful terminology. We shall say that a statement $P(x)$, where x lies in some directed set X, is true *for almost every x*, or that it is *eventually true*, if it is true for all x lying in some final segment $X(x_0)$ of X. With this, we have the following result, which will be used frequently throughout the text.

Theorem 3.2.5.
Let $(X, \succ)$ be any directed set. Suppose that $P(x)$ is true for almost every x and that $Q(x)$ is true for almost every x, and let $R(x)$ be the statement that both $P(x)$ and $Q(x)$ are true. Then $R(x)$ is true for almost every x.

Proof
There is some x_P such that $P(x)$ is true when $x \succ x_P$, and there is some x_Q such that $Q(x)$ is true when $x \succ x_Q$. As $\succ$ is a direction, there is some x_0 such that $x_0 \succ x_P$ and $x_0 \succ x_Q$. If $x \succ x_0$, then $x \succ x_P$ and $x \succ x_Q$, so that both $P(x)$ and $Q(x)$ are true. ■

Exercises

1. Verify that examples (1)–(8) are directed sets.
2. Let $(X, \succ)$ be a directed set and suppose that $x_1, \dots, x_n$ are in X. Show (by induction) that there is some y in X such that $y \succ x_j$ for $j = 1, \dots, n$.
3. Suppose that for $j = 1, \dots, n$, the statement $P_j(x)$ is true for almost every x. Let $P(x)$ be the statement that $P_1(x), \dots, P_n(x)$ are all true. Show that $P(x)$ is true for almost every x.

   Suppose that $P(x)$ is true for almost every x and that $Q(x)$ is true for almost every x. Let $R(x)$ be the statement 'either $P(x)$ is true or $Q(x)$ is true'. Is $R(x)$ true for almost every x?
4. Suppose that $(X, \succ)$ is a directed set. Show that the relation $\succeq$ defined by $x \succeq y$ if and only if $x \succ y$ or $x = y$ is a direction on X that satisfies $x \succeq x$.

5. Let X be any set and let $\mathcal{S}$ be the collection of all nonempty subsets of X. Which, if any, of the following definitions define a direction on X?
   (i) $A \succ B$ if and only if $A \subset B$;
   (ii) $A \succ B$ if and only if $A \supset B$.

6. Let $(X, \succ)$ be a directed set and let X_0 be a subset of X with the property that for all x in X there is an x_0 in X_0 such that $x_0 \succ x$. Show that $(X_0, \succ)$ is a directed set.

7. Show that the relation $<$ in an ordered field F is a direction on F (thus every ordered field is, in a natural way, a directed set).

8. Let $(A, \succ_A)$ and $(B, \succ_B)$ be directed sets. Show that each of the following is a direction on $A \times B$;
   (i) $\succ$, where $(a, b) \succ (a', b')$ if and only if $a \succ_A a'$ and $b \succ_B b'$;
   (ii) $\succ_L$, where $(a, b) \succ_L (a', b')$ if and only if either $a \succ_A a'$, or $a = a'$ and $b \succ_B b'$.

3.3 The definition of a limit

As this book is devoted to a study of a variety of limiting process, it is appropriate to begin with an informal discussion of limits. Most of us have an intuitive idea of limits, and few would disagree with the assertion that $x^2 \to 0$ as $x \to 0$. There would also be little dissent that $1/n \to 0$ as $n \to \infty$, although here the situation is less clear because ∞ is not a number, and this raises the question of what $n \to \infty$ means. The answer to this question lies at the heart of the limiting process. When we are taking the limit of $f(x)$ with respect to a variable x, we are *not* assuming that x moves towards some preassigned quantity (although this often is the case); instead, we are assuming that the variable x lies in a directed set X and that x 'moves' in the given direction on X. This allows us to consider the limit as $n \to \infty$ *without having to worry about the meaning, or the existence, of* ∞; all we need is the natural direction (given by $>$) on the set $\mathbb{N}$ so that n can 'move' along $\mathbb{N}$ in this direction.

Suppose that f is a function defined on a directed set X with direction $\succ$. The intuitive idea of f tending to the limit α with respect to the direction $\succ$ is that $f(x)$ *is arbitrarily close to* α *when* x *has moved sufficiently far in the direction* $\succ$. As we wish to be precise, there are two phrases here which need clarification, namely *arbitrarily close to* and *sufficiently far in the direction* $\succ$, and we shall consider each in turn.

The phrase *arbitrarily close to* means that we can prescribe exactly how close to α we require $f(x)$ to be; in other words, we can specify what would be acceptable error if we were to take $f(x)$ instead of α. This error is specified in terms of the distance from α; we may require, for example, that $|f(x) - \alpha| < 1$. In the definition of a limit we are being invited to prescribe the acceptable error, which, naturally, we normally think of as being some small positive number. Custom dictates that this is almost always denoted by the Greek letter ε (epsilon); thus, almost always, we shall be requiring that given any positive ε, $|f(x) - \alpha| < \varepsilon$.

The second phrase that required scrutiny is *sufficiently far in the direction* $\succ$, and this is easily dealt with. By this we simply mean 'lying in some final segment $X(x_0)$'. The definition of $f(x) \to \alpha$ should therefore read something like this: we may specify any acceptable (positive) error, say ε, and then we can claim that there is some final segment $X(x_0)$ such that if x lies in $X(x_0)$, then $f(x)$ lies within a distance ε of α. The formal definition now follows.

Definition 3.3.1.
Let X be a set directed by $\succ$, and suppose that $f : X \to \mathbb{R}$ (or $f : X \to \mathbb{C}$) is any function. We say that f *tends to the number* α *with respect to the direction* $\succ$, and write $\lim_{\succ} f = \alpha$, or simply $f(x) \to \alpha$, if given any positive number ε, there exists some x_0 in X such that $|f(x) - \alpha| < \varepsilon$ whenever $x \succ x_0$. An equivalent statement is that $f(x) \to \alpha$ if given any positive number ε, $|f(x) - \alpha| < \varepsilon$ for almost every x in X.

The rest of the text is devoted to a discussion of the limiting processes that lie at the heart of analysis, namely the convergence of sequences and infinite sums, the continuity of functions, and differentiation and integration. Before we move on to discuss these substantial matters, we pause to give four extremely simple examples of limits.

EXAMPLE 3.3.2.
The simplest function for which a limit exists is a constant function. If $(X, \succ)$ is a directed set and if f is the constant function on X with value c, then $\lim_{\succ} f = c$. To verify this, take any positive ε, and choose any x_0 in X. Then $|f(x) - c| = 0 < \varepsilon$ when $x \in X(x_0)$, so certainly $\lim_{\succ} f = c$. □

EXAMPLE 3.3.3.
Suppose that $X = \mathbb{N}$; let $\succ$ be the usual ordering $>$ (so that $m \succ n$ if and only if $m > n$), and let $f(n) = 1/n$. Then $\lim_{\succ} f = 0$. To prove this, take any positive ε, and choose an integer n_0 such that $n_0 > 1/\varepsilon$ (see Theorem 2.3.2, and the remarks following Corollary 2.3.4). If $n > n_0$, then $|f(n) - 0| = 1/n < \varepsilon$, so that $\lim_{\succ} f = 0$. We will normally write this in either of the two forms

$$\lim_{n\to\infty} \frac{1}{n} = 0, \quad 1/n \to 0.$$

□

Remark
Unless we state otherwise, we shall always assume that a direction $\succ$ on $\mathbb{N}$ is $>$.

EXAMPLE 3.3.4.
Let X be the set of *nonzero* real (or complex) numbers and let $\succ$ be the direction on X defined by $x \succ y$ if and only if $|x| < |y|$. Let $f(x) = x^2$; then $\lim_\succ f = 0$. The geometric interpretation of $\succ$ is that $x \succ y$ precisely when *x is strictly closer to 0 than y is*, so in less formal language, this example shows that $x^2 \to 0$ as $x \to 0$. The proof is easy. Given any positive number ε, let δ be any positive number satisfying $\delta < 1$ and $\delta < \varepsilon$. If $x \succ \delta$ then $|x| < |\delta| = \delta < 1$, so that

$$|x^2 - 0| = |x|^2 < |x| < \delta < \varepsilon;$$

thus $\lim_\succ f = 0$. We will normally write this in one of the forms

$$\lim_{x \to 0} x^2 = 0, \quad x^2 \to 0.$$

There is one feature in this example that is worth mentioning explicitly. We are examining the case when $x \to 0$, and we have *explicitly excluded consideration of $x = 0$*. This is *a common feature of all limits*, and the reason for this lies in the comment in **(5)** in §3.2. □

EXAMPLE 3.3.5.
We show that if $|x| < 1$, then

$$\lim_{n \to \infty} x^n = 0.$$

Here, $X = \mathbb{N}$, $\succ$ is $>$, and we are claimimg that $\lim_\succ f = 0$, where for a given x, $f(n) = x^n$. Note that x is given and that f is a function of n. The case $x = 0$ is trivial. In the remaining cases, $1/|x| > 1$, so that we may write $1/|x| = 1 + \delta$, where $\delta > 0$. Then, using the Binomial Theorem,

$$|x^n| = |x|^n = \frac{1}{(1+\delta)^n} < \frac{1}{n\delta}.$$

Now consider any positive number ε and choose an integer n_0 satisfying $n_0 > 1/(\varepsilon\delta)$. If $n \succ n_0$ (equivalently, $n > n_0$), then

$$|f(n) - 0| = |x^n| < \frac{1}{n\delta} < \varepsilon$$

as required. □

The alert reader may have noticed that Definition 3.3.1 does not assert that f cannot tend to more than one limit. We now prove that it cannot, and as this argument is typical of many of our arguments, the reader is strongly advised to master it completely now.

Theorem 3.3.6.
If $\lim_\succ f = \alpha$ *and* $\lim_\succ f = \beta$, *then* $\alpha = \beta$.

Proof
We shall suppose that $\alpha \neq \beta$ and reach a contradiction. As $\alpha \neq \beta$, the number ε defined by $\varepsilon = |\alpha - \beta|/2$ is positive. As $\lim_{\succ} f = \alpha$, there is some x_0 such that $|f(x) - \alpha| < \varepsilon$ whenever $x \succ x_0$; and as $\lim_{\succ} f = \beta$, there is some x_1 such that $|f(x) - \beta| < \varepsilon$ whenever $x \succ x_1$. Now, there is some z in X such that $z \succ x_0$ and $z \succ x_1$, and this implies that

$$2\varepsilon = |\alpha - \beta| \leq |\alpha - f(z)| + |f(z) - \beta| < 2\varepsilon,$$

which is false. This contradiction completes the proof (in fact, we have just repeated the argument used in the proof of Theorem 3.2.5 here).

Theorem 3.3.6 implies that if f tends to a limit, then the limit is uniquely determined by X, $\succ$, and f. We shall usually denote this limit by $\lim_{\succ} f$, but if $\succ$ is understood from the context, we sometimes abbreviate it to $\lim f$. If f tends to α, we write any of the expressions

$$f \underset{\succ}{\to} \alpha, \quad f \to \alpha, \quad \lim_{\succ} f = \alpha, \quad \lim f = \alpha$$

depending on our mood, but always providing that the unspecified data is clear from the context. ■

We end this section with three simple but useful results about limits. In each of these the existence of one limit is used to guarantee the existence of another.

Theorem 3.3.7.
Suppose that f is defined on the directed set $(X, \succ)$, and that α and β are constants. If $f \to \alpha$, then $f + \beta \to \alpha + \beta$.

Proof
This follows immediately from the inequality

$$\big|(f(x) + \beta) - (\alpha + \beta)\big| = |f(x) - \alpha|.$$ ■

Theorem 3.3.8.
Suppose that f and g are defined on the directed set $(X, \succ)$, and that $f \to 0$. Suppose also that for some x_1 in X there is a positive M such that $|g(x)| \leq M|f(x)|$ whenever $x \succ x_1$. Then $g \to 0$.

Proof
Given any positive number ε, let $\varepsilon_1 = \varepsilon/M$. Then there is some x_0 such that $|f(x)| < \varepsilon_1$ when $x \succ x_0$. Now take x_2 such that $x_2 \succ x_0$ and $x_2 \succ x_1$. Thus $|g(x)| < \varepsilon$ when $x \succ x_2$, so that $g \to 0$. ■

Theorem 3.3.9.
If $\lim_{\succ} f = \alpha$, then $\lim_{\succ} |f|$ exists and is $|\alpha|$.

Proof

This follows immediately from the inequality (2.4.4), namely

$$\big||f(x)| - |\alpha|\big| \leq |f(x) - \alpha|.$$

■

Exercises

1. Show that if $f(n) = 3n/(2n^2 + 1)$ on $\mathbb{N}$, then $\lim_{\succ} f = 0$.
2. Show that $\lim_{x\to 0} x^3 + 1 = 1$.
3. Show that if f is defined on the directed set $(X, \succ)$, and if $\lim f = \alpha$, then for every constant M, $\lim_{\succ} Mf = M\alpha$.
4. Let $X = \mathbb{N}$ with the usual direction $>$, and let $f(n) = (-1)^n$. Show that $\lim f$ does *not* exist.

3.4 Examples of limits

The sole purpose of this section is to give the reader an appreciation of the great variety of limits encompassed by Definition 3.3.1. Readers who are already familiar with the material should see immediately the economy and coherence arising from the definition of a limit based on a direction. We cannot emphasize too strongly that *each of the following ten definitions is a special case of Definition 3.3.1, and when we derive a result from Definition 3.3.1, it will automatically hold in each of these ten cases.* We leave the reader to verify that in each of the following definitions, $\succ$ is a direction on the given set X, and that the given definition of the limit is a direct rewording of Definition 3.3.1.

Definition 3.4.1: sequences

Let $X = \mathbb{N}$, and define $m \succ n$ if and only if $m > n$. A sequence is a function $f : X \to \mathbb{R}$ (or $\mathbb{C}$), and this converges to α if given any positive ε, there is an n_0 such that $|f(n) - \alpha| < \varepsilon$ whenever $n > n_0$. If this is so, we write either of

$$\lim_{n\to\infty} f(n) = \alpha, \quad f(n) \to \alpha.$$

When we write a sequence as $a_1, a_2, \ldots$, we are considering a function $a : \mathbb{N} \to \mathbb{R}$ and writing a_n instead of $a(n)$. A sequence may be a function defined on, say, $\{N, N+1, N+2, \ldots\}$; for example, the function $3/(n-2)$ is defined on $\{3, 4, \ldots\}$, and in this case we still write

$$\lim_{n\to\infty} \frac{3}{n-2} = 0.$$

Definition 3.4.2: infinite series
The *infinite series*

$$\sum_{n=1}^{\infty} a_n$$

of complex numbers a_n *converges to*, or *equals*, α if and only if given any positive ε, there is an n_0 such that if $n > n_0$ then

$$|(a_1 + \cdots + a_n) - \alpha| < \varepsilon.$$

This is the same as saying that the *sequence* $s_1, s_2, \ldots$ *of partial sums* given by $s_n = a_1 + \cdots + a_n$ converges to α.

As a simple example, the identity

$$1 - x^{n+1} = (1 - x)(1 + x + x^2 + \cdots + x^n)$$

shows that

$$\left|(1 + x + x^2 + \cdots + x^n) - \frac{1}{1 - x}\right| = \frac{|x|^{n+1}}{|1 - x|}.$$

Thus (see Example 3.3.5) if $|x| < 1$ then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

Definition 3.4.3: the limit of f as $x \to +\infty$
Let X be any subset of $\mathbb{R}$ that is not bounded above, and define $x \succ y$ to mean $x > y$. Then, for any function $f : X \to \mathbb{C}$, $\lim_{\succ} f$ exists and is α if given any positive ε, there is an x_0 in X such that $|f(x) - \alpha| < \varepsilon$ whenever $x > x_0$ (and, of course, $x \in X$). When this limit exists we write

$$\lim_{x \to +\infty} f(x) = \alpha.$$

Note that this definition includes Definition 3.4.1 (with $X = \mathbb{N}$) as a special case.

As an example,

$$\lim_{x \to +\infty} \frac{3x^2}{x^2 + 1} = 3,$$

because if $x > \sqrt{(3/\varepsilon)}$ then

$$\left|\frac{3x^2}{x^2 + 1} - 3\right| = \frac{3}{x^2 + 1} < \frac{3}{x^2} < \varepsilon.$$

Definition 3.4.4: the limit of f as $x \to -\infty$
This is similar to Definition 3.4.3. Let X be any subset of $\mathbb{R}$ that is not bounded below, and give X the direction $\succ$ defined by $x \succ y$ if and only if $x < y$. Then for any function $f : X \to \mathbb{C}$, $\lim_{\succ} f$ exists and is α if given

any positive ε, there is an x_0 in X such that $|f(x)-\alpha| < \varepsilon$ whenever $x < x_0$ and $x \in X$. When this limit exists we write

$$\lim_{x\to-\infty} f(x) = \alpha.$$

For example,

$$\lim_{n\to-\infty} \frac{2n^4}{n^4-2} = 2.$$

Definition 3.4.5: the limit of f as $|x| \to \infty$

This is the natural combination of Definitions 3.4.3 and 3.4.4. Let X be any unbounded subset of $\mathbb{R}$ or of $\mathbb{C}$, and define $x \succ y$ if $|x| > |y|$. Then for any function f on X, $\lim_{\succ} f$ exists and is α if given any positive ε, there is an x_0 in X such that $|f(x) - \alpha| < \varepsilon$ whenever $|x| > |x_0|$ and $x \in X$. When this limit exists we write

$$\lim_{|x|\to+\infty} f(x) = \alpha.$$

As an example,

$$\lim_{|x|\to+\infty} \frac{2x^3+1}{x^3+3} = 2$$

because

$$\left|\frac{2x^3+1}{x^3+3} - 2\right| = \frac{5}{|x^3+3|} \leq \frac{5}{|x|^3-3} \leq \frac{8}{|x|^3} < \varepsilon$$

when $|x| > 2$ and $|x|^3 > 8/\varepsilon$. This example shows at the same time that $(2z^3+1)/(z^3+3) \to 2$ as $|z| \to \infty$ in the complex plane, and also that the sequence $(2n^3+1)/(n^3+3)$ converges to 2. *It is important, and helpful, to realise that there is no essential difference between these two cases.*

Definition 3.4.6: the limit of f as $x \to a$

Let X be a subset of $\mathbb{R}$ or $\mathbb{C}$. A point a is a *limit point* of X if for every positive r, there is a point x in X that satisfies $0 < |x-a| < r$. Note that the possibility $x = a$ *is explicitly excluded* here. With this, we can define a direction $\succ$ on X by $x \succ y$ if and only if $|x-a| < |y-a|$ (see **(5)** following Definition 3.2.1), the geometric interpretation being that $x \succ y$ if and only if x is strictly closer to a than y is. Given a function $f : X \to \mathbb{C}$, $\lim_{\succ} f$ exists and equals α if given any positive ε, there is a point x_0 such that $|f(x)-\alpha| < \varepsilon$ whenever $x \in X$ and $|x-a| < |x_0-a|$. When this limit exists, we write either of

$$\lim_{x\to a} f(x) = \alpha, \quad f(x) \to \alpha \quad \text{as} \quad x \to a.$$

For example (see Example 3.3.4), $x^2 \to 0$ as $x \to 0$.

We draw the reader's attention to the fact that this definition can be rephrased as follows: the limit $\lim_{\succ} f$ exists and equals α if given any positive ε, there is a positive δ such that $|f(x) - \alpha| < \varepsilon$ whenever $x \in X$ and $0 < |x - a| < \delta$. Indeed, if a limit exists according to the given definition, for a given ε we can take δ to be $|x_0 - a|$ (which *is* positive), and so the limit exists according to this alternative definition. Conversely, if a limit exists according to this alternative definition, we can choose an x_0 with $0 < |x_0 - a| < \delta$ (such x_0 exist because a is a limit point of X), and so obtain the existence of the limit in terms of the original definition. In some sense, the conventional use of the number δ is misleading; the condition is really that x is compared with some x_0 by the inequalities $0 < |x - a| < |x_0 - a|$.

Definition 3.4.7: continuous functions
It is important to note that in the previous example, f *need not be defined at the point* a, and even if it is, the value $f(a)$ plays no role in determining whether or not the limit exists, nor (if it does exist) what its value is. When f is defined at a, the question of whether or not $\lim_{\succ} f$ exists and is $f(a)$ is a matter of continuity. Let X be any subset of $\mathbb{R}$ or $\mathbb{C}$. Suppose that a is a limit point of X, and let f be any function defined on the set $X \cup \{a\}$ (if $a \in X$ this set is just X). Then we say that f is *continuous* at a if and only if $\lim_{x \to a} f(x)$ exists and equals $f(a)$.

According to the remarks in Definition 3.4.6, we can rewrite this in the following form: f is continuous at a if for every positive ε, there is a positive δ such that $|f(x) - f(a)| < \varepsilon$ whenever $0 < |x - a| < \delta$. However, in this case the inequality $|f(x) - f(a)| < \varepsilon$ is trivially true when $x = a$, so that in the definition of *continuity* (and only in this), we may replace the double inequality $0 < |x - a| < \delta$ by the single inequality $|x - a| < \delta$. Thus f is continuous at a if and only if given any positive ε, there is a positive δ such that if $|x - a| < \delta$ and $x \in X$, then $|f(x) - f(a)| < \varepsilon$. For example, the function x^2 is continuous at 0 because (as we have seen) $x^2 \to 0$ as $x \to 0$.

Definition 3.4.8: derivatives
A function f defined on $\{x \in \mathbb{R} : |x - a| < r\}$, where $r > 0$, is *differentiable* at a if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists according to Definition 3.4.6. If this is so, then its value is the *derivative* $f'(a)$ *of* f *at* a. The same definition applies if f is defined on the subset $\{z \in \mathbb{C} : |z - a| < r\}$ of the complex plane, where $a \in \mathbb{C}$.

Let us confirm that if $f(x) = x^4$, then $f'(a) = 4a^3$ (here, a and x may be real or complex). Using the identity $u^2 - v^2 = (u - v)(u + v)$ twice, we see that

$$\left|\frac{x^4 - a^4}{x - a} - 4a^3\right| = |(x + a)(x^2 + a^2) - 4a^3|$$

$$= |(x - a)(x^2 + 2ax + 3a^2)|.$$

Imposing the condition $|x - a| < 1$ on x (if it is not already so restricted), we have $|x| \leq |a| + 1$, and so for these x,

$$\left|\frac{x^4 - a^4}{x - a} - 4a^3\right| \leq 6(|a| + 1)^2|a - x|.$$

The result now follows easily (or by Theorem 3.3.8). We have considered this example in detail, but soon we will become much more efficient, and distracting estimates such as these can usually be avoided.

One matter concerning derivatives is worth mentioning now. It is immediate (and totally trivial) that if f is a constant function on $\mathbb{R}$, say, then $f'(x) = 0$ for every x. It is not so obvious, however, that if $f'(x) = 0$ at every point x of $\mathbb{R}$, then f is constant on $\mathbb{R}$. This is true, and it will be proved in the next chapter.

Definition 3.4.9: the limit of f as $x \to a+$
Let $X = \{x \in \mathbb{R} : a < x < b\}$, where $a < b$, and define $x \succ y$ if $x < y$ (again, $x \succ y$ if x is closer to a than y is). Then a is a limit point of X, and for any function f on X, if $\lim_{\succ} f$ exists and equals α we write

$$\lim_{x \to a+} f(x) = \alpha.$$

The + sign here indicates that the limit is being taken with the restriction that x is larger than a. Explicitly, this limit exists if given any positive ε, there is a positive δ such that $|f(x) - \alpha| < \varepsilon$ whenever $a < x < a + \delta$ and $x \in X$. We call this a *one-sided limit* of f at a, and we say that this is the limit of f as x *tends to a from above.*

Once again, if f is defined at a (and it need not be) the value of $f(a)$ plays no role in determining this limit. If $f(a)$ is defined and if $f(x)$ tends to $f(a)$ as $x \to a+$, f is continuous at a, and conversely.

Of course, there is a corresponding notion of a limit from below, and as this presents no new ideas we omit the details; the notation used for this is

$$\lim_{x \to a-} f(x).$$

As an example of these limits, let $f(x) = x/|x|$ when $x \neq 0$. Then

$$\lim_{x \to 0+} f(x) = 1, \quad \lim_{x \to 0-} f(x) = -1.$$

EXAMPLE 3.4.10: THE INTEGRAL.

Let f be any real-valued function defined on $\{x \in \mathbb{R} : a \le x \le b\}$, and suppose that $|f(x)| \le M$ there. We define the integral

$$\int_a^b f(x)\,dx \tag{3.4.1}$$

as the limit of sums of the form

$$S_f(X,T) = \sum_{j=0}^{n} f(t_j)(x_{j+1} - x_j), \tag{3.4.2}$$

where $X = \{x_0, x_1, \ldots x_n\}$, $T = \{t_0, t_1, \ldots, t_{n-1}\}$, and

$$a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b, \quad x_j \le t_j \le x_{j+1}. \tag{3.4.3}$$

To be more explicit, we consider the set $\mathcal{P}$ of all pairs (X, T) of this type (with varying n) and define a direction $\succ$ on $\mathcal{P}$ by $(X_1, T_1) \succ (X, T)$ if and only if $X_1 \supset X$ (notice that T does not enter into this definition). It is clear that $\succ$ is a direction on $\mathcal{P}$. Indeed, if $(X_1, T_1) \succ (X_2, T_2)$ and $(X_2, T_2) \succ (X_3, T_3)$ then $X_1 \supset X_3$, so that $(X_1, T_1) \succ (X_3, T_3)$. Also, given any pair (X_1, T_1) and (X_2, T_2), each is dominated by $(X_1 \cup X_2, T)$ for any appropriate T.

For a given f, the integral in (3.4.1) is now defined as the limit of the function $S_f(X,T)$ with respect to the direction $\succ$ on $\mathcal{P}$. If we rewrite this without making an explicit reference to the direction $\succ$ we see that the integral in (3.4.1) exists and equals I if given any positive ε, there is a pair (X_0, T_0) such that

$$\left| \sum_{j=0}^{n} f(t_j)(x_{j+1} - x_j) - I \right| < \varepsilon$$

whenever $X \supset X_0$. This simply says that the sums $S_f(X,T)$ should be close to some number I whenever the subdivision X of $\{x : a \le x \le b\}$ is 'sufficiently fine'. □

Exercises

1. Show that $(2n + 4)/(3n^2 + 1) \to 0$ as $n \to +\infty$.
2. Show that if $|x| < 1$, then $nx^n \to 0$ as $n \to +\infty$ (see Example 3.3.5). By considering

$$(1 - x)^2(1 + 2x + 3x^2 + \cdots + nx^{n-1}),$$

show that

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}.$$

3. Show that f defined for $-1 < x < 1$ by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous at 0.

4. Show that $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

is continuous at 0, but not at any other point of $\mathbb{R}$.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x)| \le x^2$ on $\mathbb{R}$. Show that f is differentiable at 0 and that $f'(0) = 0$.

6. Let $f(x) = x/(x^2 + 1)$, where $x \in \mathbb{R}$. Show that $\lim_{|x| \to +\infty} f(x) = 0$.

7. Suppose that f is defined on $\{x \in \mathbb{R} : a \le x \le b\}$. Show that f is continuous at a if and only if $\lim_{x \to a+} f(x)$ exists and equals $f(a)$.

8. Let $f(x) = |x|$, where $x \in \mathbb{R}$. Show that f is not diffentiable at 0. Show, however, that the *one-sided derivative*

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0}$$

exists and equals 1. [More generally, if the graph of a function has a 'corner', the function will not be differentiable there, but sometimes the two one-sided derivatives will exist.]

9. Show that a is a limit point of the set X (see Definition 3.4.6) if and only if there is a sequence $x_1, x_2, \ldots$ of *distinct* points in X such that $x_n \to a$.

10. Show by induction that

$$(1 + x)(1 + x^2)(1 + x^{2^2}) \cdots (1 + x^{2^n}) = 1 - x^{2^{n+1}}.$$

Deduce that if $|x| < 1$, then

$$\prod_{n=0}^{\infty} (1 + x^{2^n}) = \frac{1}{1 - x},$$

in the sense that the left-hand side is the limit as $n \to \infty$ of

$$\prod_{n=0}^{N} (1 + x^{2^n}) = (1 + x)(1 + x^2)(1 + x^{2^2}) \cdots (1 + x^{2^N}).$$

11. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 0$ for all x, and

$$f(x) = \begin{cases} 0 & \text{if } x = 0; \\ 1 & \text{if } x \neq 0. \end{cases}$$

Show that

$$\lim_{x \to 0} g(x) = 0, \quad \lim_{x \to 0} f(x) = 1,$$

but that

$$\lim_{x\to 0} f\big(g(x)\big) = 0.$$

This example shows that the limit of a composition of functions may not be what you might expect.

3.5 Sums, products, and quotients of limits

In Example 3.3.3 we showed that $1/n \to 0$ as $n \to \infty$. By using the same method, but with just a little more work, we could show, for example, that $(2/n) - (7/n^2) \to 0$, and the reader could easily go on to create many more examples of this type in which the argument is conceptually the same but in which the details become increasingly more complicated. There must be advantages, then, in developing a machinery that would enable us to operate at the same simple conceptual level as the example $1/n$ while avoiding the more complicated numerical details. The key to this lies in understanding how limits combine with the arithmetic operations, and this is our task now. We consider the sum and product first, and then the quotient.

The sum $f + g$ of two given functions f and g on X is defined by

$$(f + g)(x) = f(x) + g(x).$$

The addition on the right is the addition of real (or complex) numbers. As the addition on the left is the addition *of functions*, we should, strictly speaking, use a different symbol (such as $\oplus$) for this, but convention, and simplicity, demand otherwise. Similarly, we define the product $f{\cdot}g$ of the functions f and g by

$$(f{\cdot}g)(x) = f(x)g(x).$$

The results concerning the limits of the sum and product of functions are exactly what we would expect.

Theorem 3.5.1.

Suppose that f and g are defined on the directed set $(X, \succ)$ and that $f \to \alpha$ and $g \to \beta$. Then $f + g \to \alpha + \beta$ and $f{\cdot}g \to \alpha\beta$.

Proof

First, we consider $f + g$. For any x,

$$\begin{aligned}\big|(f + g)(x) - (\alpha + \beta)\big| &= \big|(f(x) - \alpha) + (g(x) - \beta)\big| \\ &\le |f(x) - \alpha| + |g(x) - \beta|. \qquad (3.5.1)\end{aligned}$$

Given a positive ε, there is some x_1 such that $|f(x)-\alpha| < \varepsilon/2$ when $x \succ x_1$, and there is some x_2 such that $|g(x) - \beta| < \varepsilon/2$ when $x \succ x_2$. Now take w with $w \succ x_1$ and $w \succ x_2$; then $x \succ w$ implies that the right-hand side in (3.5.1) is less than ε, so that $f + g \to \alpha + \beta$.

We now consider the product function $f{\cdot}g$. Let $F(x) = f(x) - \alpha$ and $G(x) = g(x) - \beta$, so that (directly from the definition) $F \to 0$ and $G \to 0$. Because $G \to 0$, there is some x_1 such that $|G(x)| < 1$ when $x \succ x_1$. It follows that $|F(x)G(x)| \leq |F(x)|$ when $x \succ x_1$, so that by Theorem 3.3.8, $F{\cdot}G \to 0$. As

$$f(x)g(x) = F(x)G(x) + \beta F(x) + \alpha G(x) + \alpha\beta,$$

we see now that $f{\cdot}g \to \alpha\beta$. ■

Obviously, Theorem 3.5.1 extends (by induction) to any finite sum of functions; if $f_j \to \alpha_j$ for $j = 1, \dots, n$ then

$$f_1 + \cdots + f_n \to \alpha_1 + \cdots + \alpha_n. \tag{3.5.2}$$

Similarly, it extends to finite products of functions, and also to the following result.

Theorem 3.5.2.
Suppose that the functions $f_1, f_2, \dots, f_n$ are defined on $(X, \succ)$, and that for each j, $f_j \to \ell_j$. Then, for any numbers $a_1, \dots, a_n$,

$$(a_1 f_1 + \cdots + a_n f_n) \to a_1\ell_1 + \cdots + a_n\ell_n.$$

There are many important corollaries of Theorems 3.5.1 and 3.5.2. As a simple application, let $g(z) = z$. Then $g(z) \to g(a)$ as $z \to a$. Applying Theorem 3.5.1 to the product of g with itself k times, we see that (in less formal language) $z^k \to a^k$ as $z \to a$. Now applying Theorem 3.5.2, we see that for any polynomial, say

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

$p(z) \to p(a)$ as $z \to a$. Recalling the definition of continuity, we see that we have now proved the following result.

Theorem 3.5.3.
Every polynomial is continuous at every point of $\mathbb{C}$.

Likewise, Theorem 3.5.2 implies the following results on continuity and on differentiability.

Theorem 3.5.4.
Let s and t be any (real or complex) numbers, and suppose that the functions f and g are continuous at a. Then so is the linear combination $sf + tg$, and also the product function $f{\cdot}g$.

Theorem 3.5.5.
Let s and t be any (real or complex) numbers, and suppose that the functions f and g are differentiable at a. Then so is the linear combination $sf + tg$, with derivative $sf'(a) + tg'(a)$, and also the product function $f \cdot g$, with derivative given by

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).$$

The assertion concerning the linear combination $sf + tg$ in both of these results is a direct consequence of Theorem 3.5.2. The continuity of the product function $f \cdot g$ is an immediate consequence of Theorem 3.5.1, for this shows that if $f \to f(a)$ and $g \to g(a)$, then $f \cdot g \to f(a)g(a)$. Next, using the identity

$$\frac{f(x)g(x) - f(a)g(a)}{x - a} = \left(\frac{f(x) - f(a)}{x - a}\right)g(x) + f(a)\left(\frac{g(x) - g(a)}{x - a}\right)$$

and Theorem 3.5.1, we obtain the existence of, and the given formula for, the derivative of $f \cdot g$.

The formula for the derivative of the product function provides an easy proof (by induction) that if $f(z) = z^n$, where n is a positive integer, then $f'(z) = nz^{n-1}$. In particular, we now know that if

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

then

$$p'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots + na_n z^{n-1};$$

here the a_j and z may be real or complex.

As Theorems 3.5.1 and 3.5.2 are valid for all limits, similar statements apply to sequences and series. Suppose, for example, that the sequences a_n and b_n converge to a and b, respectively; then, from Theorems 3.5.1 and 3.5.2, the sequence $a_n b_n$ converges to ab, and for any constants s and t, the sequence $sa_n + tb_n$ converges to $sa + tb$. From this, we see immediately that

$$s\sum_{n=0}^{\infty} a_n + t\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty}(sa_n + tb_n) \tag{3.5.3}$$

in the sense that if the two sums on the left converge, then so does the sum on the right, and equality holds.

Let us move on now to consider how the limiting process interacts with division. By writing the quotient $g(x)/f(x)$ as the product of $g(x)$ and $1/f(x)$, and using Theorem 3.5.1, we can immediately simplify the situation a little and confine our attention to the function $1/f$ defined by

$$(1/f)(x) = 1/f(x).$$

We are surely anticipating that if $f \to \alpha$ then

$$1/f \to 1/\alpha, \tag{3.5.4}$$

but there are two problems with this. First, α may be 0, and second, the function $1/f$ is not defined at any x for which $f(x) = 0$. The way round the first difficulty is clear: we assume at the outset that $\alpha \neq 0$ (indeed, this is forced upon us, as otherwise (3.5.4) has no meaning). Given this assumption, the remaining difficulty disappears, for suppose that $f \to \alpha$, and $\alpha \neq 0$. Then there is some x_0 such that $|f(x)-\alpha| < |\alpha|/2$ when $x \succ x_0$, and as this inequality implies that

$$|f(x)| \geq |\alpha|/2 > 0, \tag{3.5.5}$$

there is no difficulty if we restrict $1/f$ to the final segment $X(x_0)$. We now state and prove the result.

Theorem 3.5.6.

Suppose that f is defined on $(X, \succ)$, and that $f \to \alpha$, where $\alpha \neq 0$. Then there is some x_0 such that $f(x) \neq 0$ when $x \succ x_0$, and if we restrict $1/f$ to be defined on $X(x_0)$, then $\lim_{\succ} (1/f)$ exists and equals α^{-1}.

Proof

The existence of x_0 that leads to (3.5.5) has already been established. Now let F be the function $1/f$ restricted to $X(x_0)$, and let $\beta = 1/\alpha$. Using (3.5.5), if $x \succ x_0$ then

$$|F(x) - \beta| = \left|\frac{\alpha - f(x)}{\alpha f(x)}\right| < \frac{2|f(x) - \alpha|}{|\alpha|^2},$$

and $F \to \beta$ is now a direct consequence of Theorem 3.3.8. ∎

With Theorem 3.5.6 available, we can now derive the familar formula for the derivative of a quotient.

Theorem 3.5.7.

Suppose that f and g are differentiable at a, and that $g(a) \neq 0$. Then f/g is differentiable at a, and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof

This follows directly from Theorem 3.5.6 and the identity

$$\frac{1}{x-a}\left(\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}\right) = \left(\frac{1}{g(x)}\right)\frac{f(x)-f(a)}{x-a} - \left(\frac{f(a)}{g(x)g(a)}\right)\frac{g(x)-g(a)}{x-a}.$$

∎

Exercises

1. Show that as $n \to \infty$,
$$\frac{3n^3+n-1}{4n^3+7n^2+2} \to \frac{3}{4}.$$
2. Suppose that f and g are continuous at a, and that $g(a) \neq 0$. Show that f/g is continuous at a.
3. Suppose that a set X contains the point a, and that f, defined on X, is continuous at a. Suppose also that g is defined on $f(X)$ and is continuous at $f(a)$. Prove that $x \mapsto g(f(x))$ is continuous at a.
4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at every point of $\mathbb{R}$, and that $f(a) > 0$. Show that there is some positive δ such that $f(x) > 0$ whenever $|x-a| < \delta$. Now define $g(x) = \sqrt{f(x)}$ (so that $g(x) > 0$ when $|x-a| < \delta$. Show that g is continuous at a.
[*Hint*: $u^2 - v^2 = (u-v)(u+v)$.]
5. Let $f(x) = x^2 + 2ax + b$, where x, a, and b are real with $a^2 > b$. Show that the equation $f(x) = 0$ has two solutions, and let x_1 be the larger of these. Formulate (precisely) and prove a statement that justifies the assertion that 'x_1 varies continuously with a and b'.

3.6 Limits and inequalities

In this section we consider how limits interact with inequalities. It is *not* true that if $f(x) > 0$ for all x, and $\lim_{\succ} f = \alpha$, then $\alpha > 0$; for example, if $f(n) = 1/n$, then $f(n) > 0$ for every n, yet $\lim_{\succ} f = 0$. The corresponding result for $\geq$ is true, and this is our first result.

Theorem 3.6.1.
Let $(X, \succ)$ be a directed set and suppose that the real-valued functions f and g are defined on X. If the limits $\lim_{\succ} f$ and $\lim_{\succ} g$ exist, and if $g(x) \geq f(x)$ for all x in X, then $\lim_{\succ} g \geq \lim_{\succ} f$.

Proof
Let $\lim_{\succ} f = \alpha$ and $\lim_{\succ} g = \beta$. We suppose that $\alpha > \beta$ and seek a contradiction. Let $\varepsilon = (\alpha - \beta)/2$, so that ε is positive. There is some x_1 in X such that $|f(x) - \alpha| < \varepsilon$ when $x \succ x_1$, and there is some x_2 in X such that $|g(x) - \beta| < \varepsilon$ when $x \succ x_2$. As $\succ$ is a direction, there is some x_3 that dominates both x_1 and x_2, so that
$$|f(x_3) - \alpha| < \varepsilon, \quad |g(x_3) - \beta| < \varepsilon.$$
It follows that $0 \leq g(x_3) - f(x_3) < (\beta + \varepsilon) - (\alpha - \varepsilon) = 0$ (so that $0 < 0$), and this is the contradiction we are seeking. ∎

The following special cases are worthy of special mention.

Corollary 3.6.2.

Suppose that f is defined on the directed set $(X, \succ)$, and suppose that $\lim_{\succ} f$ *exists. If $f(x) \geq \alpha$ for all x, then* $\lim_{\succ} f \geq \alpha$. *If $f(x) \leq \beta$ for all x, then* $\lim_{\succ} f \leq \beta$.

We come now to one of the most useful criteria that guarantee that a limit exists. Given a real-valued function f defined on a directed set $(X, \succ)$, we say that f is

(i) *increasing* if $x \succ y$ implies $f(x) \geq f(y)$, and
(ii) *decreasing* if $x \succ y$ implies $f(x) \leq f(y)$.

While it might seem more natural to use *nondecreasing* and *nonincreasing* here, in common with many authors we prefer the shorter phrase. We shall also say that f is

(iii) *strictly increasing* if $x \succ y$ implies $f(x) > f(y)$, and
(iv) *strictly decreasing* if $x \succ y$ implies $f(x) < f(y)$.

A real-valued function f is *bounded above* on X if there is some constant M such that for all x in X, $f(x) \leq M$; that is, if the subset $\{f(x) : x \in X\}$ of $\mathbb{R}$ is bounded above. Likewise, the function f is *bounded below* if this set is bounded below.

The following result is arguably the most fundamental result of all about limits. It is a powerful criterion for the *existence of a limit*, yet despite this, its proof is extremely simple.

Theorem 3.6.3.

Suppose that f is a real-valued function defined on the directed set $(X \succ)$.

(i) *If f is increasing and bounded above on X, then* $\lim_{\succ} f$ *exists and*

$$\lim_{\succ} f = \sup\{f(x): x \in X\}. \tag{3.6.1}$$

(ii) *If f is decreasing and bounded below on X, then* $\lim_{\succ} f$ *exists and*

$$\lim_{\succ} f = \inf\{f(x): x \in X\}. \tag{3.6.2}$$

Proof

The hypotheses in (i) imply that the subset $\{f(x) : x \in X\}$ of $\mathbb{R}$ is nonempty and bounded above; thus it has a least upper bound, say α. It follows that for any positive ε, $\alpha - \varepsilon$ is not an upper bound for this set, so that there is some x_0 in X with $f(x_0) > \alpha - \varepsilon$. If $x \succ x_0$, then

$$\alpha - \varepsilon < f(x_0) \leq f(x) \leq \alpha < \alpha + \varepsilon,$$

so that for these x, $|f(x) - \alpha| < \varepsilon$. This means that $\lim_{>} f$ exists and equals α. The proof of (ii) is similar (using the greatest lower bound) and is omitted. ■

As an application of this result, we see that if a_n is a sequence of real numbers with the properties

(i) $a_n \leq a_{n+1}$ for all n, and
(ii) $a_n \leq M$ for all n,

then the sequence a_n converges. As this is so important, and as we shall refer to it often, we state it as a separate result.

Theorem 3.6.4.
If a sequence $a_1, a_2, \ldots$ *of real numbers is increasing and bounded above, then it converges to* $\sup\{a_n : n = 1,2,\ldots\}$.

To illustrate the general applicability of Theorem 3.6.3 we also give another corollary of it.

Theorem 3.6.5.
Suppose that $f:\mathbb{R} \to \mathbb{R}$ *is increasing and bounded above. Then* $\lim_{x\to+\infty} f(x)$ *exists and is* $\sup\{f(x):x \in \mathbb{R}\}$.

Exercises

1. Suppose that each of the numbers $a_1, a_2, \ldots$ is in $\{0, 1, 2, \ldots, 9\}$. Show how to give a value to the infinite decimal expansion $0{\cdot}a_1a_2\cdots$. Express the decimal expansion $0{\cdot}123123123\cdots$ as a rational number.

2. Use Theorem 3.6.3 to show that
$$\lim_{x\to+\infty} \frac{x^2+1}{x^2+2} = 1.$$

3. Show that if $x > 1$, the series
$$\sum_{n=0}^{\infty} \frac{1}{x^n + x^2 + 1}$$
converges to a value not greater than $x/(x-1)$.

4. Show that if the series $\sum_{n=0}^{\infty} a_n$ of positive terms a_n converges, then $a_n \to 0$. Deduce that if a and b are positive, then the series
$$\sum_{n=0}^{\infty} \frac{1}{a^n + b^n}$$
converges if and only if $a > 1$ or $b > 1$.

5. Show that if a real-valued function f is increasing and bounded on the set $\{x \in \mathbb{R} : a < x < b\}$, where $a < b$, then the limits

$$\lim_{x\to a+} f(x), \quad \lim_{x\to b-} f(x)$$

exist. Which is the smaller of these limits and why?

3.7 Functions tending to infinity

Consider the two sequences given by $1, 2, 3, \ldots$ and $-1, -2, -3, \ldots$. Although we have not yet defined what '$a_n \to +\infty$' and '$a_n \to -\infty$' mean, there must surely be some sense in which the first sequence tends to $+\infty$, and the second sequence tends to $-\infty$, as $n \to \infty$. A real sequence $a_1, a_2, \ldots$ is a map $a : \mathbb{N} \to \mathbb{R}$, and the phrase '$n \to \infty$' is meaningful precisely because of the existence of the ordering $>$ on $\mathbb{N}$. It is more or less self-evident, then, that if we wish to give a meaning to the phrase '$a_n \to +\infty$', we must use the fact that $\mathbb{R}$ is also a directed set ordered by $>$. This leads naturally to the following definition.

Definition 3.7.1.
The real sequence a_n tends to $+\infty$ if for every real number k, there is an integer n_0 such that $a_n > k$ whenever $n > n_0$. The sequence a_n tends to $-\infty$ if for every real number k, there is an n_0 such that $a_n < k$ whenever $n > n_0$.

There are other rather obvious generalisations of this that accommodate other situations and we mention just a few now. In all of these cases we are guided by the fact that we are considering a function *mapping one directed set into another*. With these available, the reader should be able to invent more definitions of this type.

Definition 3.7.2.
Let $X = \mathbb{N}$ with the ordering $>$; let $\mathbb{C}$ be ordered by the relation $z \succ w$ if and only if $|z| > |w|$. A complex sequence a_n converges to ∞, and we write $a_n \to \infty$, if given any positive number r, there is an integer n_0 such that $|a_n| > r$ whenever $n > n_0$.

Definition 3.7.3.
(i) A function $f : \mathbb{R} \to \mathbb{R}$ tends to $+\infty$ as $x \to +\infty$ if for every real number k, there is a real number x_0 such that $f(x) > k$ whenever $x > x_0$. If this is so, we write

$$\lim_{x\to+\infty} f(x) = +\infty.$$

In this definition we have given both the domain and the image of f the direction $>$.

(ii) The function f tends to $-\infty$ as $x \to +\infty$ if for every real number k, there is an x_0 such that $f(x) < k$ whenever $x > x_0$. Here we have given the domain of f the direction $>$ and the image of f the direction $<$.

Definition 3.7.4.
A real-valued function f defined on $\{x \in \mathbb{R} : x > 0\}$ tends to $+\infty$ as $x \to 0+$ if for every real number k, there is a positive number δ such that $f(x) > k$ whenever $0 < x < \delta$. If this is so, we write

$$\lim_{x \to 0+} f(x) = +\infty.$$

Exercises

1. Show that if $f(x) = 1/x$ when $x > 0$, then
$$\lim_{x \to 0+} f(x) = +\infty.$$
2. Let z_n be a sequence of complex numbers. Show that $z_n \to \infty$ if and only if the sequence of real numbers $|z_n| \to +\infty$.
3. Consider the directed sets $(\mathbb{N}, >)$ and $(\mathbb{C}, \succ)$ as in Definition 3.7.2. A complex sequence (a_n) is a map from $(\mathbb{N}, >)$ to $(\mathbb{C}, \succ)$. Show that $a_n \to \infty$ if and only if given any number w in $\mathbb{C}$, there is an integer n_0 in $\mathbb{N}$ such that $a_n \succ w$ whenever $n > n_0$.
4. Give $\mathbb{N}$ the usual ordering $>$, and define a direction $\succ$ on $\mathbb{R}$ by $m \succ n$ if and only if $|m| > |n|$. Describe carefully which real unbounded sequences $a_1, a_2, \ldots$ have a limit in the sense of Definition 3.7.1.

CHAPTER 4 Bisection Arguments

Abstract
This chapter contains three important results: (1) *the Intermediate Value Theorem (that if a continuous function takes the values a and b, then it takes all values between a and b);* (2) *a function whose derivative is identically zero is constant;* (3) *the General Principle of Convergence (which gives a necessary and sufficient condition for a limit to exist). All of these are proved by the repeated bisection of an interval.*

4.1 Nested intervals

An *interval* is a subset I of $\mathbb{R}$ with the property that if a and b are in I, then all points between a and b are also in I. Intervals are used constantly in analysis, and the following notation for the four possible types of bounded intervals is standard:

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\},$$

$$(a, b) = \{x \in \mathbb{R} : a < x < b\},$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\},$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

We say that $[a, b]$ is a *closed interval*, (a, b) is an *open interval*, and that $[a, b)$ and $(a, b]$ are *half-open half-closed intervals*. Notice that a closed interval contains both of its endpoints, whereas an open interval contains neither of them. Also, $[a, a] = \{a\}$, and $(a, a) = \varnothing$.

Next we list the five possible types of unbounded intervals:

$$(a, +\infty) = \{x \in \mathbb{R} : x > a\},$$
$$[a, +\infty) = \{x \in \mathbb{R} : x \geq a\},$$
$$(-\infty, b) = \{x \in \mathbb{R} : x < b\},$$
$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\},$$
$$(-\infty, +\infty) = \mathbb{R}.$$

Notice the introduction of the symbols $+\infty$ and $-\infty$ here. We are *not* asserting that either is a real number, and they are used *as a notational convenience* only. We say that the intervals $(a, +\infty)$ and $(-\infty, b)$ are *open* (they do not contain their single endpoint), whereas the intervals $[a, +\infty)$ and $(-\infty, b]$ (which do contain their single endpoint) are *closed*. Notice also that $(a, +\infty)$ and $(-\infty, b)$ are final segments with respect to the directions $>$ and $<$ on $\mathbb{R}$.

A sequence of intervals $I_1, I_2, \ldots$ is *decreasing* if $I_1 \supset I_2 \supset \cdots$. Our single concern in this section is whether or not the intersection $I_1 \cap I_2 \cap \ldots$ of a decreasing sequence $I_1, I_2, \cdots$ of intervals is nonempty. The example

$$(0, \tfrac{1}{2}) \cap (0, \tfrac{1}{3}) \cap (0, \tfrac{1}{4}) \cap \cdots \tag{4.1.1}$$

shows that the intersection may be empty when each I_n is a bounded open interval, and the example

$$[1, +\infty) \cap [2, +\infty) \cap [3, +\infty) \cap \cdots \tag{4.1.2}$$

shows that the intersection may be empty when each I_n is an unbounded closed interval. It is of fundamental importance that the intersection is *nonempty* when the I_n are *bounded closed intervals.*

Theorem 4.1.1.

Suppose that $I_1, I_2, I_3, \ldots$ is a sequence of closed, bounded, nonempty intervals, and that $I_1 \supset I_2 \supset I_3 \supset \cdots$. Then

$$I_1 \cap I_2 \cap I_3 \cap \cdots \neq \varnothing.$$

Proof

We write $I_n = [a_n, b_n]$, where $a_n \leq b_n$. For every m and n, $I_{m+n} \subset I_n$ and $I_{m+n} \subset I_m$, so that $a_n \leq a_{m+n} \leq b_{m+n} \leq b_m$. Thus, for all m and n, $a_n \leq b_m$. Keeping m fixed, we see that the sequence a_n is increasing and bounded above by b_m; thus by Theorem 3.6.4, $a_n \to \alpha$, where $\alpha \leq b_m$. Now, this inequality holds for every m; thus the sequence b_n, which is decreasing and bounded below by α, converges to some number β, where $\alpha \leq \beta$. We deduce that $a_n \leq \alpha \leq \beta \leq b_n$, so that $[\alpha, \beta] \subset I_n$ for every n, and this shows that $I_1 \cap I_2 \cap I_3 \cap \cdots$ contains the nonempty interval $[\alpha, \beta]$. In fact, this intersection is $[\alpha, \beta]$. ■

We now explain what we mean by the *repeated bisection* of the interval $[a, b]$ of *length* $b - a$ and midpoint c, where $c = (b + a)/2$. We say that the intervals $[a, c]$ and $[c, b]$ are *obtained from* $[a,b]$ *by bisection*. Suppose now that the intervals $I_1, I_2, \ldots$ in Theorem 4.1.1 are such that each I_{n+1} is obtained from I_n by bisection. If ℓ_n is the length of I_n, then $\ell_n = \ell_1/2^{n-1}$, so that $\ell_n \to 0$, and this implies that in the proof above, $\alpha = \beta$. This proves the following refinement of Theorem 4.1.1.

Theorem 4.1.2.
Suppose that each interval I_{n+1} *in Theorem 4.1.1 is obtained from* I_n *by bisection. Then* $I_1 \cap I_2 \cap I_3 \cap \cdots$ *consists of a single point.*

Exercises

1. Show that in the proof of Theorem 4.1.1, $I_1 \cap I_2 \cap I_3 \cap \cdots = [\alpha, \beta]$.
2. Show that any interval in $\mathbb{R}$ must be one of the nine given types.
3. Verify that the intersections in (4.1.1) and in (4.1.2) are empty.
4. The four (congruent) rectangles obtained by dividing a rectangle S_1 by the two lines through its centre (in the obvious way) are obtained *by subdivision* of S_1. Formulate and prove a version of Theorem 4.1.2 for rectangles in the plane.

4.2 The Intermediate Value Theorem

It is intuitively clear that a continuous function cannot pass between positive and negative values without passing through the value zero. We shall now prove this and thereby obtain a sufficient condition for a continuous function to have a zero.

Theorem 4.2.1: the Intermediate Value Theorem.
Let f *be a real valued function that is continuous at each point of the bounded closed interval* $[a,b]$. *If* y *lies between* $f(a)$ *and* $f(b)$, *then there is some point* c *in* $[a,b]$ *with* $f(c) = y$.

Proof
We may suppose that $f(a) < y < f(b)$, as the proof in the case where the inequalities are reversed is similar. By considering $f(x) - y$ instead of $f(x)$, we may also suppose that $y = 0$, $f(a) < 0$, and $f(b) > 0$

Now write $a_0 = a$, $b_0 = b$, and let c_0 be the midpoint of $[a_0, b_0]$. Next, we define the numbers a_1 and b_1 by

(i) $a_1 = a_0$ and $b_1 = c_0$ if $f(c_0) \geq 0$, and
(ii) $a_1 = c_0$ and $b_1 = b_0$ if $f(c_0) < 0$.

Regardless of which of these holds, we then find that $[a_1, b_1] \subset [a, b]$ and that $f(a_1) < 0 \leq f(b_1)$. We may continue this argument using $[a_1, b_1]$ instead of $[a_0, b_0]$ and c_1 the midpoint of $[a_1, b_1]$, and so on, and in this way we obtain a decreasing sequence of intervals $[a_n, b_n]$, each of which is obtained from the previous one by bisection, with the property that $f(a_n) < 0 \leq f(b_n)$.

By Theorem 4.1.2, the intervals $[a_n, b_n]$ have exactly one point, say c, in common, and $a_n \to c$ and $b_n \to c$ as $n \to \infty$. As f is continuous at c, $f(a_n) \to f(c)$, and as $f(a_n) < 0$, we have $f(c) \leq 0$. Similarly, as $b_n \to c$ we have $f(c) \geq 0$, so that $f(c) = 0$ as required. ■

We give one example to illustrate this result. The reader may care to compare the proof of this with the (longer) proof of (the weaker) Theorem 2.3.3 that any positive number has a square root.

Theorem 4.2.2.
Every positive number a has a positive nth root.

Proof
Suppose that $a > 0$, and n is a positive integer. The polynomial p given by $p(x) = x^n$ is continuous at every point of $\mathbb{R}$, and $p(0) < a < p(1 + a)$. The Intermediate Value Theorem now implies that there is some c in $[0, 1 + a]$ with $c^n = a$; thus c is a positive nth root of a. ■

Exercises

1. Show that every positive a has a *unique* positive nth root.
2. Show that every real polynomial of odd degree has at least one real root. Use the bisection argument (on a computer) to find a real root of $x^3 + 3x + 7$ correct to 3 decimal places.
3. Use a computer to find the positive cube root of 5 correct to three decimal places.
4. Show that the real polynomial $x^2 + ax + b$ has two real roots if $a^2 > 4b$.
5. Use the Intermediate Value Theorem to show that if a real-valued function f is continuous at every point of an interval I, then $f(I)$ is an interval. Deduce that if f is continuous on an interval I, and if $f(x)$ is rational for every x in I, then f is constant on I. Is the same true if $f(x)$ is irrational for every x in I?

6. Give an example of an open interval I and a continuous function $f : I \to \mathbb{R}$ such that $f(I) = [0, 1)$.
7. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous at every point of $[0, 1]$.
   (i) Show that if $f([0, 1]) \subset [0, 1]$ then f has a fixed point in $[0, 1]$; that is, there is some x in $[0, 1]$ such that $f(x) = x$.
   [Hint: Consider $f(x) - x$ at 0 and at 1.]
   (ii) Show that if $f([0, 1]) \supset [0, 1]$ then f has a fixed point in $[0, 1]$.

4.3 The Mean Value Inequality

We prove that if the derivative of a function is nonnegative on an interval, then the function is increasing on that interval. As a corollary, we see that a function whose derivative is identically zero is constant.

Theorem 4.3.1.

Suppose that the function $f : I \to \mathbb{R}$ is differentiable at each point of an open interval I, and that $a \in I$, $b \in I$, and $a < b$.

(i) *If $f'(x) < M$ for every x in I, then $f(b) - f(a) < M(b - a)$.*
(ii) *If $f'(x) > M$ for every x in I, then $f(b) - f(a) > M(b - a)$.*
(iii) *If $f'(x) \le M$ for every x in I, then $f(b) - f(a) \le M(b - a)$.*
(iv) *If $f'(x) \ge M$ for every x in I, then $f(b) - f(a) \ge M(b - a)$.*

Proof

We suppose that $f'(x) < M$ throughout I but that the conclusion in (i) fails; then there are points a and b in I with $a < b$ and

$$f(b) - f(a) \ge M(b - a). \tag{4.3.1}$$

Let c be the midpoint of $[a, b]$; then one of the inequalities

$$f(b) - f(c) \ge M(b - c), \quad f(c) - f(a) \ge M(c - a)$$

must hold, since if neither of these holds, then

$$\begin{aligned} f(b) - f(a) &= [f(b) - f(c)] + [f(c) - f(a)] \\ &< M(b - c) + M(c - a) \\ &= M(b - a), \end{aligned}$$

contrary to (4.3.1). This shows that the inequality (4.3.1) is transmitted from the interval $[a, b]$ to one of the intervals $[a, c]$ and $[c, b]$ obtained from $[a, b]$ by bisection; we denote this interval by $[a_1, b_1]$. By repeating this argument, we obtain an increasing sequence a_n and a decreasing sequence b_n, where $a_n < b_n$, both converging to a common limit x_0 in I, such that for every n,

$$f(b_n) - f(a_n) \ge M(b_n - a_n). \tag{4.3.2}$$

If $a_n = x_0$ for infinitely many n, or if $b_n = x_0$ for infinitely many n, we can let $n \to \infty$ in (4.3.2) and obtain $f'(x_0) \geq M$, which is contrary to our assumption.

If $a_n < x_0 < b_n$ for $n > n_0$, say, then for each n, one of the inequalities

$$f(b_n) - f(x_0) \geq M(b_n - x_0), \quad f(x_0) - f(a_n) \geq M(x_0 - a_n) \qquad (4.3.3)$$

holds, and clearly one of these must hold for infinitely many n. Letting $n \to \infty$ we again obtain $f'(x_0) \geq M$. The proof of (i) is complete, and (ii) follows by applying (i) to the function $-f(x)$ and using $-M$.

If $f'(x) \leq M$ for x in I, then for every positive ε and every x, $f'(x) < M + \varepsilon$. Applying (i) and then letting $\varepsilon \to 0$ gives (iii), and (iv) is proved similarly. ∎

Theorem 4.3.1 has the following useful extension. We recall that f is increasing on an interval I if $x \leq y$ implies that $f(x) \leq f(y)$, and that f is *strictly increasing* if $x < y$ implies $f(x) < f(y)$.

Theorem 4.3.2.

Suppose that the function $f: I \to \mathbb{R}$ is differentiable at each point of an open interval I, and that $a \in I$, $b \in I$, and $a < b$.

(i) *If $f'(x) > 0$ at each point of I, then f is strictly increasing on I.*
(ii) *If $f'(x) < 0$ at each point of I, then f is strictly decreasing on I.*
(iii) *If $f'(x) \geq 0$ at each point of I, then f is increasing on I.*
(iv) *If $f'(x) \leq 0$ at each point of I, then f is decreasing on I.*
(v) *If $f'(x) = 0$ at each point of I, then f is constant on I.*

Proof

Parts (i), (ii), (iii), and (iv) follow directly from Theorem 4.3.1 with $M = 0$. Finally, (v) follows from (iii) and (iv). ∎

We end this section with the complex version of Theorem 4.3.2(v).

Theorem 4.3.3.

Suppose that $f: \mathbb{C} \to \mathbb{C}$ is differentiable and that $f'(z) = 0$ at every point z of $\mathbb{C}$. Then f is constant on $\mathbb{C}$.

Proof

We write $f = u + iv$, where u and v are real-valued functions of a complex variable $x + iy$. Now, for any real x and x_0,

$$0 \leq \left| \frac{u(x) - u(x_0)}{x - x_0} \right| \leq \left| \frac{f(x) - f(x_0)}{x - x_0} \right|$$

(because for any complex number ζ we have $|\operatorname{Re}[\zeta]| \leq |\zeta|$), and by letting $x \to x_0$, we see that the real-valued function $u(x)$ is differentiable on $\mathbb{R}$

with $u'(x) = 0$ for every x. Thus, by Theorem 4.3.2, u is constant on $\mathbb{R}$. An entirely similar argument holds for the function $x \mapsto u(x + iy_0)$, for any fixed y_0; thus u is constant on each horizontal line in $\mathbb{C}$.

Now let $g(z) = -if(z) = v(z) - iu(z)$; then $g'(z) = -if'(z) = 0$ throughout $\mathbb{C}$, so by what we have just proved, v is constant on every horizontal line in $\mathbb{C}$. This shows that f is constant on every horizontal line in $\mathbb{C}$. Finally, let $h(z) = f(iz)$. Then $h'(z) = 0$ for every z, so that h is constant on every horizontal line. This means that f is constant on every vertical line, so that finally, f is constant on $\mathbb{C}$. ■

Exercises

1. Use the function x^3 to show that the derivative of a strictly increasing function may be zero at some point.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$ and $f'(x) = 1$ for every x. Show that for every x, $f(x) = x$.

3. Let $f(x) = (1+x)^n - 1 - nx$. Show that f is decreasing on $(-1, 0)$ and increasing on $(0, +\infty)$. Deduce that if $x > -1$ then $f(x) \geq f(0)$. Now consider Exercise 2.3.6.

4. Let f be defined by $f(0) = 0$, and for $x \neq 0$, $f(x) = x + x^2 \sin(1/x)$ [you may assume familiarity with the function $\sin x$ here]. Show that $f'(0) = 1$, *but that* f *is not strictly increasing on any interval that contains* 0.

5. Suppose that the complex-valued function f is differentiable on the set D, where

$$D = \{x + iy : x > 0 \text{ or } y \neq 0\},$$

and that $f'(z) = 0$ for every z in D. Prove that f is constant on D.

However, let u be defined in D by

$$u(x+iy) = \begin{cases} x^2 & \text{if } x > 0; \\ x^2 & \text{if } x \leq 0 \text{ and } y > 0; \\ -x^2 & \text{if } x \leq 0 \text{ and } y < 0. \end{cases}$$

Show (assuming a knowledge of partial derivatives) that

$$\frac{\partial u}{\partial y} = 0$$

at every point of D, yet $u(x + iy)$ is *not* independent of y.

4.4 The Cauchy Criterion

Suppose that f is defined on a directed set $(X, \succ)$ and that we wish to show that $\lim_{\succ} f$ exists. If we can guess the value, say α, of the limit, it is usually easiest to estimate $|f(x) - \alpha|$ directly and so prove that $f(x) \to \alpha$. However, there are many cases in which it is impossible to guess the limit but in which we still wish to know that the limit exists; for example, we may need to show that an infinite series converges even though we have no idea of its value. In these cases *we need a test for convergence that does not depend on knowing the limit.* The Cauchy Criterion is such a test, and roughly speaking, it says that $\lim_{\succ} f$ exists if and only if the variation of f is arbitrarily small on some final segment $X(x_0)$.

Theorem 4.4.1: the Cauchy Criterion.
Suppose that $f:X \to \mathbb{C}$ is defined on the directed set $(X, \succ)$. Then

(i) $\lim_{\succ} f$ *exists*
if and only if
(ii) *for every positive ε, there is some x_0 in X such that $|f(x) - f(y)| < \varepsilon$ whenever $x \succ x_0$ and $y \succ x_0$.*

Proof
Suppose first that $\lim_{\succ} f$ exists and equals α. Then, given any positive ε, there is an x_0 in X such that $|f(x) - \alpha| < \varepsilon/2$ whenever $x \succ x_0$. It follows that if $x \succ x_0$ and $y \succ x_0$, then

$$|f(x) - f(y)| \le |f(x) - \alpha| + |\alpha - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so that (ii) holds.

Now suppose that (ii) holds, and suppose also, for the moment, that f is real-valued. First, we apply (ii) with $\varepsilon = 1$, so there is some x_1 such that $|f(x) - f(y)| \le 1$ when $x \succ x_1$ and $y \succ x_1$. Select any y with $y \succ x_1$, and let $a_1 = f(y) - 1$, $b_1 = f(y) + 1$. Thus if $x \succ x_1$, then $f(x) \in [a_1, b_1]$, or equivalently,

(a) f maps $X(x_1)$ into $[a_1, b_1]$ of length ℓ, say.

As (ii) holds, there is some x_2 such that $|f(x) - f(y)| < \ell/3$ whenever $x \succ x_2$ and $y \succ x_2$, and we may clearly assume that $x_2 \succ x_1$. If we *trisect* the interval $[a_1, b_1]$ into three intervals $[a_1, c]$, $[c, d]$, $[d, b_1]$ of equal length, then we cannot have $x \succ x_2$, $y \succ x_2$, $f(x) \in [a_1, c]$, and $f(y) \in [d, b_1]$ (for then $|f(x) - f(y)| \ge \ell/3$), so that f maps $X(x_2)$ into one of the two intervals $[a_1, d]$, and $[c, b_1]$. Thus there is a subinterval $[a_2, b_2]$ of $[a_1, b_1]$ such that $x_2 \succ x_1$, and

(b) f maps $X(x_2)$ into $[a_2, b_2]$ of length $2\ell/3$.

The argument used to get from (a) to (b) can be repeated starting at (b), and this yields a sequence x_n in X such that f maps $X(x_n)$ into $[a_n, b_n]$, where

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots \le b_n \le \cdots \le b_2 \le b_1,$$

and where $\lim a_n = \lim b_n = c$, say. Given any positive ε, there is some n such that $c - \varepsilon < a_n \le b_n < c + \varepsilon$. It follows that if $x \succ x_n$, then $|f(x) - c| < \varepsilon$; thus $\lim_{\succ} f$ exists and equals c.

It remains to show that (ii) implies (i) for complex-valued functions, so suppose that $f = u + iv$, where u and v are real-valued functions. Now if f satisfies (ii), then because

$$|u(x) - u(y)| \le |f(x) - f(y)|, \quad |v(x) - v(y)| \le |f(x) - f(y)|,$$

we see that both u and v satisfy (ii). Thus $\lim_{\succ} u$ and $\lim_{\succ} v$ exist, and so too does $\lim_{\succ} u + iv$. ■

We have stated and proved the Cauchy Criterion, which is also known as the **General Principle of Convergence**, for functions defined on an arbitrary directed set, and we end this chapter by applying it to sequences and infinite series. First, we rewrite the criterion explicitly in terms of infinite series, then for sequences. We shall see in Chapter 5 that the Cauchy Criterion provides a powerful test for the convergence of infinite series.

Theorem 4.4.2: the Cauchy Criterion for infinite series.

The series $\sum_{n=1}^{\infty} a_n$ of complex numbers converges if and only if given any positive ε, there is an integer n_0 such that $|a_{m+1} + \cdots + a_n| < \varepsilon$ whenever $n > m \ge n_0$.

Proof

This is a direct rewording of Theorem 4.4.1 when $X = \mathbb{N}$ and $f(n) = a_1 + \cdots + a_n$. ■

Theorem 4.4.3: the Cauchy Criterion for sequences.

The sequence a_n of complex numbers converges if and only if given any positive ε, there is an integer n_0 such that $|a_n - a_m| < \varepsilon$ whenever $n > m \ge n_0$.

Proof

Again, this is a direct rewording of Theorem 4.4.1. ■

Exercises

1. Use the Cauchy Criterion to prove Theorem 3.6.4, that if a sequence of real numbers is increasing and bounded above, then it is convergent.

2. Use the Cauchy Criterion to prove that if $f : (a, b) \to \mathbb{R}$ is increasing and bounded, then the limits

$$\lim_{x \to a+} f(x), \quad \lim_{x \to b-} f(x)$$

both exist.

3. Suppose that f is a function from $(0, +\infty)$ to $\mathbb{R}$. Show that

$$\lim_{x \to +\infty} f(x)$$

exists if and only if given any positive ε, there is a positive number R such that $|f(x) - f(y)| < \varepsilon$ whenever $x > y > R$.

4. Suppose that for $n = 1, 2, 3, \ldots, 0 \leq a_n \leq 1/2^n$. Prove that $\sum_n a_n$ converges. Now suppose that $|b_n| \leq 1/2^n$. Show that $\sum_n b_n$ converges.

5 CHAPTER Infinite Series

Abstract
This chapter is devoted to a study of infinite series (both ordered and unordered), absolute convergence, rearrangements of infinite sums, and double series.

5.1 Infinite series

The definition of convergence of the *infinite series*

$$\sum_{n=0}^{\infty} a_n \tag{5.1.1}$$

was given in Definition 3.4.2, and we begin by rewriting this without mentioning a direction.

Definition 5.1.1.
The infinite series (5.1.1) *converges to* the number α if given any positive ε, there is some integer n_0 such that

$$|(a_0 + a_1 + \cdots + a_n) - \alpha| < \varepsilon$$

whenever $n > n_0$.

If the series (5.1.1) converges to α, we write either of the expressions

$$a_0 + a_1 + a_2 + \cdots = \alpha, \quad \sum_{n=0}^{\infty} a_n = \alpha.$$

Of course, a series may be the sum over any subset of the integers that is bounded below, and providing that the range of summation is understood, we often use the abbreviated form $\sum_n a_n$, or even just $\sum a_n$, for the series (5.1.1). An infinite series *diverges*, or is *divergent*, if it not convergent. We say that s_n given by

$$s_n = a_0 + a_1 + \cdots + a_n$$

is the nth *partial sum* of the series, and the series converges to α if and only if its sequence of partial sums converges to α.

It may come as a surprise to those meeting infinite series for the first time that *while we frequently need to know whether or not a given series converges, we often have little or no interest in the actual value of the infinite sum*; in other words, the *existence* of the infinite sum is usually more important than its value. Indeed, examples where we can obtain an explicit form for the value of an infinite sum are very rare, and in most cases where the value of the sum is important, we have to turn to computational methods to obtain estimates of it.

It is obvious that if the series (5.1.1) converges to α, then the series $a_1 + a_2 + \cdots$ converges to $\alpha - a_0$. Repeating this, we see that the convergence of an infinite series is unaffected by the addition, or the deletion, of a finite number of terms. The inclusion, or the removal, of brackets in an infinite sum is a little more complicated, and the problem here is probably best explained by an example. Consider the series $\sum_n a_n$, where $a_n = [1 + (-1)] = 0$ for every n. Clearly this series converges to 0. However, if we write down the series as

$$a_1 + a_2 + a_3 + \cdots = [1 + (-1)] + [1 + (-1)] + [1 + (-1)] + \cdots$$

and then remove the brackets [. . .], we obtain the series

$$1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots,$$

which does not converge (for the sequence of partial sums is $1, 0, 1, 0, \ldots$). In short, *we cannot remove brackets from an infinite series without some justification*. It should be clear, however, that we can always introduce brackets into a convergent series without destroying the convergence or altering the value of the infinite sum. Indeed, the introduction of brackets simply means that we are passing from a convergent sequence of partial sums to a sequence containing only some of the terms of the original sequence, and this will necessarily converge to the same value.

Quite generally, if $\sum a_n$ converges to α, then $a_1 + \cdots - a_n \to \alpha$, so that

$$a_{n+1} = (a_1 + \cdots + a_{n+1}) - (a_1 + \cdots + a_n) \to \alpha - \alpha = 0.$$

This shows that $a_n \to 0$ is a *necessary condition* for the convergence of $\sum a_n$. Thus, for example, $\sum x^n$ diverges when $|x| \geq 1$.

As the definition of convergence of the sum (5.1.1) is just a special case of the general limit defined in Definition 3.3.1, it follows that the general theorems about limits proved in Chapters 3 and 4 are applicable to infinite series. For example, rewriting Theorem 3.5.2 in the context of infinite series, we have the following result.

Theorem 5.1.2.
Suppose that $\sum a_n$ and $\sum b_n$ converge to α and β, respectively. Then for any numbers s and t, the series $\sum (sa_n + tb_n)$ converges to $s\alpha + t\beta$.

As an example,

$$\sum_{n=0}^{\infty}\left(\frac{2}{3^n}+\frac{1}{5^n}\right) = 2\sum_{n=0}^{\infty}\frac{1}{3^n}+\sum_{n=0}^{\infty}\frac{1}{5^n} = \frac{17}{4}.$$

If we apply Theorem 3.6.4 (that an increasing sequence of real numbers that is bounded above converges) to the partial sums of an infinite series $\sum_n a_n$ of nonnegative terms, we obtain the following primitive, yet powerful, test for the convergence of an infinite series.

Theorem 5.1.3.
Suppose that for each n, $a_n \geq 0$. Then $\sum a_n$ converges if and only if there is a positive M such that for each n,

$$a_1 + a_2 + \cdots + a_n \leq M. \tag{5.1.2}$$

Further, if this is so, then $\sum_n a_n \leq M$.

This result justifies the use of the decimal expansion of a real number, for it implies that any series of the form

$$a_0 + \sum_{n=1}^{\infty}\frac{a_n}{10^n}, \tag{5.1.3}$$

where $a_0 \in \mathbb{Z}$ and the remaining a_j are in $\{0, 1, \ldots, 9\}$, converges. This follows from Theorem 5.1.3 because

$$a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} \leq a_0 + 9\sum_{k=1}^{\infty}\frac{1}{10^k} = a_0 + 1,$$

and the value of the series in (5.1.3) is the value that we assign to the infinite decimal expansion $a_0 \cdot a_1 a_2 \ldots$.

We shall often refer to Theorem 5.1.3 by saying that *the partial sums of the infinite series are bounded above.* The following result illustrates the use of Theorem 5.1.3, and it is an important example in its own right. In this example, we shall assume that we know what is meant by n^k, where $k > 0$ (this will be defined in Section 6.4).

Theorem 5.1.4.
The series

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \tag{5.1.4}$$

converges if $k > 1$ *and diverges if* $0 < k \leq 1$.

Proof
The proof of divergence is based on the simple inequality

$$\frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} \geq (2^{n+1} - 2^n)\left(\frac{1}{2^{n+1}}\right) = \frac{1}{2}.$$

This shows that

$$\sum_{n=1}^{2^m} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{m-1} + 1} + \cdots + \frac{1}{2^m}\right) > \frac{m}{2},$$

and hence that $\sum 1/n$ diverges. If $0 < k < 1$, then

$$\frac{1}{n} < \frac{1}{n^k},$$

so that (by Theorem 5.1.3) the series (5.1.4) also diverges for these k. For an alternative proof of this part, see Exercise 5.1.7.

The argument for convergence is similar, for when $k > 1$,

$$\frac{1}{(2^n + 1)^k} + \cdots + \frac{1}{(2^{n+1})^k} \leq 2^n \left(\frac{1}{(2^n + 1)^k}\right) \leq \frac{2^n}{2^{nk}} \leq \left(\frac{2}{2^k}\right)^n,$$

so that the series converges when $2^k > 2$, that is, when $k > 1$. ■

We have seen that if $\sum a_n$ converges, then $a_n \to 0$. Example 5.1.4 shows that the converse is false, for $\sum n^{-1}$ is an example of a divergent series $\sum a_n$ in which $a_n \to 0$. Explicitly,

(i) if $\sum a_n$ converges, then $a_n \to 0$, but
(ii) $a_n \to 0$ *does not imply that* $\sum a_n$ converges.

We shall now state and prove several corollaries of Theorem 5.1.3. Although these are used frequently, each is only a small step away from the fundamental Theorem 5.1.3.

Theorem 5.1.5: the Comparison Test.
Suppose that for each n, $0 \leq a_n \leq b_n$, *and that* $\sum b_n$ *converges to some value* B. *Then* $\sum a_n$ *converges to some value* A, *where* $A \leq B$.

Proof

For each n, $a_1 + \cdots + a_n \le b_1 + \cdots + b_n \le B$, and the result follows immediately from Theorem 5.1.3. A particularly simple case of this is when there are constants ℓ and M with $0 \le \ell < 1$ and $M \ge 0$ such that for each n, $0 \le a_n \le M\ell^n$. In these circumstances, $\sum a_n$ converges. ■

Theorem 5.1.6: the Ratio Test.

Suppose that $\sum a_n$ is a series of positive terms, and that there is some ℓ with $0 < \ell < 1$ such that for each n, $a_{n+1}/a_n \le \ell$. Then $\sum a_n$ converges.

Proof

The hypotheses imply that $a_n \le \ell^n a_0$, and convergence follows by the remark at the end of the proof of Theorem 5.1.5. Of course, as the alteration of a finite number of terms in a series does not affect its convergence, it suffices to assume that $a_{n+1}/a_n \le \ell$ for all but a finite set of n. ■

The following two examples illustrate these simple tests.

Example 5.1.7.

We show that the series

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2} + (-1)^n \frac{1}{3}\right)^n$$

converges. Writing a_n for the nth term, and noting that $a_n > 0$, we have

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n}\left(\frac{1}{2} + (-1)^{n+1}\frac{1}{3}\right) \le \frac{5(n+1)}{6n} \le \frac{6}{7}$$

when $n \ge 35$. The Ratio Test now implies convergence. □

Example 5.1.8.

Consider the series

$$\sum_{n=1}^{\infty} a_n, \quad a_n = \frac{n!}{n^n}.$$

Clearly, for $n \ge 2$ we have $0 < a_n \le 2/n^2$, so that from Theorems 5.1.4 and 5.1.5, the series converges. Of course, we can if we wish obtain a far better estimate of a_n than $a_n \le 2/n^2$, but if our only objective is to establish convergence there is no need to do so. If we attempt to prove convergence here by applying the Ratio Test, we obtain

$$\frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n}\right)^{-n},$$

and the limiting behaviour of this will be discussed in Theorem 7.2.3. □

Tests for the convergence or divergence of infinite series in which *some of the terms are positive and some are negative* are much more delicate. We already have one such test, namely Theorem 4.4.2, which for emphasis we restate here.

Cauchy's Criterion for infinite series.

The series $a_0 + a_1 + a_2 + \cdots$ of complex numbers converges if and only if given any positive ε, there is an integer n_0 such that $|a_{m+1} + \cdots + a_n| \leq \varepsilon$ whenever $n > m \geq n_0$.

The next test, known as the Alternating Series Test, depends crucially on the terms in the series having alternating signs.

Theorem 5.1.9: the Alternating Series Test.

Suppose that the real numbers a_n satisfy $a_0 \geq a_1 \geq a_2 \geq \cdots \geq 0$, and that $a_n \to 0$ as $n \to \infty$. Then the series $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

Proof

Let $s_m = a_0 - a_1 + a_2 + \cdots + (-1)^m a_m$. It is clear that

$$s_1 \leq s_3 \leq s_5 \leq \cdots, \qquad s_0 \geq s_2 \geq s_4 \geq \cdots ;$$

for example, $s_7 - s_5 = a_6 - a_7 \geq 0$ and $s_6 - s_4 = a_6 - a_5 \leq 0$. Next, observe that

$$s_1 \leq s_{2p+1} = s_{2p} - a_{2p+1} \leq s_{2p} \leq s_0,$$

so that the two sequences $s_0, s_2, s_4, \ldots$ and $s_1, s_3, s_5, \ldots$ both converge. As $s_{2p} - s_{2p+1} \to 0$, the two sequences have a common limit α, say, so that $s_n \to \alpha$ as $n \to \infty$. The proof is complete. ■

We end with an example to illustrate the Alternating Series Test.

Example 5.1.10.

The Alternating Series Test shows that for each positive k the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^k}$$

converges. □

Exercises

1. Show that if $|a_n| \le M$ for each n, then the following two series are convergent:

$$\sum_{n=1}^{\infty} \frac{a_n}{2^n}, \quad \sum_{n=1}^{\infty} \frac{a_n}{n!}.$$

2. By considering a linear combination of the two series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}, \quad \sum_{n=1}^{\infty} \frac{1}{2^n},$$

show that the first series converges to 2.

3. Show that the first of the three series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}, \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)}$$

converges to 1. Evaluate the second and the third series.

4. Show that if $|x| < 1$ then

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^4} + \cdots = \begin{cases} 0 & \text{if } x = 0; \\ 1 + x^2 & \text{if } 0 < |x| < 1. \end{cases}$$

This shows that an infinite sum of continuous functions need not be a continuous function!

5. Show that for any complex numbers d and z,

$$1 + (1+d)z + (1+2d)z^2 + \cdots + (1+nd)z^n = \frac{1-(1+nd)z^{n+1}}{1-x} + \frac{dz(1-z^n)}{(1-z)^2}.$$

Deduce that if $|z| < 1$ then

$$1 + 2z + 3z^2 + 4z^3 + \cdots = \frac{1}{(1-z)^2}.$$

6. Use Cauchy's Criterion to prove the Alternating Series Test.

7. Use Cauchy's Criterion to prove Pringsheim's Theorem, namely that if $a_1 \ge a_2 \ge \cdots \ge 0$ and if $\sum a_n$ converges, then $na_n \to 0$. Deduce that $\sum n^{-1}$ diverges.

   Show, more generally, that if $a_1 \ge a_2 \ge \cdots \ge 0$ and if $a_n > 1/n$ for infinitely many n, then $\sum a_n$ diverges.

8. Suppose that $0 \le x_n < 1$ for $n = 1, 2, \ldots$. Show that if $\sum x_n$ converges, then so do $\sum x_n^2$ and $\sum x_n/(1-x_n)$. Are either of the converse statements true?

9. By finding a formula for the partial sums, show that if $|x| < 1$, then

$$\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \cdots = \begin{cases} \dfrac{x}{1-x} & \text{if } |x| < 1; \\ \dfrac{1}{1-x} & \text{if } |x| > 1. \end{cases}$$

5.2 Unordered sums

As the definition of convergence of an infinite series $\sum_n a_n$ includes the phrase '$n > n_0$', it depends explicitly on the usual ordering of $\mathbb{N}$, and it would perhaps be more natural (and honest) to refer to it as an *ordered sum*. Unordered sums occur in probability theory, where one considers a set Ω of 'events', each with a probability, such that the the probabilities sum to 1. Now usually there is no natural ordering on Ω, so before we can sum the probabilities, we have to construct an order on Ω. This usually presents no problem, except that it is then necessary to check that any other ordering of Ω would yield the same sum (as otherwise, we would have achieved nothing).

As we are concerned with elegance as well as truth, it seems unsatisfactory to have to introduce an order and then immediately show that the order is irrelevant, and we prefer an alternative approach, which we shall now develop, in which order plays no role at all. Given a nonempty (unordered) set X and a function f defined on X, we shall define the *unordered sum*

$$\sum_{x \in X} f(x).$$

The properties of addition of complex numbers guarantee that we can add any *finite* set E of complex numbers, in any order, to obtain a unique value. It follows that if f is a complex-valued function defined on a set X and if E is a finite subset of X, then we can attach a unique value to the finite sum

$$\sum_{x \in E} f(x).$$

Now let $\mathcal{S}(X)$ be the collection of all *finite* subsets of X, and for any pair A and B of finite subsets of X, define $A \succ B$ if and only if $A \supset B$. Then, as is easily checked, $\succ$ is a direction on $\mathcal{S}(X)$ and we can define the function $F : \mathcal{S}(X) \to \mathbb{C}$ by

$$F(A) = \sum_{x \in A} f(x).$$

Definition 5.2.1a.
The *unordered sum* $\sum_{x \in X} f(x)$ of f over X is the limit of F with respect to $\succ$ whenever this limits exists.

It is instructive to rewrite this definition without explicit mention of the direction $\succ$.

Definition 5.2.1b.
The *unordered sum* $\sum_{x \in X} f(x)$ exists and equals α if given any positive ε, there is a finite subset E of X such that

$$\left| \sum_{x \in A} f(x) - \alpha \right| < \varepsilon$$

whenever A is a finite subset of X and $A \supset E$.

Immediately we see that if the two sums

$$\sum_{x \in X} f(x), \quad \sum_{x \in X} g(x)$$

exist, then for every pair of constants α and β,

$$\sum_{x \in X} [\alpha f(x) + \beta g(x)] = \alpha \sum_{x \in X} f(x) + \beta \sum_{x \in X} g(x)$$

in the sense that the sum on the left exists and equals the expression on the right.

The reader will no doubt have noticed that we now have two different definitions of infinite sums over the set $\mathbb{N}$ of positive integers, namely the sum

$$\sum_{n=1}^{\infty} a_n \tag{5.2.1}$$

defined in terms of the sequence of partial sums (and explicitly depending on the order $>$ on $\mathbb{N}$) and the unordered sum

$$\sum_{n \in \mathbb{N}} a_n \tag{5.2.2}$$

in which we are summing over the set $\mathbb{N}$ but deliberately ignoring the order $>$ on this set. We shall compare these two sums (which are *not* the same; see Exercise 5.2.1) in the next section, but we ask the reader to be patient, for this will be much easier after we have proved a few simple results about unordered sums. We do emphasize, however, the difference between the notation (5.2.1) for the ordered sum and the notation (5.2.2) for the unordered sum.

We end with three results about unordered sums, and we shall apply these frequently in the rest of this chapter. Loosely speaking, the first of these confirms in a very general way that if we do order X, then the corresponding ordered sum is independent of the chosen order.

Theorem 5.2.2.

Suppose that the unordered sum $\sum_{x \in X} f(x)$ *exists and equals* α. *Suppose also that* $E_1, E_2, \ldots$ *is a sequence of finite subsets of* X *such that*

$$E_1 \subset E_2 \subset E_3 \subset \cdots, \quad \bigcup_{n=1}^{\infty} E_n = X.$$

Then

$$\lim_{n \to \infty} \sum_{x \in E_n} f(x) \tag{5.2.3}$$

exists and equals α. *In particular, if* $F_1, F_2, \ldots$ *satisfies the same properties as the* E_n, *then*

$$\lim_{n \to \infty} \left(\sum_{x \in E_n} f(x) \right) = \lim_{m \to \infty} \left(\sum_{x \in F_m} f(x) \right). \tag{5.2.4}$$

Proof

Given any positive ε, there is a finite subset K of X such that

$$\left| \sum_{x \in A} f(x) - \alpha \right| < \varepsilon$$

whenever A is a finite subset of X with $A \supset K$. Clearly, there is some integer N such that $E_N \supset K$. Thus, for $n > N$, $E_n \supset K$, and hence

$$\left| \sum_{x \in E_n} f(x) - \alpha \right| < \varepsilon.$$

This shows that the limit in (5.2.3) exists and equals α. Given this, (5.2.4) is obvious. ■

The next result gives a sufficient condition for an unordered sum to exist.

Theorem 5.2.3.

Let X *be any set, and let* $f: X \to \mathbb{C}$ *be any function. Then the following are equivalent*:

(i) *the sum* $\sum_{x \in X} f(x)$ *exists;*
(ii) *the sum* $\sum_{x \in X} |f(x)|$ *exists;*
(iii) *there exists a constant* M *such that for every finite subset* A *of* X, $\sum_{x \in A} |f(x)| \leq M$.

It is clear that this result has the following corollary.

Corollary 5.2.4.
Suppose that the unordered sum $\sum_{x\in X} f(x)$ exists. Then for any subset Y of X, the unordered sum $\sum_{x\in Y} f(x)$ exists.

Proof of Theorem 5.2.3.
First, the equivalence of (ii) and (iii) follows from the general Theorem 3.6.3 on limits of functions that are monotonic with respect to a direction $\succ$. To be explicit, write

$$F^*(A) = \sum_{x\in A} |f(x)|.$$

If $B \succ A$ then $B \supset A$, so that $F^*(B) \geq F^*(A)$. Thus, from Theorem 3.6.3, the limit of F^* exists (that is, the sum in (ii) converges) if and only if the function F^* is bounded above, and this is (iii).

We note that as (i) and (ii) are the same statement when f is nonnegative ($f \geq 0$), the theorem has been proved for nonnegative functions. This observation will enable us to show that (iii) implies (i).

We suppose now that f satisfies (iii) and write $f = u + iv$. As $|u| \leq |f|$ and $|v| \leq |f|$, it is clear that u and v satisfy (iii). Next, define the functions u^+ and u^- by

$$u^+(x) = \frac{1}{2}\left(|u(x)| + u(x)\right), \quad u^-(x) = \frac{1}{2}\left(|u(x)| - u(x)\right),$$

so that $u(x) = u^+(x) - u^-(x)$ and

$$0 \leq u^+(x) \leq |u(x)|, \quad 0 \leq u^-(x) \leq |u(x)|.$$

As u satisfies (iii), so do u^+ and u^-. And as (i) and (iii) are equivalent for nonnegative functions, we can now conclude that u^+ and u^- satisfy (i), and hence that

$$\sum_{x\in X} u^+(x) - \sum_{x\in X} u^-(x) = \sum_{x\in X} u(x)$$

exists. The same argument shows that the coresponding sum for v exists. Thus, finally,

$$\sum_{x\in X} u(x) + i\sum_{x\in X} v(x) = \sum_{x\in X} f(x)$$

exists. This proves that (iii) implies (i).

Next, we show that (i) implies (iii). From (i), there is a number α (the value of the sum) and a finite subset K of X such that

$$\left|\sum_{x\in B} f(x) - \alpha\right| < 1$$

when B is finite and $B \supset K$. Now take any finite subset A of X and let

$$A^+ = \{x \in A : x \notin K, f(x) \geq 0\},$$
$$A^- = \{x \in A : x \notin K, f(x) < 0\}.$$

As $f(x) \geq 0$ when $x \in A^+$, and as A^+ and K are disjoint, we have

$$\begin{aligned}\sum_{x\in A^+} |f(x)| &= \Big|\sum_{x\in A^+} f(x)\Big| \\ &= \Big|\sum_{x\in A^+\cup K} f(x) - \sum_{x\in K} f(x)\Big| \\ &\leq \Big|\sum_{x\in A^+\cup K} f(x) - \alpha\Big| + \Big|\alpha - \sum_{x\in K} f(x)\Big| \\ &\leq 2.\end{aligned}$$

A similar argument holds for A^-, so finally, we have

$$\begin{aligned}\sum_{x\in A} |f(x)| &\leq \sum_{x\in K\cup A} |f(x)| \\ &= \sum_{x\in A^+} |f(x)| + \sum_{x\in K} |f(x)| + \sum_{x\in A^-} |f(x)| \\ &\leq 4 + \sum_{x\in K} |f(x)| \\ &\leq 5 + |\alpha|.\end{aligned}$$

This shows that (i) implies (iii), and the proof is complete. ■

Exercises

1. Let $X = \mathbb{N}$ and let $f : X \to \mathbb{R}$ be defined by $f(n) = (-1)^n/n$. Show that the sum (5.2.1) exists but that the sum (5.2.2) does not.

2. Suppose that $a_1, a_2, \ldots$ are such that $\sum_n |a_n|$ converges. By appealing to Theorem 5.2.3 and then taking $E_n = \{1, 2, \ldots, n\}$ in Theorem 5.2.2, show that the sum $\sum_n a_n$ converges.

   Show that

$$a_1 + a_2 + a_4 + a_3 + a_6 + a_8 + a_5 + \cdots$$

converges and equals $a_1 + a_2 + a_3 + \cdots$. Formally, we define a function $\varphi : \mathbb{N} \to \mathbb{N}$ by

$$\varphi(3n) = 4n;$$
$$\varphi(3n+1) = 2n+1;$$
$$\varphi(3n+2) = 4n+2.$$

Show that φ is a bijection of $\mathbb{N}$ onto itself, and convince yourself that the series given above is $a_{\varphi(1)} + a_{\varphi(2)} + a_{\varphi(3)} + \cdots$.

3. Show that if $0 < a < 1$ and $0 < b < 1$, then the unordered sum

$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a^m b^n$$

exists. What is its value?

4. Let $\mathbb{Z}$ denote the set of integers. Show that the unordered sum

$$\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} \frac{1}{m^2+n^2+1}$$

diverges.

5. Show that the sum

$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} \frac{1}{(m^2+n^2)^k}$$

converges if and only if $k > 1$. What can you say about the sum

$$\sum_{(m,n,p)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \frac{1}{(m^2+n^2+p^2)^k}.$$

5.3 Absolute convergence and rearrangements

Many tests for convergence of a series apply only to series with nonnegative terms. If some of the terms in a series $\sum a_n$ are negative, we can apply these tests to the modified series $\sum_{n=1}^{\infty} |a_n|$, and in this section we shall see why this is a useful thing to do.

Definition 5.3.1.
We say that the infinite series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

We begin by comparing the following three forms of convergence of infinite sums over the set $\mathbb{N}$ of positive integers. First, we have the *ordered* sum

$$\sum_{n=1}^{\infty} a_n. \tag{5.3.1}$$

This is defined in terms of the sequence of partial sums and depends explicitly on the ordering $>$ on the integers. Next, we have the *ordered sum of absolute values*, namely

$$\sum_{n=1}^{\infty} |a_n|, \tag{5.3.2}$$

which is used in the definition of absolute convergence. Finally, we have the *unordered sum*

$$\sum_{n\in\mathbb{N}} a_n \tag{5.3.3}$$

in which we are summing over the set $\mathbb{N}$ but deliberately ignoring the order $>$. The relation between the three series is given in the next result.

Theorem 5.3.2.

The two sums

$$\sum_{n\in\mathbb{N}} a_n, \quad \sum_{n=1}^{\infty} |a_n| \tag{5.3.4}$$

are both convergent or both divergent. If they are both convergent, then $\sum_{n=1}^{\infty} a_n$ *is also convergent,*

$$\sum_{n=1}^{\infty} a_n = \sum_{n\in\mathbb{N}} a_n, \tag{5.3.5}$$

and

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|. \tag{5.3.6}$$

Proof

Theorems 5.1.3 and 5.2.3 show that the sums in (5.3.4) converge or diverge together, for they are both equivalent to the existence of some constant M such that for all n,

$$|a_1| + |a_2| + \cdots + |a_n| \leq M.$$

Suppose now that the sums converge. Then we may apply Theorem 5.2.2 with

$$X = \mathbb{N}, \quad E_n = \{1, 2, \ldots, n\}, \quad f(n) = a_n,$$

and this gives (5.3.5).

Finally, for any numbers $a_1, a_2, \ldots, a_n$,

$$|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|.$$

Now by definition,

$$|a_1| + \cdots + |a_n| \to \sum_{n=1}^{\infty} |a_n|,$$

and from Theorem 3.3.9,

$$|a_1 + \cdots + a_n| \to \left|\sum_{n=1}^{\infty} a_n\right|$$

The inequality (5.3.6) now follows because inequalities involving $\leq$ are preserved under limits (see Theorem 3.6.1). ■

The following result is an immediate consequence of Theorem 5.3.2 (see also Exercise 5.3.5).

Theorem 5.3.3.
An absolutely convergent series is convergent.

An ordered series *may be convergent but not absolutely convergent*, and such a series is said to be *conditionally convergent*. The reason for this name is that if $\sum a_n$ is conditionally convergent, then the unordered sum associated with it does not converge; thus the convergence of $\sum a_n$ is conditional on the particular order in which the a_n are summed. The following example illustrates this point.

EXAMPLE 5.3.4.
Consider the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots.$$

We know that this series converges and that it is not absolutely convergent (see Example 5.1.10 and Theorem 5.1.4). According to Theorem 5.3.2, the unordered sum

$$\sum_{n \in \mathbb{N}} \frac{(-1)^n}{n}$$

must diverge, and it is easy to see directly that this is so. Indeed, as $\sum 1/n$ diverges, given any positive K, there is a finite set of (positive) terms of the series whose sum exceeds K, so that the unordered sum does not exist.

Suppose that we now form the ordered sum *but take the terms in a different order*; for example, let us form the series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \cdots.$$

Let s_n be the sum of the first n terms of the original series, and let σ_n be the sum of the first n terms of the modified series. Then

$$\begin{aligned}\sigma_{3n} &= \sum_{k=0}^{n-1}\left(\frac{1}{2k+1} - \frac{1}{4k+2} - \frac{1}{4k+4}\right)\\ &= \sum_{k=0}^{n-1}\left(\frac{1}{2(2k+1)} - \frac{1}{2(2k+2)}\right)\\ &= \frac{1}{2}s_{2n}.\end{aligned}$$

As $\sigma_{n+1} - \sigma_n \to 0$, it is now easy to see that the modified series converges to a value that is half that of the original series! This is a general phenomenon; if we sum the terms of a conditionally convergent series in a different order, we may well get a different answer! □

We turn now to a more careful discussion of this phenomenon and discuss what we mean by a rearrangement of an infinite series. Informally, a rearrangement of the infinite series

$$a_1 + a_2 + a_3 + \cdots \tag{5.3.6}$$

is simply the ordered infinite sum obtained by placing the terms of the series in a different order. For example, if for every odd integer n we interchange n and $n+1$, we arrive at the rearranged series

$$a_2 + a_1 + a_4 + a_3 + \cdots. \tag{5.3.7}$$

We emphasize that in a rearrangement, we are summing *the same function* $n \mapsto a_n$ on $\mathbb{N}$, but we are placing *a different direction* (or order) on $\mathbb{N}$; in the case of (5.3.7), this direction is $2 \prec 1 \prec 4 \prec 3 \prec \cdots$. There is no reason to suppose that a series and a rearrangement of it converge or diverge together, nor that if they both converge their sums are equal. In fact, neither of these need be true; there are examples of a convergent series with a divergent rearrangement, and examples of a series and a rearrangement of it that converge to different values. The good news is that *if a series is absolutely convergent (equivalently, and preferably, if the unordered sum exists), then the series, and every rearrangement of it, converge to the same value, namely the value of the unordered sum.*

The conventional treatment of this topic in texts on analysis avoids reference to unordered sums, but as we have just seen, these do have a part to play. Indeed, it is the value of the convergent unordered sum that plays the key role here, and an ordered sum, taken over any order whatsoever, always converges to this fundamental value.

A formal definition of a rearrangement of a series follows. After that, we state and prove the invariance of the sum under rearrangements.

Definition 5.3.5

A series $b_1 + b_2 + b_3 + \cdots$ is a *rearrangement* of the series $a_1 + a_2 + a_3 + \cdots$ if there is a bijection ϕ of $\mathbb{N}$ onto itself such that $b_n = a_{\phi(n)}$.

Theorem 5.3.6.

If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then every rearrangement of it converges to the same value, namely the value of the unordered sum.

Proof

Let the rearrangement be given by the bijection $\phi : \mathbb{N} \to \mathbb{N}$, and let

$$E_n = \{1, 2, \ldots, n\}, \quad F_n = \{\phi(1), \phi(2), \ldots, \phi(n)\}.$$

As these sets satisfy the conditions of Theorem 5.2.2, the desired result follows immediately from this theorem. ■

We end this section by mentioning a result that shows the striking effect of rearranging conditionally convergent series. We omit the proof (which is not difficult).

Theorem 5.3.7.

Suppose that the series $\sum a_n$ is convergent but not absolutely convergent. Given any real number α, there is some rearrangement of $\sum a_n$ whose sum is α.

In short, by suitably rearranging a conditionally convergent series (for example, $\sum(-1)^n/n$), we can force the sum to be π, or $\sqrt{2}^{\sqrt{3}}$, or whatever we want! An example of this behaviour was given in Example 5.3.4, and another occurs in Exercise 5.3.2.

Exercises

1. Let s be the sum of the series
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots.$$
The point of Example 5.3.4 would be lost if $s = 0$. Show that $s \neq 0$.

2. Show that the series
$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$$

converges to the value $3s/2$, where s is given in Exercise 5.3.1.

3. Construct a rearrangement of the series in Exercise 5.3.1 that diverges.

4. We know that the series $\sum(-1)^n/n$ is convergent but not absolutely convergent. Show that the series $\sum(-1)^n/n^t$ is
   (i) divergent if and only if $t \leq 0$;
   (ii) convergent if and only if $t > 0$;
   (ii) absolutely convergent if and only if $t > 1$.

5. Give a direct proof of Theorem 5.3.3 (that an absolutely convergent series is convergent) by applying Cauchy's Criterion (Theorem 4.4.2) to the inequality

$$|a_m + a_{m+1} + \cdots + a_n| \leq |a_m| + |a_{m+1}| + \cdots + |a_n|.$$

6. Let $\sum_n a_n$ be a series of real numbers, and define

$$a_n^+ = \frac{1}{2}(|a_n| + a_n), \quad a_n^- = \frac{1}{2}(|a_n| - a_n).$$

Show that $\sum_n a_n$ is absolutely convergent if and only if both of the ordered sums $\sum_n a_n^+$ and $\sum_n a_n^-$ converge.

5.4 The Cauchy Product

We are concerned here with the question of whether or not

$$(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$
$$= (a_0 b_0) + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots, \quad (5.4.1)$$

or, more generally, whether for given series $\sum_n a_n$ and $\sum_n b_n$,

$$(a_0 + a_1 + a_2 + a_3 + \cdots)(b_0 + b_1 + b_2 + b_3 + \cdots)$$
$$= (a_0 b_0) + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots. \quad (5.4.2)$$

The series on the right-hand side is the Cauchy Product of the two series on the left.

Definition 5.4.1.
The *Cauchy Product* of the series $a_0 + a_1 + a_2 + \cdots$ and $b_0 + b_1 + b_2 + \cdots$ is the series

$$\sum_{n=0}^{\infty} c_n, \quad c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0. \quad (5.4.3)$$

We shall discuss (5.4.1) later in the text and concentrate now on (5.4.2). However plausible (5.4.2) may seem, it may be false, and we begin with an example of this type.

EXAMPLE 5.4.2.
Let a_n and b_n be given by

$$a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}, \qquad n = 0, 1, 2, \dots .$$

The Alternating Series Test shows that the series $\sum a_n$ and $\sum b_n$ converge. However, for each m we have

$$a_0 b_m + a_1 b_{m-1} + \cdots + a_{m-1} b_1 + a_m b_0 = (-1)^m \sum_{k=0}^{m} \frac{1}{\sqrt{k+1}\sqrt{m-k+1}}.$$

The Arithmetic-Geometric Mean inequality (namely that if $x > 0$ and $y > 0$, then $(\sqrt{x} - \sqrt{y})^2 \geq 0$, so that $x + y \geq 2\sqrt{xy}$) shows that

$$2\sqrt{k+1}\sqrt{m-k+1} \leq m + 2.$$

Thus

$$|a_0 b_m + a_1 b_{m-1} + \cdots + a_{m-1} b_1 + a_m b_0| \geq \frac{2(m+1)}{m+2} \geq 1,$$

and it follows from this that the Cauchy Product diverges. □

The example in Exercise 5.4.1 is one in which the two series $\sum_n a_n$ and $\sum_n b_n$ diverge but their Cauchy Product converges. The same example gives an instance in which the Cauchy Product converges, yet the series

$$a_0 b_0 + a_0 b_1 + a_1 b_0 + a_0 b_2 + a_1 b_1 + a_2 b_0 + \cdots \tag{5.4.4}$$

diverges, and this emphasises the fact that the series (5.4.4) is *not* the Cauchy product of the two series (the reader may recall earlier comments about removing 'brackets' from convergent series). Despite these negative examples, we expect (5.4.2) to hold in *some* circumstances, and indeed it does.

Theorem 5.4.3.
Suppose that the series

$$\sum_{n=0}^{\infty} a_n, \quad \sum_{n=0}^{\infty} b_n \tag{5.4.5}$$

are absolutely convergent and that their sums are A and B, respectively. Then their Cauchy Product

$$\sum_n c_n, \quad c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0,$$

is absolutely convergent and its sum is AB.

Proof
We consider the set

$$X = \{(m, n) : m, n = 0, 1, 2, \dots\}$$

and the function $f : X \to \mathbb{C}$ defined by $f(m, n) = a_m b_n$. Because the series in (5.4.5) are absolutely convergent, Theorem 5.2.3(iii) implies that $\sum_{x \in X} f(x)$ exists. Now let

$$R_p = \{(m, n) : 0 \le m \le p,\ 0 \le n \le p\}, \quad T_q = \{(m, n) : 0 \le m + n \le q\}.$$

It is clear that these sets satisfy the hypotheses placed on the sets E_n in Theorem 5.2.2, so that

$$\lim_{p \to \infty} \sum_{(m,n) \in R_p} f(m, n) = \lim_{q \to \infty} \sum_{(m,n) \in T_q} f(m, n).$$

The desired result now follows, as

$$\sum_{(m,n) \in R_p} f(m, n) = (a_0 + \cdots + a_n)(b_0 + \cdots + b_n),$$

which tends to AB, while

$$\sum_{(m,n) \in T_q} f(m, n)$$

is just the partial sum of the Cauchy Product of $\sum a_n$ and $\sum b_n$. ■

We end with a simple application of Theorem 5.4.3. We know that if $|z| < 1$, then

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.$$

It follows that

$$\left(\frac{1}{1 - z}\right)^2 = 1 + 2z + 3z^2 + \cdots,$$

because the expression on the left must be the Cauchy Product $\sum c_n$ of $\sum z^n$ with itself, and

$$c_n = z^0 z^n + z^1 z^{n-1} + \cdots + z^{n-1} z^1 + z^n z^0 = (n + 1)z^n.$$

Exercises

1. Show that if $a_0 = 2$, $b_0 = -1$, and, for $n \ge 1$, $a_n = 2^n$ and $b_n = 1$, then $\sum a_n$ and $\sum b_n$ diverge, whereas their Cauchy product converges. Show also that in this case the series (5.4.4) diverges.
2. Express $(1 - z)^{-3}$, where $|z| < 1$, as a series $a_0 + a_1 z + a_2 z^2 + \cdots$.
3. Show that (5.4.1) holds if $|x| < 1$ and if for some constant M and all n, $|a_n| \le M$ and $|b_n| \le M$.

4. Assuming that both of the series converge, show that

$$(1-x)\sum_{n=0}^{\infty} s_n x^n = \sum_{n=0}^{\infty} a_n x^n,$$

where $s_n = a_0 + a_1 + \cdots + a_n$. This shows that if $0 < a_n \le 1$ and $0 < x < 1$ then

$$\frac{\sum_{n=0}^{\infty}(a_0 + a_1 + \cdots + a_n)x^n}{\sum_{n=0}^{\infty} a_n x^n} = \frac{1}{1-x}$$

and so is independent of the choice of the a_n.

Assuming now that the series are absolutely convergent, verify this result by considering the Cauchy Product of $\sum a_n x^n$ and $\sum x^n$.

5. Find the Cauchy Product of $\sum_{n=0}^{\infty} 1/(2^n\, n!)$ with itself.

5.5 Iterated sums

Suppose that a_{mn} is defined when m and n are in $\mathbb{N}$. It will be helpful to visualise these numbers arranged in an array as follows:

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \tag{5.5.1}$$

Leaving aside questions of convergence for the moment, we ask whether or not the two *iterated*, or *repeated, sums*

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{mn}\right), \quad \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{mn}\right) \tag{5.5.2}$$

are equal. This is the same as asking whether or not summing over the rows in (5.5.1) and then adding the results is the same as summing over the columns and then adding these results. To persuade the reader that there is work to be done, we begin with an example in which the two sums in (5.5.2) are not equal.

EXAMPLE 5.5.1.
Let the infinite array of the a_{mn} be as follows:

$$\begin{array}{rrrrc} 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & \cdots \\ 0 & -1 & 0 & 1 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \end{array}$$

The sums over the first, second, ... columns are $-1, 0, 0, \ldots$, respectively, so that the second sum in (5.5.2) is -1. Likewise, the first sum in (5.5.2) is 1. Thus in this case, and *despite the fact that all of the first sums reduce to finite sums*, the two sums in (5.5.2) are not equal. □

We come now to a positive result.

Theorem 5.5.2.
Given numbers a_{mn}, where $m,n \in \{1,2,\ldots\}$, suppose that there exists a positive number M such that for all finite sets of pairs (m,n), say $\{(m_1,n_1),\ldots,(m_k,n_k)\}$, we have

$$|a_{m_1,n_1}| + \cdots + |a_{m_k,n_k}| \le M. \tag{5.5.3}$$

Then

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{m,n}\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{m,n}\right). \tag{5.5.4}$$

Proof
By Theorem 5.2.3, (5.5.3) is equivalent to the convergence of the unordered sum

$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{m,n}.$$

Thus, if this sum is α, say, then given any positive ε, there is a finite subset, say $K = \{(u_1, v_1), \ldots (u_r, v_r)\}$, of $\mathbb{N} \times \mathbb{N}$ such that

$$\left|\sum_{(m,n)\in A} a_{m,n} - \alpha\right| < \varepsilon$$

whenever A is a finite subset of $\mathbb{N} \times \mathbb{N}$ with $A \supset K$. It follows that if p and q exceed each u_i and each v_j, then

$$\{(m,n) : 0 \le m \le p,\ 0 \le n \le q\} \supset K,$$

so that

$$\left|\sum_{n=1}^{p}\left(\sum_{m=1}^{q} a_{m,n}\right) - \alpha\right| < \varepsilon.$$

For each n in the *finite* set $\{1, 2, \ldots, p\}$, the bracketed term converges (as $q \to \infty$) to the infinite sum $\sum_{m=1}^{\infty} a_{m,n}$, so by the linearity of limits over finite sums, we obtain

$$\left|\sum_{n=1}^{p}\left(\sum_{m=1}^{\infty} a_{m,n}\right) - \alpha\right| \le \varepsilon.$$

This shows that the sum on the right-hand side of (5.5.4) converges to the value α. The same argument shows that the other sum does too, and the proof is complete. ∎

Exercises

1. Show that if $|z| < 1$, then
$$\sum_{n=1}^{\infty} \frac{nz^n}{1-z^n} = \sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)^2}.$$
[Hint: express one series as a repeated sum and justify the interchange of order of summation.]

2. Show that if $|z| < 1$, then
$$\frac{8z}{(1-z)} + \frac{16z^2}{1+z^2} + \frac{24z^3}{1-z^3} + \cdots = \frac{8z}{(1-z)^2} + \frac{8z^2}{(1+z^2)^2} + \frac{8z^3}{(1-z^3)^2} + \cdots.$$

3. Let $a_{mn} = (-1)^{m+n}/(mn)$, where $m, n = 1, 2, \ldots$. Show that the unordered sum
$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{mn}$$
fails to exist, but that
$$\lim_{N\to\infty} \sum_{m=1}^{N}\sum_{n=1}^{N} a_{mn}, \quad \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{mn}\right), \quad \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{mn}\right)$$
all exist.

4. Let $\succ$ be the lexicographic order on $\mathbb{N}\times\mathbb{N}$, so that $(m, n) \succ (p, q)$ if and only if $m > p$, or $m = p$ and $n > q$. Show that the function $f : \mathbb{N}\times\mathbb{N} \to \mathbb{C}$ given by $f(m, n) = a_{m,n}$ tends to a limit with respect to $\succ$ if and only
$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{m,n}\right)$$
exists.

6 Periodic Functions

CHAPTER

Abstract
In this chapter we combine the ideas developed so far to put the theory of the exponential, logarithm and trigonometric functions on a firm analytic foundation. With these, we can define the argument and the logarithm of a complex number.

6.1 The exponential function

We define the function exp : $\mathbb{C} \to \mathbb{C}$ by the series

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \tag{6.1.1}$$

where by definition, $0! = 1$. If $z \neq 0$ and if n is sufficiently large, then

$$0 \leq \frac{|z|^{n+1}/(n+1)!}{|z|^n/n!} = \frac{|z|}{n+1} < \frac{1}{2},$$

so that by the Ratio Test, this series is absolutely convergent, and hence convergent, for each z in $\mathbb{C}$. The number e is defined by

$$e = \exp 1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots,$$

and $e = 2{\cdot}7182\ldots$. The series (6.1.1) is sometimes denoted by e^z. However, later we shall define what is meant by a^b (a 'to the power' b), and this definition will then assign a value to e^z. It is then a *theorem* that (in a certain sense) $e^z = \exp z$, so it seems best not to confuse the issue now by referring to the series in (6.1.1) as e^z.

Our first result is that the function exp is differentiable, and therefore continuous, at every point of $\mathbb{C}$.

Theorem 6.1.1.

The function exp *is differentiable, and hence also continuous, at each point of* $\mathbb{C}$. *Further, for each* z, $\exp'(z) = \exp z$.

Proof

We need a simple inequality that is based on the Binomial Theorem. Take any complex numbers a and z and suppose that $|z| < 1$. Then, using the Binomial Theorem,

$$\begin{aligned} |(a+z)^n - a^n - nza^{n-1}| &= \left|\sum_{k=2}^{n} \binom{n}{k} z^k a^{n-k}\right| \\ &\leq |z|^2 \sum_{k=2}^{n} \binom{n}{k} |a|^{n-k} \\ &\leq |z|^2 (1+|a|)^n. \end{aligned}$$

This inequality implies that if $z \neq 0$ and $|z| < 1$, then

$$\begin{aligned} \left|\frac{\exp(a+z) - \exp a}{z} - \exp a\right| &= \left|\sum_{n=1}^{\infty} \frac{(a+z)^n - a^n}{n!z} - \sum_{n=1}^{\infty} \frac{a^{n-1}}{(n-1)!}\right| \\ &= \left|\sum_{n=1}^{\infty} \frac{1}{n!} \left\{\frac{(a+z)^n - a^n - nza^{n-1}}{z}\right\}\right| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \left|\frac{(a+z)^n - a^n - nza^{n-1}}{z}\right| \\ &\leq |z| \sum_{n=1}^{\infty} \frac{1}{n!} (1+|a|)^n \\ &< |z| \exp(1+|a|). \end{aligned}$$

As $|z|\exp(1+|a|) \to 0$ as $z \to 0$, the proof is complete. ∎

The next result in this section is one of the really important results in analysis, and we give two quite different proofs.

Theorem 6.1.2.

For all a *and* b *in* $\mathbb{C}$, $\exp(a+b) = \exp a \, \exp b$.

The first proof.
We define a function F by

$$F(z) = \exp(a + b + z)\exp(-z).$$

A calculation shows that $F'(z) = 0$ for all z, so that by Theorem 4.3.3, F is constant on $\mathbb{C}$. As $F(0) = \exp(a + b)$, we see that for all z,

$$\exp(a + b + z)\exp(-z) = \exp(a + b),$$

and the result follows by letting $z = -b$. ■

The second proof.
As the series for $\exp a$ and $\exp b$ are absolutely convergent, we have (from Theorem 5.4.3)

$$\exp a \ \exp b = \sum_{n=0}^{\infty} c_n,$$

where $\sum c_n$ is the Cauchy Product of the two series. As

$$c_n = \sum_{k=0}^{n} \left(\frac{a^k}{k!}\right)\left(\frac{b^{n-k}}{(n-k)!}\right) = \frac{(a+b)^n}{n!},$$

the stated result follows. ■

Corollary 6.1.3.
For all complex z, $\exp z \exp(-z) = 1$. *In particular*, $\exp z \neq 0$.

Proof
The identity follows from Theorem 6.1.2 when $a = z$ and $b = -z$, and it shows that $\exp z \neq 0$. ■

It is clear that if x is real, then $\exp x$ is real, and the next result summarises the main properties of the function $\exp : \mathbb{R} \to \mathbb{R}$.

Theorem 6.1.4.
The function $\exp : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing map of* $\mathbb{R}$ *onto* $(0, +\infty)$.

Proof
The definition of exp shows that if $x > 0$, then $\exp x > 1$. This in turn shows that if $t > 0$, then

$$\exp(x + t) = \exp x \exp t > \exp x,$$

so that exp is strictly increasing on $\mathbb{R}$. Next, for all real x,

$$\exp x = \exp(x/2)\exp(x/2) = [\exp x/2]^2 > 0$$

because $\exp(x/2) \neq 0$. Finally, if $y > 0$, then

$$\exp(-1/y) < y < \exp y$$

(see Exercise 6.1.1), so that from the Intermediate Value Theorem, there is some x in $[-1/y, y]$ such that $\exp x = y$. This completes the proof. ■

The complex version of Theorem 6.1.4 is deeper, and we state this now although we shall have to defer completion of the proof until later.

Theorem 6.1.5.
The function exp *maps* $\mathbb{C}$ *onto* $\{z \in \mathbb{C} : z \neq 0\}$.

Proof
Let $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$. We have seen that for any z, $\exp z \neq 0$, so that exp certainly maps $\mathbb{C}$ into $\mathbb{C}^*$. Further, it is easy to see that if $w \in \exp(\mathbb{C})$, then $\{tw : t > 0\} \subset \exp(\mathbb{C})$. Indeed, $w = \exp z_0$, say, and by Theorem 6.1.4, there is some r with $\exp r = t$, and then

$$\exp(r + z_0) = \exp r \exp z_0 = tw.$$

It would be sufficient, then, to show that the circle C given by $|w| = 1$ lies in the image of exp, but this is not so easy. ■

It is not hard to see that $|\exp z| = 1$ if and only if $z = i\theta$ for some real θ, so that exp maps the imaginary axis into C. We have to work harder to show that exp actually maps the imaginary axis *onto* C, and again this is a problem about the argument of a complex number. However, we record that the proof will be complete once we know that exp maps the imaginary axis onto the circle C.

Exercises

1. The definition of $\exp y$ shows that if $y > 0$ then $y < \exp y$. Deduce that if $t > 0$ then $t > \exp(-1/t)$.
2. Show that if x is real, then $\exp x > 1$ if and only if $x > 0$.
   Suppose that $z = x + iy$. Show that
   $$|\exp z| = \exp x, \quad \overline{\exp z} = \exp \bar{z}.$$
   Deduce that $|\exp z| = 1$ if and only if $z = i\theta$ for some real θ.
3. Let k be any positive integer. Show that there is some x_k such that $\exp x > x^k$ when $x > x_k$. Deduce that $x^{-k} \exp x \to +\infty$ as $x \to +\infty$.
4. Suppose that $y > 0$, and let x be the unique real number such that $\exp x = y$ (see Theorem 6.1.4). For $t > 0$ define $y^t = \exp(tx)$. Show that

(i) $y^t y^s = y^{t+s}$;
(ii) $(y^t)^s = y^{ts}$;
(iii) $y^0 = 1$;
(iv) if $t_n \to t$, where $t > 0$, then $y^{t_n} \to y^t$ as $n \to \infty$.

6.2 The trigonometric functions

The elementary approach to trigonometric functions in terms of angles is unsatisfactory for a variety of reasons. For example, it requires us first to define what we mean by an angle and then learn how to measure angles (which we would normally do by using calculus to measure lengths of curves). We shall now consider the trigonometric functions sin and cos defined analytically (on the complex plane, and without any reference to geometry) by the infinite series

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \tag{6.2.1}$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots. \tag{6.2.2}$$

These series converge absolutely for all z (by comparison with the series for $\exp |z|$). One of the great advantages of using complex numbers here is that we have the following identities available to assist us in our study of trigonometric functions of a *real* variable.

Theorem 6.2.1.
For all complex numbers z,

$$\sin z = \frac{1}{2i}[\exp(iz) - \exp(-iz)]; \tag{6.2.3}$$

$$\cos z = \frac{1}{2}[\exp(iz) + \exp(-iz)]; \tag{6.2.4}$$

$$\exp(iz) = \cos z + i \sin z. \tag{6.2.5}$$

Proof
Clearly, (6.2.3) and (6.2.4) follow immediately from the definition of sin, cos, and exp as series, and these imply (6.2.5). For example,

$$\frac{\exp(iz) + \exp(-iz)}{2} = \sum_{n=0}^{\infty} \frac{z^n i^n}{n!} \left[\frac{1 + (-1)^n}{2}\right] = \cos z.$$

■

Many of the familiar properties (from elementary accounts) of the real functions sin and cos can now be proved without effort and also for complex values.

Theorem 6.2.2.
The functions cos *and* sin *are differentiable on* $\mathbb{C}$, *and for all* z, $\sin'(z) = \cos z$ *and* $\cos'(z) = \sin z$.

Theorem 6.2.3.
For all complex numbers z *and* w,

$$\sin(w + z) = \sin w \cos z + \cos w \sin z$$

$$\cos(w + z) = \cos w \cos z - \sin w \sin z.$$

Also, for all z, $\cos^2 z + \sin^2 z = 1$.

The proofs are by algebraic manipulation. Theorem 6.2.2 follows directly from Theorem 6.1.1, (6.2.3), and (6.2.4). The addition formulae in Theorem 6.2.3 are immediate consequences of the addition formula for exp and (6.2.5). For example,

$$\begin{aligned} &4i(\sin w \cos z + \cos w \sin z) \\ &\quad = [\exp(iw) - \exp(-iw)][\exp(iz) + \exp(-iz)] \\ &\qquad + [\exp(iw) + \exp(-iw)][\exp(iz) - \exp(-iz)], \end{aligned}$$

and this readily simplifies to $4i \sin(w + z)$. Finally, the last statement in Theorem 6.2.3 follows directly from (6.2.3) and (6.2.4). For an alternative proof of this, see Exercise 6.2.4.

In the next section we shall discuss the zeros of sin, a definition of π, and the periodicity of sin, cos, and exp.

Exercises

1. Show that for all complex z, $\cos(-z) = \cos z$, $\sin(-z) = -\sin z$, and

$$\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z.$$

2. Prove Theorem 6.2.3 by considering the Cauchy products of the series.
3. Prove Theorem 6.2.2 directly by using the same argument as in the proof of Theorem 6.1.1.
4. Show that $\sin^2 z + \cos^2 z$ has derivative zero throughout $\mathbb{C}$, and deduce that for all z, $\sin^2 z + \cos^2 z = 1$.
5. Prove De Moivre's Theorem that for every complex z,

$$(\cos z + i \sin z)^n = \cos nz + i \sin nz$$

Deduce that if θ is real, then

$$\cos n\theta = \operatorname{Re}[(\cos\theta + i\sin\theta)^n].$$

By using the Binomial Theorem, show that

$$\cos n\theta = P_n(\cos\theta),$$

for some polynomial P_n of degree n.

Express $\cos 4\theta$ as a polynomial in $\cos\theta$.

6. Suppose that θ is real, and let $w = \exp(i\theta)$. Show that

$$\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \operatorname{Re}[w + w^2 + \cdots + w^n].$$

By summing the series on the right, show that

$$1 + 2[\cos\theta + \cos 2\theta + \cdots + \cos n\theta] = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}.$$

7. Exercises 6.2.5 and 6.2.6 give information about the function cos. Find and prove the corresponding results for the function sin.

8. To what extent are the statements in Exercises 6.2.5, 6.2.6, and 6.2.7 true for complex values of θ?

9. The *hyperbolic functions* sinh and cosh are defined as follows:

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots, \qquad \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots.$$

Prove that both of these series are absolutely convergent for all real z and that for all real z,

(i) $\cosh z + \sinh z = \exp z$;
(ii) $\cosh(-z) = \cosh z$, and $\sinh(-z) = -\sinh z$;
(iii) $\sinh'(z) = \cosh z$ and $\cosh'(z) = \sinh z$;
(iv) $\cosh^2 z - \sinh^2 z = 1$.

10. Show that if $z = x + iy$, then

$$|\cos z|^2 = \cos^2 x + \cosh^2 y;$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y.$$

Deduce that given any positive k, there is a positive y_0 such that if $|y| > y_0$, then $|\sin(x+iy)| > k$ and $|\cos(x+iy)| > k$.

6.3 Periodicity and π

How should we define π? We could use either of the known formulae

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \qquad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

(see Chapter 11), or we could use geometry and define π as the angle sum of a triangle, or perhaps as the length of the circumference of a circle divided by the length of its diameter. In fact, the best way is to avoid lengths and angles altogether and to define π *in terms of the zeros of the function* sin. As the zeros of sin are intimately connected with the periodicity of the functions sin, cos, and exp, we shall at the same time establish their periodicity. Before following this path, we remark that it is not at all obvious that sin has any zeros (apart from 0).

Theorem 6.3.1.

There is a positive number, which is denoted by π, *such that for all real* x, $\sin x = 0$ *if and only if* x *is an integral multiple of* π. *Further, for all real* x, $\sin(x + 2\pi) = \sin x$ *and* $\cos(x + 2\pi) = \cos x$.

Proof

If $0 < x < \sqrt{6}$, then

$$\sin x = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots > 0,$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \left(\frac{x^6}{6!} - \frac{x^8}{8!}\right) - \cdots < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

This shows that $\cos\sqrt{6} \leq -1/2$, and as $\cos 0 = 1$, the Intermediate Value Theorem implies that cos has at least one zero in the interval $(0, \sqrt{6})$. However, as

$$\cos' x = -\sin x < 0$$

when $0 < x < \sqrt{6}$, cos is strictly decreasing in this interval, so that cos *has exactly one zero,* say η, *in the interval* $(0, \sqrt{6})$, and $\cos x > 0$ when $0 \leq x < \eta$. As $\sin\eta > 0$ and $\sin^2\eta + \cos^2\eta = 1$, we see that $\sin\eta = 1$, and the addition formulae now show that

$$\cos 2\eta = -1, \quad \sin 2\eta = 0, \quad \cos 4\eta = 1, \quad \sin 4\eta = 0. \qquad (6.3.1)$$

Now, if $0 < x < \eta$, then $\sin x > 0$ and

$$\sin(x + \eta) = \cos x > 0.$$

It follows that $\sin x > 0$ when $0 < x < 2\eta$ and $\sin 2\eta = 0$; thus 2η *is the smallest positive zero of* sin.

We now define π by $\pi = 2\eta$. Then the periodicity of cos and sin follows immediately from the addition formulae and (6.3.1), and it is easy to prove by induction that for every integer n,

$$\sin n\pi = 0, \quad \cos n\pi = (-1)^n.$$

Finally, if $n\pi < x < (n+1)\pi$, then $\sin x \neq 0$, for

$$0 < \sin(x - n\pi) = \sin x \cos n\pi = (-1)^n \sin x,$$

and this completes the proof. ∎

We remark that the familiar features of the graphs of sin and cos are now available to us. Indeed, on the interval $[0, \pi/2]$, $\sin x$ increases from 0 to 1, and $\cos x$ decreases from 1 to 0. With this information, and the fact that $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, the addition formulae provide the general features of the graphs in the range $[\pi/2, 2\pi]$, and the rest follows by periodicity.

We turn now to a discussion of the corresponding properties for cos and sin as functions of a complex variable. First we show that *the zeros of* sin *in* $\mathbb{C}$ *are precisely those that we have already found in* $\mathbb{R}$, and after that we examine the periodicity of these and the exponential function. In general, the complex number w is a *period* of the function f if $f(z + w) = f(z)$ for all complex numbers z. It is clear that 0 is a period of every function and that the set $\mathcal{P}(f)$ of periods of f is an additive group (that is, the sum and difference of periods of f are also periods of f).

Theorem 6.3.2.
The zeros of sin *in* $\mathbb{C}$ *are* $n\pi$, *where* $n \in \mathbb{Z}$.

Theorem 6.3.3.
The set of periods of the functions sin *and* cos *is* $\{2n\pi : n \in \mathbb{Z}\}$. *The set of periods of* exp *is* $\{2n\pi i : n \in \mathbb{Z}\}$.

Proof of Theorem 6.3.2.
It is sufficient to show that if $z \in \mathbb{C}$ and $\sin z = 0$, then z is real. To do this we introduce the hyperbolic functions sinh and cosh on $\mathbb{C}$ by

$$\sinh z = -i\sin(iz) = \frac{\exp(z) - \exp(-z)}{2},$$

and

$$\cosh z = \cos(iz) = \frac{\exp(z) + \exp(-z)}{2},$$

and note that $\cosh^2 z - \sinh^2 z = 1$. Now, if $z = x + iy$, then

$$\begin{aligned}
|\sin z|^2 &= |\sin x\cos(iy) + \cos x\sin(iy)|^2 \\
&= |\sin x\cosh y + i\cos x\sinh y|^2 \\
&= \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y \\
&= \sin^2 x(1 + \sinh^2 y) + (1 - \sin^2 x)\sinh^2 y \\
&= \sin^2 x + \sinh^2 y.
\end{aligned}$$

As $\sinh y$ is real when y is real, this shows that if $\sin z = 0$, then $\sinh y = 0$. Thus $\exp(2y) = 1$, and as exp is strictly increasing on $\mathbb{R}$, this shows that $y = 0$ as required. ■

Proof of Theorem 6.3.3.
First, the addition formulae (which are valid for complex numbers) show that each integral multiple of 2π is a period of sin and of cos. Suppose now that w is a period of sin. Then

$$0 = \sin 0 = \sin(0 + w) = \sin w,$$

so that $w = n\pi$ for some integer n. It follows that

$$1 = \sin(\pi/2) = \sin(\pi/2 + w) = \cos w = (-1)^n,$$

so that n is even and w is an integral multiple of 2π. A similar proof shows that each period of cos is an integral multiple of 2π.

Finally, the addition formula for exp shows that exp is never zero and that w is a period of exp if and only if $\exp w = 1$. Now, this is so if and only if

$$\exp(w/2) = \exp(-w/2),$$

and this is equivalent to $\sin(w/2i) = 0$. Theorem 6.3.2 now shows that w is a period of exp if and only if w is an integral multiple of $2\pi i$. ∎

Exercises

1. Verify that sin is increasing on $[0, \pi/2]$, decreasing on $[\pi/2, 3\pi/2]$, and increasing again on $[3\pi/2, 2\pi]$.
2. Show that $\sin(\pi/6) = 1/2$ and that $\cos(\pi/4) = 1/\sqrt{2}$.
3. Show that for all real x, $\cosh x \geq 1$.

6.4 The argument of a complex number

We begin with a definition.

Definition 6.4.1.
Suppose that z is a nonzero complex number. Then θ is a *value*, or *choice*, of $\arg z$ if and only if $z = |z| \exp(i\theta)$.

We are not assuming here that θ is real; it is, but this follows from this definition.

Theorem 6.4.1.
The set of values of $\arg z$ *is* $\{\theta_0 + 2n\pi : n \in \mathbb{Z}\}$ *for some real* θ_0.

Proof
First, if θ_1 and θ_2 are any two values of $\arg z$, then

$$1 = \frac{\exp(i\theta_1)}{\exp(i\theta_2)} = \exp i(\theta_1 - \theta_2),$$

so that from Theorem 6.3.3, $\theta_1 - \theta_2 = 2n\pi$ for some integer n. It remains to show that there is one real choice of $\arg z$. As $-1 \leq x/|z| \leq 1$, the properties of cos guarantee that there is a unique φ in $[0, \pi]$ such that $\cos\varphi = x/|z|$. From this we obtain

$$\sin\varphi = \sqrt{1 - \frac{x^2}{|z|^2}} = \frac{|y|}{|z|},$$

so that

$$z = \begin{cases} |z|\exp(i\varphi) & \text{if } y > 0; \\ |z|\exp(-i\varphi) & \text{if } y < 0. \end{cases}$$

It follows that either φ or $-\varphi$ is a choice of $\arg z$, and the proof is complete. ■

We end by showing that it is impossible to define a choice of the argument that varies continuously with z on the unit circle. This is geometrically self-evident, and it is of great significance in the theory of functions of a complex variable.

Theorem 6.4.2.
There is no continuous choice of $\arg z$ *on the circle given by* $|z| = 1$.

Proof
We suppose that for each z on the circle we can make a choice, say $\theta(z)$, of $\arg z$ that varies continuously with z, and we shall reach a contradiction. For each z with $|z| = 1$, we then have $z = \exp i\theta(z)$.

For each real t we write $z(t) = \exp(it)$, so that

$$\exp(it) = z(t) = \exp\left[i\theta(z(t))\right],$$

and from this and Theorem 6.4.1 we deduce that

$$t = \theta(z(t)) + 2\pi N(t),$$

where $N(t)$ is an integer that may depend on t. Now as $t - \theta(z(t))$ is a continuous function of t, so too is $N(t)$. As $N(t)$ takes only integer values, the Intermediate Value Theorem implies that it is constant, say with value N, and this shows that

$$\pi - \theta(z(\pi)) = 2\pi N = -\pi - \theta(z(-\pi)).$$

This is false, as $z(\pi) = z(-\pi)$, and the proof is complete. ■

Exercises

1. Show that $\arg(1+i) = \pi/4$.

2. By considering $(5-i)^4(1+i)$ show (with a suitable interpretation of $\tan^{-1}$) that
$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right),$$
where $\tan z = \sin z / \cos z$. Give another proof of this using the formula
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$
Prove also that
$$\frac{\pi}{4} = 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right).$$

3. Suppose that $z = x + iy$ and that $x \neq 0$. Show that if θ is a value of $\arg z$, then $\tan\theta = y/x$, but that *not every solution of the equation* $\tan\varphi = y/x$ need be a value of $\arg z$.

4. Suppose that θ_1 and θ_2 are choices of $\arg z$ and $\arg w$, respectively. Show that $\theta_1 + \theta_2$ is a choice of $\arg zw$.

5. Let n be an integer. Show that $z^n = 1$ if and only if $z = \exp(2\pi ik/n)$ for some integer k.

6. Show that if n is a positive integer and if $w \neq 0$, then w has exactly n nth roots.

7. Show (in the proof of Theorem 6.4.2) that $t - \theta(z(t))$ is a continuous function of t.

6.5 The logarithm

We now define the logarithm, and the real powers, of a positive number. We know that exp is a strictly increasing map of $\mathbb{R}$ onto the interval $(0, +\infty)$.

Definition 6.5.1.
The function $\log : (0, +\infty) \to \mathbb{R}$ is the inverse of the function $\exp : \mathbb{R} \to (0, +\infty)$.

Theorem 6.5.2.
(i) *The function* log *is a strictly increasing differentiable map of* $(0, +\infty)$ *onto* $\mathbb{R}$, *and at each positive* a, $\log' a = 1/a$. *Further, for all positive* x *and* y,
$$\log(xy) = \log(x) + \log(y). \qquad (6.5.1)$$

Proof

We know that log maps $(0, +\infty)$ onto $\mathbb{R}$ and that log is strictly increasing because exp is. If x and y are positive, then

$$\exp[\log x + \log y] = \exp[\log x]\exp[\log y] = xy = \exp(\log xy),$$

so that (6.5.1) holds.

We know that exp is differentiable on $\mathbb{R}$ with $\exp'(x) = \exp x > 0$. Thus, by the Mean Value Inequality (Theorem 4.3.1), if $a < 0 < b$ and $a \leq t \leq b$, then

$$\exp a \leq \exp t \leq \exp b,$$

so that

$$\exp a \leq \frac{\exp t - \exp 0}{t - 0} \leq \exp b. \tag{6.5.2}$$

Given any positive ε, decrease ε if necessary so that $0 < \varepsilon < 1$, and then suppose that $|x - 1| < \varepsilon$. We apply (6.5.2) with

$$a = \log(1 - \varepsilon), \quad t = \log x, \quad b = \log(1 + \varepsilon)$$

and obtain

$$1 - \varepsilon \leq \frac{x - 1}{\log x} \leq 1 + \varepsilon,$$

and this shows that $(x - 1)/\log x \to 1$ as $x \to 0$. As this implies that $\log x/(x - 1) \to 1$ as $x \to 0$, we see that log is differentiable at the point 1, and $\log'(1) = 1$. Now consider any positive a. Then

$$\frac{\log(a + x) - \log a}{x} = \frac{\log(1 + x/a)}{x} = \frac{1}{a}\left(\frac{\log(1 + x/a)}{x/a}\right)$$

and as the term on the right tends to $1/a$ as $x \to 0$, we see that log is differentiable at a with derivative $1/a$ there. ■

Definition 6.5.3.

If $a > 0$ and $b \in \mathbb{R}$, then

$$a^b = \exp(b \log a). \tag{6.5.3}$$

This definition justifies the familiar (but perhaps puzzling) formula $a^0 = 1$, and it also shows that $e^1 = \exp(1)$ and, more generally, that $e^x = \exp x$. At last, the series in (6.1.1) may now be denoted by e^x.

Exercises

1. We have used a^p for the product of a with itself p times, and it is necessary to show that this does not conflict with the definition of a^p in (6.5.3). Suppose that

$a > 0$ and that p is a positive integer. Show that a^p defined by (6.5.3) satisfies $a^1 = a$ and $a^{p+1} = aa^p$ (hence the two definitions are consistent). Show also that a^{-p} defined by (6.5.3) is the multiplicative inverse of a^p.

2. Suppose that n is a positive integer and that $a > 0$. Show that

$$\left[\exp\left(\frac{\log a}{n}\right)\right]^n = a.$$

This shows that $a^{1/n}$ defined by (6.5.3) is indeed the positive nth root of a.

3. Suppose that $a > 0$ and that b and c are real. Show that

$$\log a^b = b\log a, \quad a^b a^c = a^{b+c}, \quad \left(a^b\right)^c = a^{bc}.$$

4. Show that if $a > 0$ and $b > 0$ then $a^{\log b} = b^{\log a}$.

5. We define the logarithm $\operatorname{Log} z$ of a nonzero complex number z as *the set of values* $\log|z| + i\theta$, where θ is any value of the argument of z (see Exercises 6.3). Show that

$$\operatorname{Log} 1 = \{2n\pi i : n \in \mathbb{Z}\},$$

$$\operatorname{Log}(-1) = \{(2n+1)\pi i : n \in \mathbb{Z}\},$$

$$\operatorname{Log} i = \{(2n + \tfrac{1}{2})\pi i : n \in \mathbb{Z}\},$$

$$\operatorname{Log}(1+i) = \{\log\sqrt{2} + (2n + 1/4)\pi i : n \in \mathbb{Z}\}.$$

6. For any nonzero complex number z and any complex number w, we define z^w as *the set of values*

$$\exp\Big(w[\log|z| + i\theta]\Big),$$

where θ is any value of the argument of z. Show that, for example, $z^{1/2}$ is a set with exactly two elements, say u and v, where $u^2 = z = v^2$.

Show that

$$i^i = \{\exp(2n - 1/2)\pi : n \in \mathbb{Z}\},$$

$$e^{i\pi} = \{-\exp(2n\pi^2) : n \in \mathbb{Z}\}.$$

Note that $e^{i\pi} = -1$ only in the sense that -1 is *one of the infinitely many values taken by* $e^{i\pi}$.

PART III

Analysis

Analysis is the study of limiting processes, and in Chapters 7 to 10 we discuss sequences, continuous functions, and differentiation and integration. The material here is standard, as are most of the proofs. In Chapter 11, we collect together a variety of results that are concerned with π, e, and $n!$.

7 CHAPTER Sequences

Abstract
We discuss convergent sequences, give some examples, and then prove the fundamental result that every bounded sequence has a convergent subsequence. This is then used to prove that every nonconstant complex polynomial has a complex root. Finally, we discuss what is meant by a sequence converging to ∞.

7.1 Convergent sequences

We recall the definition of a convergent sequence from Chapter 3.

Definition 7.1.1.
A *sequence* is a function (real- or complex-valued) defined on the set $\mathbb{N}$ of positive integers ordered by the relation $>$. The sequence f *converges* to the number a if given any positive ε, there is an integer n_0 such that $|f(n) - a| < \varepsilon$ whenever $n > n_0$.

A variety of different notations for sequences are in common use. Sequences are often written, for example, as x_n, a_n, or s_n, these simply being an alternative notation for $f(n)$. Other notations are $x_1, x_2, x_3, \ldots$, and $(x_n : n \geq 1)$. The notation $\{x_n : n \geq 1\}$ is, however, unacceptable because this defines a *set* and not a sequence (if $x_n = 1$ for every n, then $\{x_n : n \geq 1\}$ is the set containing the single element 1). Of course, we can equally well consider sequences defined on any set of the form $\{N, N+1, N+2, \ldots\}$.

Again, we recall the following result from Chapter 3.

Theorem 7.1.2.

Suppose that $a_n \to a$, and that $b_n \to b$. Then for all constants α and β,

$$\alpha a_n + \beta b_n \to \alpha a + \beta b, \quad a_n b_n \to ab.$$

In particular, if p is a polynomial, then $p(a_n) \to p(a)$. Further, if $b \neq 0$, then $a_n/b_n \to a/b$.

Note that more generally, if f is defined in some interval $(a-r, a+r)$ and if f is continuous at a, then $f(x_n) \to f(a)$ for every sequence x_n that converges to a. This is true because given any positive ε, there is a positive δ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$, and also $|x_n - a| < \delta$ for all sufficiently large n.

We also mention Theorem 3.6.4 (on bounded monotonic sequences) and Theorem 4.4.3 (the Cauchy Criterion), each of which gives a criterion for a sequence to be convergent. The reader should review these two important theorems now.

We begin our discussion with some simple examples.

Example 7.1.3.

We show that

$$\lim_{n\to\infty} \frac{4n^3 + 2n - 1}{2n^3 + n^2} = 2$$

(for when n is very large, this ratio is approximately $4n^3/2n^3$). Now,

$$\frac{4n^3 + 2n - 1}{2n^3 + n^2} = \frac{4 + 2/n^2 - 1/n^3}{2 + 1/n} = \frac{p(1/n)}{q(1/n)},$$

where $p(x) = 4 + 2x^2 - x^3$ and $q(x) = 2 + x$, so that from Theorem 7.1.2,

$$\frac{4n^3 + 2n - 1}{2n^3 + n^2} = \frac{p(1/n)}{q(1/n)} \to \frac{p(0)}{q(0)} = \frac{4}{2} = 2.$$

□

Example 7.1.4.

For another example of this type, observe that

$$\lim_{n\to\infty} \frac{3n^2 + 4n + 7}{n^2 - 4n + 4} = \lim \frac{p(1/n)}{q(1/n)} = \frac{p(0)}{q(0)} = 3,$$

where $p(x) = 3 + 4x + 7x^2$ and $q(x) = 1 - 4x + 4x^2 = (2x-1)^2$. In this example, $q(1/n) = 0$ when $n = 2$, so that the original sequence is only defined on $\{n : n \geq 3\}$. □

EXAMPLE 7.1.5.
We show that

$$\lim_{n\to\infty} \frac{2^n + n^5}{2^n + n} = 1. \tag{7.1.1}$$

In this case,

$$\frac{2^n + n^5}{2^n + n} = \frac{1 + n^5/2^n}{1 + n/2^n} \to 1$$

providing that

$$\lim_{n\to\infty} \frac{n^5}{2^n} = 0, \quad \lim_{n\to\infty} \frac{n}{2^n} = 0.$$

The Binomial Theorem shows that if $n \geq 6$, then

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} > \binom{n}{6} > \frac{(n-5)^6}{6!},$$

so that

$$0 < \frac{n}{2^n} \leq \frac{n^5}{2^n} < \frac{6!\, n^5}{(n-5)^6}.$$

As this upper bound tends to 0, this establishes (7.1.1). □

EXAMPLE 7.1.6.
We shall show that the sequence a_n, defined inductively by

$$a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + \sqrt{a_n}},$$

converges to some real number α, and we shall show how to find α.

First, a straightforward argument by induction shows that for all n, $\sqrt{2} \leq a_n < 2$. Next,

$$a_{n+1}^2 - a_n^2 = \sqrt{a_n} - \sqrt{a_{n-1}},$$

so another argument by induction shows that

$$\sqrt{2} = a_1 < a_2 < \cdots < a_n < \cdots < 2.$$

It follows from this that $a_n \to \alpha$, say, where $\sqrt{2} < \alpha \leq 2$, and it only remains to identify α. As

$$(a_{n+1}^2 - 2)^2 = a_n,$$

we see that $(\alpha^2 - 2)^2 = \alpha$, and a little algebra shows that this is equivalent to

$$(\alpha - 1)(\alpha^3 + \alpha^2 - 3\alpha - 4) = 0.$$

As $\alpha \neq 1$, we have $f(\alpha) = 0$, where $f(x) = x^3 + x^2 - 3x - 4$. If $x \geq \sqrt{2}$, then

$$f'(x) = 3x^2 + 2x - 3 \geq 3 + 2\sqrt{2} > 0,$$

so that f is strictly increasing for $x \geq \sqrt{2}$. As $f(\sqrt{2}) < 0$, there is a unique root of f in the interval $[\sqrt{2}, 2]$, and this root must be α. We have not found α explicitly, but it can be estimated numerically to within any prescribed error by the methods described in Section 4.2. □

Exercises

1. Show that if k is a positive integer and if $0 < x < 1$, then $n^k x^n \to 0$ as $n \to \infty$.

2. Use the identity $x^2 - y^2 = (x+y)(x-y)$ to show that if a and b are positive, then $\sqrt{n^2+a} - \sqrt{n^2+b} \to 0$ as $n \to \infty$. Discuss the case when a or b is negative.

   Find the limit of $\sqrt{n^2+n} - n$ as $n \to \infty$.

3. Show that
$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1.$$

4. Prove that the sequence $a_1, a_2, \ldots$ given by
$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$
converges to some number α, where $\frac{1}{2} < \alpha \leq 1$.

5. Suppose that the sequence $x_1, x_2, \ldots$ converges to α. Show that the sequence $y_1, y_2, \ldots$ defined by
$$y_n = \frac{x_1 + \cdots + x_n}{n}$$
also converges to α. Give an example in which the sequence y_n converges but the sequence x_n does not.

   Show also that if each x_n is positive, then $(x_1 x_2 \cdots x_n)^{1/n} \to \alpha$.

6. Suppose that $a_1 = 1$, $b_1 = 2$ and that
$$b_{n+1} = \frac{1}{2}\left(a_n + b_n\right), \quad a_{n+1}b_{n+1} = 2.$$
Show that $a_1 < a_2 < a_3 < \cdots < b_3 < b_2 < b_1$, and deduce that both sequences a_n and b_n converge to $\sqrt{2}$.

7. Suppose that $p > 0$, $a_1 > 0$, and
$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{p}{a_n}\right).$$
Show that $a_n \geq \sqrt{p}$, and deduce that $a_n \to \sqrt{p}$.

8. Suppose that $a_1 > 0$, $a_2 > 0$ and that $a_{n+2} = \sqrt{a_n a_{n+1}}$. Show that if $a_n \to \ell$, then $\ell = (a_1 a_2^2)^{1/3}$. Now prove that $a_n \to (a_1 a_2^2)^{1/3}$.

9. Suppose that $a_1 = 3$ and that $a_{n+1} = (3a_n + 3)/(a_n + 3)$. By considering

$$\frac{a_{n+1} - \sqrt{3}}{a_{n+1} + \sqrt{3}},$$

find a formula for a_n in terms of a_1 and n, and deduce that $a_n \to \sqrt{3}$.

10. Show that for each real x,

$$\lim_{m\to+\infty}\left(\lim_{n\to+\infty}\left[\cos\left(m!\pi x\right)\right]^{2n}\right) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

7.2 Some important examples

This section contains some useful and interesting examples of sequences that involve nth powers, nth roots and $n!$. For more results of this nature, see Chapter 11. Our first result concerns the nth root $a^{1/n}$ of a.

Theorem 7.2.1.
The sequence $n^{1/n}$ converges to 1. Also, for any positive a, $a^{1/n} \to 1$.

Proof
As $n^{1/n} > 1$, we can write $n^{1/n} = 1 + t_n$, where $t_n > 0$. The Binomial Theorem gives

$$n = (1 + t_n)^n > \frac{n(n-1)}{2} t_n^2,$$

so that

$$0 < t_n < \sqrt{\frac{2}{n-1}}.$$

We deduce that $t_n \to 0$, and hence that $n^{1/n} \to 1$.

Suppose now that $a \geq 1$. If $n > a$, then $1 \leq a^{1/n} < n^{1/n}$, so that $a^{1/n} \to 1$. If $0 < a < 1$, then $(1/a)^{1/n} \to 1$, so that

$$a^{1/n} = \frac{1}{(1/a)^{1/n}} \to 1,$$

and the proof is complete. ■

Theorem 7.2.3.
For each real x,

$$\left(1 + \frac{x}{n}\right)^n \to \exp x.$$

Proof
This is clearly true when $x = 0$, so we may suppose that $x \neq 0$. We know from Theorem 6.5.2 that the function $y \mapsto \log y$ is differentiable at $y = 1$ with derivative 1 there. Thus, as $y \to 0$,

$$\frac{\log(1+y)}{y} \to 1.$$

We write $y = x/n$; then

$$\frac{\log(1+y)}{y} = \frac{n}{x}\log\left(1+\frac{x}{n}\right) = \frac{1}{x}\log\left(1+\frac{x}{n}\right)^n,$$

so that

$$\lim_{n\to\infty}\log\left(1+\frac{x}{n}\right)^n = x.$$

As the function exp is continuous at x, the result follows. ■

The next result is a little more complicated.

Theorem 7.2.4.
We have

$$\lim_{n\to\infty}\frac{n}{(n!)^{1/n}} = e. \tag{7.2.1}$$

Proof
We write $a_n = n^n/n!$, so that from Theorem 7.2.3 (with $x = 1$),

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^n \to e.$$

It is sufficient, then, to prove the following more general result. ■

Theorem 7.2.5.
Suppose that $a_n > 0$ and that $a_{n+1}/a_n \to M$. Then $a_n^{1/n} \to M$.

Proof
Clearly $M \geq 0$. Given any positive ε, there is an integer N such that if $n > N$, then $a_{n+1} \leq (M+\varepsilon)a_n$, and hence that

$$a_n \leq c(M+\varepsilon)^n,$$

where c depends on N, M, and ε. Suppose now that $a_n^{1/n} \geq M + 2\varepsilon$. Then

$$c(M+\varepsilon)^n \geq a_n \geq \left([M+\varepsilon]+\varepsilon\right)^n \geq n\varepsilon(M+\varepsilon)^{n-1},$$

so that $n \leq c(M+\varepsilon)/\varepsilon$. It follows that given ε, if $n > c(M+\varepsilon)/\varepsilon$, then $a^{1/n} < M + 2\varepsilon$.

If $M = 0$, then for these n, $0 \le a_n^{1/n} < M + 2\varepsilon = 2\varepsilon$, and the result follows. If $M > 0$, we can write $b_n = 1/a_n$ and then from the argument above,

$$b_n^{1/n} \le \frac{1}{M} + \varepsilon = \frac{1 + M\varepsilon}{M} < \frac{1}{M - M^2\varepsilon}$$

for all sufficiently large n. For these n, $a_n^{1/n} > M - M^2\varepsilon$, and the proof is complete. ∎

Theorem 7.2.4 is a weaker version of a famous result known as Stirling's formula, which is discussed (with Theorem 7.2.4) in Section 11.4. We say that the two sequences a_n and b_n of positive terms are *asymptotic*, and write $a_n \sim b_n$ as $n \to \infty$ if $a_n/b_n \to 1$. *Stirling's formula* is that

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}. \tag{7.2.2}$$

Exercises

1. Show that if the sequence $x_1, x_2, \ldots$ satisfies $n^{-k} \le x_n \le n^k$ for some positive number k, then $x_n^{1/n} \to 1$.

2. Let $\varphi_n(x) = x^n e^{-x}$. Show that φ_n is increasing for $0 \le x \le n$ and decreasing for $x \ge n$. By comparing $\varphi_n(n)$ with $\varphi_n(n+1)$, and $\varphi_{n+1}(n)$ with $\varphi_{n+1}(n+1)$, show that

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$$

Deduce that $(1 + 1/n)^n \to e$.

The Mean Value Inequality implies that if $0 < a < b$, then

$$b^{n+1} - a^{n+1} \le (n+1)(b-a)b^n.$$

Use this to show that $(1 + 1/n)^n$ is increasing with n.

3. Show that the inequality $n^{1/n} > (n+1)^{1/(n+1)}$ is equivalent to the inequality

$$n > \left(1 + \frac{1}{n}\right)^n,$$

and deduce that the sequence $n^{1/n}$ is eventually decreasing.

4. Show that if $a_n \sim b_n$, then $a_n^{1/n} \sim b_n^{1/n}$. By taking $a_n = (n+1)^n$ and $b_n = n^n$, show that the converse is false. Show that Stirling's formula (7.2.2) implies (7.2.1).

5. Show that $(n!)^{1/n^2} \to 1$ and $n^{1/\sqrt{n}} \to 1$.

7.3 Bounded sequences

We begin with the definition of a bounded sequence.

Definition 7.3.1.
A (real or complex) sequence $x_1, x_2, \ldots$ is *bounded* if there is some real number M such that for every n, $|x_n| \leq M$.

Every convergent sequence is bounded, for all but a finite number of terms in the sequence lie within a distance 1, say, of the limit. Evidently, not every bounded sequence converges, but a bounded increasing sequence does converge, as does a bounded decreasing sequence. We now ask, What can we say about a sequence that is bounded but neither increasing nor decreasing?

To answer this question we introduce the notion of a subsequence of a sequence $(x_n : n \geq 1)$. The sequence x_n is a function on $\mathbb{N}$, and a *subsequence* of this is the same function, but restricted to some infinite subset $\{n_1, n_2, \ldots\}$ of $\mathbb{N}$. For example, the sequence $2, 4, 6, \ldots$ is a subsequence of the sequence $1, 2, 3, 4, \ldots$, as is the sequence $1^2, 2^2, 3^3, \ldots$. It is clear that if the sequence x_n converges to α, then so does every subsequence of x_n. However, there may be many subsequences of x_n that converge (to different values), while the sequence x_n itself does not converge. For example, the sequence $0, 1, 0, 1, 0, 1, \ldots$ has the convergent subsequences $0, 0, 0, \ldots$ and $1, 1, 1, \ldots$, and also the subsequence $0, 0, 1, 0, 0, 1, \ldots$, which does not converge. Of course, any subsequence of a subsequence of x_n is itself a subsequence of x_n.

We can now state the main result of this section, and as this is so important, we give two proofs.

The Bolzano-Weierstrass Theorem.
Every bounded complex sequence has a convergent subsequence.

Proof
We prove this first for real sequences. Let x_n be a bounded sequence with, say, $a \leq x_n \leq b$ for every n. The proof is based on the method of bisection discussed in detail in Chapter 4. Starting with the interval $[a, b]$, we let the midpoint be c. If there are infinitely many values of m for which $x_m \in [a, c]$, then we write $a_1 = a$ and $b_1 = c$; if not, the interval $[c, b]$ must have this property, and we write $a_1 = c$ and $b_1 = b$. We have now constructed an interval $[a_1, b_1]$, obtained from $[a, b]$ by bisection, such that $x_m \in [a_1, b_1]$ for infinitely many values of m. We can repeat this process starting now with the interval $[a_1, b_1]$ to obtain an interval $[a_2, b_2]$ with similar properties; then again starting with $[a_2, b_2]$ and so on. In this way we can construct a sequence of intervals I_n, $I_n = [a_n, b_n]$, that

satisfies the hypotheses of Theorems 4.1.1 and 4.1.2 and such that for all n, $x_m \in [a_n, b_n]$ for infinitely many values of m.

The sequence a_n is increasing and bounded above, the sequence b_n is decreasing and bounded below, and both converge to a common limit α, say. It is now easy to construct a subsequence of x_n that converges to α. We select an integer n_1 such that $x_{n_1} \in [a_1, b_1]$. Next, as $x_m \in [a_2, b_2]$ for infinitely many choices of m, there is certainly one such choice of m, say n_2, such that $n_2 > n_1$. For the same reason, there is an integer n_3 such that $n_3 > n_2$ and $x_{n_3} \in [a_3, b_3]$, and so on. This provides a subsequence $(x_n : n = n_1, n_2, n_3, \ldots)$ of the sequence x_n with the property that

$$|x_{n_k} - \alpha| < (b - a)/2^k.$$

It is now clear that the subsequence $(x_n : n = n_1, n_2, \ldots)$ converges to α, and the proof for real sequences is complete.

Now let z_n be a complex sequence with, say, $|z_n| \leq M$ for all n. We write $z_n = x_n + iy_n$ and observe that $|x_n| \leq M$ and $|y_n| \leq M$. We now apply the Bolzano-Weierstrass Theorem (for real sequences) to the sequence x_n and conclude that there is a subsequence, say $(x_n : n = n_1, n_2, \ldots)$, of x_n that converges to some value x^*.

Now consider the sequence $(y_n : n = n_1, n_2, \ldots)$ (which is a subsequence of the sequence $y_1, y_2, \ldots$). We apply the Bolzano-Weierstrass Theorem (for real sequences) to this and conclude that it has a subsequence, say $(y_n : n = p_1, p_2, \ldots)$, that converges to some y^*. As $(x_n : n = p_1, p_2, \ldots)$ is a subsequence of the convergent sequence $(x_n : n = n_1, n_2, \ldots)$, it converges to x^*, so that finally,

$$z_{p_j} = x_{p_j} + iy_{p_j} \to x^* + iy^*.$$

This completes the first proof of the Bolzano-Weierstrass Theorem. ■

A real sequence is said to be *monotonic* if it is either an increasing sequence or a decreasing sequence. As any bounded monotonic sequence converges, the Bolzano-Weierstrass Theorem (for real sequences) is an immediate corollary of the following result, which is of interest in its own right.

Theorem 7.3.2.
Every real sequence has a monotonic subsequence.

Proof
Let $a_1, a_2, \ldots$ be a real sequence. We suppose first that for every m, the set

$$\{a_m, a_{m+1}, a_{m+2}, \ldots\} \tag{7.3.1}$$

has a greatest member (which need not be unique). Pick any one of these, and let it be $a_{\mu(m)}$. Clearly, μ is a map of $\mathbb{N}$ into itself, and by construction,

$\mu(m) \geq m$, so that $\mu(m) \to +\infty$ as $m \to \infty$. Thus there is a sequence n_j of integers such that

$$n_1 < n_2 < n_3 < \cdots, \quad \mu(n_1) < \mu(n_2) < \mu(n_3) < \cdots,$$

and it follows that $a_{\mu(n_1)}, a_{\mu(n_2)}, \ldots$ is a subsequence of the original sequence. Moreover, this subsequence is decreasing because if $p > q$, then $a_{\mu(p)} \leq a_{\mu(q)}$ for

$$a_{\mu(p)} = \max\{a_p, a_{p+1}, a_{p+2}, \ldots\} \leq \max\{a_q, a_{q+1}, a_{q+2}, \ldots\} = a_{\mu(q)}.$$

The remaining case is when one of the sets (7.3.1) has no greatest member, and by discarding a finite number of the a_j, we may assume that this is so when $m = 1$. In this case, given any integer r, there is an integer s with $s > r$ and

$$a_s > \max\{a_1, \ldots, a_r\}.$$

It is now obvious that the sequence $a_1, a_2, \ldots$ has an increasing subsequence. ■

Exercises

1. Our proof of the Bolzano-Weierstrass Theorem uses the result for real sequences to derive the result for complex sequences. Show how the following gives a direct proof for complex sequences (which therefore includes the case for real sequences as a special case). Let z_n be a bounded complex sequence with $|x_n| \leq M$ and $|y_n| \leq M$, say. Subdivide this square (centred at the origin with sides of length $2M$) into four congruent squares. Show that one of these smaller squares must contain z_n for infinitely many values of n. Repeat this subdivision process to obtain a sequence of squares $Q_1, Q_2, \ldots$ such that (a) $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$, and (b) the length d_n of the diagonal of Q_n tends to 0. Deduce that $Q_1 \cap Q_2 \cap Q_3 \cdots$ consists of a single point, say w, and that there is a subsequence of z_n that converges to w.

   You have now proved a two-dimensional version of the Bolzano-Weierstrass Theorem. A similar proof will be valid in $\mathbb{R}^3$, and indeed in any dimension.

2. Use Theorem 7.3.2 to give a proof of the Bolzano-Weierstrass Theorem for sequences in Euclidean 3-space.

3. We can arrange the set of rational numbers in a sequence in the following way. We start with 0, then list the finite number of rationals p/q with $|p| + |q| = 1$, then those with $|p| + |q| = 2$, and so on. This construction provides us with a sequence $r_1, r_2, \ldots$ of rational numbers. Show that for each real number α there is a subsequence of $r_1, r_2, \ldots$ that converges to α.

4. Let x_n be any sequence in $[0, 1]$, and let E be the set of points y such that there is some subsequence of x_n that converges to y. Suppose that $y_1, y_2, \ldots$ are in E and that $y_n \to y$. Show that $y \in E$.

Deduce that there is no sequence x_n in $[0, 1]$ for which $E = (0, 1/2]$. Construct a sequence x_n in $[0, 1]$ for which $E = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.

7.4 The Fundamental Theorem of Algebra

Let p be the polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n, \tag{7.4.1}$$

where the a_j are complex numbers. If $a_n \neq 0$ the *degree* of p is n, and p is constant if and only if $n = 0$. The polynomial p is a *real polynomial* if each a_j is real; otherwise, we say that p is a *complex polynomial*. There are real polynomials, for example $x^2 + 1$, that have no real zeros, and one of the great advantages of working with complex numbers is that there, the situation is completely satisfactory, for all nonconstant polynomials have zeros. This powerful result is known as the Fundamental Theorem of Algebra.

Fundamental Theorem of Algebra.
Every nonconstant polynomial has a zero.

It is very easy to see that this result implies that every complex polynomial factorises into a product of linear factors. Let p be given by (7.4.1). The Fundamental Theorem of Algebra guarantees that there is some z_1 such that $p(z_1) = 0$, and then

$$p(z) = p(z) - p(z_1) = \sum_{k=0}^{n} a_k(z^k - z_1^k).$$

As

$$z^k - z_1^k = (z - z_1)(z^{k-1} + z^{k-2}z_1 + \cdots z z_1^{k-2} + z_1^{k-1}),$$

we can now express p in the form

$$p(z) = (z - z_1)q(z),$$

where q is a polynomial of degree $n - 1$. We now apply this argument to q, and then continue this process until we have found that

$$p(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n)\ ;$$

in other words, every complex polynomial of degree n is the product of a constant factor and n linear factors of the form $z - z_j$. In this sense, every complex polynomial of degree n has n complex zeros, although of course, they may not be distinct.

Our proof of the Fundamental Theorem of Algebra uses the number

$$\beta = \text{glb}\,\{|p(z)| : z \in \mathbb{C}\}.$$

It is clear that $\beta \geq 0$, and that if p has a zero, then $\beta = 0$. However, it is not at all clear that if $\beta = 0$, then p has a zero (for we do not yet know that there exists any z with $|p(z)| = \beta$). Our proof depends on the following three propositions in which $n > 0$ and $a_n \neq 0$ in (7.4.1).

Proposition 7.4.1.
If $|z| > 2(|a_0| + |a_1| + \cdots + |a_n|)/|a_n|$, *then*

$$|p(z)| > |a_n z^n|/2.$$

Proposition 7.4.2.
There is some ζ *such that* $|p(\zeta)| = \beta$.

Proposition 7.4.3.
If $|p(z_0)| = \beta$, *then* $p(z_0) = 0$.

Proposition 7.4.1 confirms that $|p(z)|$ is large when $|z|$ is large, so that if $|p(z)|$ is close to β, then z must lie in some bounded part of the complex plane. This enables us to use the Bolzano-Weierstrass Theorem to prove Proposition 7.4.2, and the proof is completed by Proposition 7.4.3.

Proof of Proposition 7.4.1.
If z satisfies the given inequality, then $|z| > 1$, so that

$$\begin{aligned} |a_n z^n| &= |(a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}) - p(z)| \\ &\leq (|a_0| + |a_1| + \cdots + |a_{n-1}|)|z^{n-1}| + |p(z)| \\ &< |a_n z^n|/2 + |p(z)|. \end{aligned}$$

The given inequality for $|p(z)|$ now follows. ■

Proof of Proposition 7.4.2.
The definition of β implies that for each positive integer k there is a complex number z_k such that

$$\beta \leq |p(z_k)| \leq \beta + \frac{1}{k} \leq \beta + 1.$$

This defines a sequence $z_1, z_2, z_3, \ldots$ of complex numbers, and by Proposition 7.4.1, there is some positive number R such that for each k, $|z_k| \leq R$. The Bolzano-Weierstrass Theorem guarantees that there is subsequence $z_{k_1}, z_{k_2}, \ldots$ of the sequence z_k that converges to some point ζ. As p is continuous at ζ,

$$|p(\zeta)| = \lim_{j\to\infty} |p(z_{k_j})| = \beta.$$

as required. ■

Proof of Proposition 7.4.3.
We suppose that $p(z_0) \neq 0$ and shall show that $|p(z_0)| > \beta$. We begin by performing several elementary transformations so as to simplify the analysis that follows. First, let $q(z) = p(z_0 + z)/p(z_0)$. Then q is a polynomial of degree n with $q(0) = 1$ so, that for some constants b_j,

$$q(z) = 1 + b_k z^k + \cdots + b_n z^n,$$

where $b_k \neq 0$ and $b_n \neq 0$. Next, choose σ such that $\sigma^k = 1/b_k$, and let $Q(z) = q(\sigma z)$. Then

$$Q(z) = 1 + z^k + c_{k+1}z^{k+1} + \cdots + c_n z^n,$$

say, where $c_n \neq 0$. We shall show that there is some w with $|Q(w)| < 1$, and then

$$\beta \leq |p(z_0 + \sigma w)| = |Q(w)|.|p(z_0)| < |p(z_0)|$$

as required.

To construct w, choose any t satisfying $t(|c_{k+1}| + \cdots + |c_n|) < 1$ and $0 < t < 1$, and let $w = t\exp(\pi i/k)$. Then $w^k = -t^k < 0$, $|w| = t < 1$, and

$$\begin{aligned}|Q(w)| &\leq |1 + w^k| + |c_{k+1}w^{k+1} + \cdots + c_n w^n| \\ &\leq |1 - t^k| + t^{k+1}(|c_{k+1}| + \cdots + |c_n|) \\ &< 1 - t^k + t^k \\ &= 1\end{aligned}$$

as required. ■

Exercises

1. Show that every real polynomial of odd degree has at least one real zero. For every positive odd integer n construct a real polynomial p with only one real zero (that is not a multiple zero of p).
2. Suppose that a polynomial $p(x)$ is divisible by $x - a$ and that its derivative $p'(x)$ is divisible by $(x - a)^m$. Show that $p(x)$ is divisible by $(x - a)^{m+1}$.
3. Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $a_0 > a_1 > \cdots > a_n > 0$. By considering $(1 - z)P(z)$, show that P has no zeros in $\{z : |z| \leq 1\}$.
4. Show that $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$. Find a cubic polynomial p such that for all n,

$$p(n) = 1^2 + 2^2 + \cdots + n^2.$$

Show more generally that if k is a positive integer, then $1^k + 2^k + \cdots + n^k$ is a polynomial p_k in n of degree $k+1$.

7.5 Unbounded sequences

We recall the definition of a real sequence tending to $+\infty$ (or $-\infty$) from Section 3.7. We shall now extend the real line $\mathbb{R}$ by adjoining two points $+\infty$ and $-\infty$. Formally, we take any two objects not in $\mathbb{R}$, label these points $+\infty$ and $-\infty$, and form the union

$$\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty, -\infty\}.$$

We call $\mathbb{R}_\infty$ the *extended real line.* It is important to realise that we are *not* saying that $+\infty$ and $-\infty$ are real numbers (they are not), nor are we saying that we can perform algebraic operations with them. However, *we can extend the ordering of* $\mathbb{R}$ *to* $\mathbb{R}_\infty$ by defining $+\infty$ to be larger than every other point of $\mathbb{R}_\infty$, and $-\infty$ to be smaller than every other point of $\mathbb{R}_\infty$. These conventions allow us to restate earlier results in this new setting. For example, we have the following theorem.

Theorem 7.5.2.
Any monotonic sequence in $\mathbb{R}_\infty$ *converges to some point of* $\mathbb{R}_\infty$.

Proof
The reader should note that this proof depends only on the ordering of $\mathbb{R}_\infty$ and does not involve any algebraic operations at all. We may suppose that $a_1 \leq a_2 \leq \cdots$. If a_n is bounded above by some real number, then as we have seen, a_n converges to some point in $\mathbb{R}$. If a_n has no upper bound in $\mathbb{R}$, then given any k, there is some a_p with $a_p > k$. Then $a_n > k$ when $n > p$, so that $a_n \to +\infty$. ■

We emphasize that there is nothing new here; we are simply introducing and adopting a more flexible terminology. In this new setting, the Bolzano-Weierstrass Theorem reads as follows.

Theorem 7.5.3.
Every sequence in $\mathbb{R}_\infty$ *has a convergent subsequence.*

This result implies that any real sequence has a subsequence that converges to some real number or a subsequence that converges to $+\infty$ or a subsequence that converges to $-\infty$.

Proof
We begin with a sequence a_n of points in $\mathbb{R}_\infty$. If $a_n = +\infty$ for infinitely many n, then we have a subsequence in which all terms are $+\infty$, and

this subsequence clearly converges to $+\infty$. The same is true if $a_n = -\infty$ for infinitely many n, so we may now assume that for some n_0, $a_n \in \mathbb{R}$ whenever $n \geq n_0$. Since the convergence or divergence of a subsequence is unaffected by the addition or deletion of a finite number of terms in the sequence, we may suppose that $a_n \in \mathbb{R}$ for every n. In this case, Theorem 7.3.2 guarantees the existence of a monotonic subsequence, and by Theorem 7.5.2, this converges to some point in $\mathbb{R}_\infty$. The proof is complete. ∎

The introduction of 'infinity' into the complex plane is a little different because $\mathbb{C}$ is not ordered. In this case, we introduce a *single* 'point' ∞ and form the union $\mathbb{C}_\infty$, which we call the *extended complex plane.*

Definition 7.5.4.
The complex sequence z_n tends to ∞ in $\mathbb{C}_\infty$, and we write $z_n \to \infty$, if for every positive number R, there is an integer n_0 such that $|z_n| > R$ when $n > n_0$.

Note that this definition is based on the direction $\succ$ in $\mathbb{C}$ defined by $z \succ w$ if and only if $|z| > |w|$.

We end with a warning. Before this new terminology is used, we (and those we are communicating with!) must be clear about which space we are working in. The example $a_n = (-1)^n n$ will clarify this. As a real sequence, this does not converge to any point in $\mathbb{R}_\infty$, though it does have a subsequence converging to $+\infty$ and another subsequence converging to $-\infty$. However, as a sequence in $\mathbb{C}_\infty$, it converges to ∞. This example serves to emphasize the fact that ∞ in $\mathbb{C}_\infty$ is quite distinct from either of the points $+\infty$ and $-\infty$ in $\mathbb{R}_\infty$.

Exercises

1. Show that $\log n \to +\infty$, and that $\log 1/n \to -\infty$, as $n \to \infty$.

2. Let $\sum a_n$ be a series of nonnegative terms. Show that this series converges to a point of $\mathbb{R}_\infty$ in the sense that the sequence of partial sums necessarily converges to a point in $\mathbb{R}_\infty$.

3. For $n = 1, 2, \ldots$, define

$$a_n = \Big(1 - \cos(\log n)\Big) \log n.$$

Show that a_n has a subsequence converging to 0, and also a subsequence converging to $+\infty$. Show also that $a_{n+1} - a_n \to 0$, and deduce that given any positive number α, there is a subsequence of a_n that converges to α.

7.6 Upper and lower limits

Let a_n be a real sequence, and suppose that for all n, $|a_n| \leq M$. For all n, let

$$\begin{aligned} b_n &= \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}, \\ c_n &= \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}. \end{aligned} \tag{7.6.1}$$

Then, for all n, $c_n \leq b_n$ and

$$b_1 \geq b_2 \geq b_3 \geq \cdots \geq -M,$$

$$c_1 \leq c_2 \leq c_3 \leq \cdots \leq M,$$

so that $b_n \to b$ and $c_n \to c$, say. As $c_n \leq c_{m+n} \leq b_{m+n} \leq b_n$ for every m and n, we see that $c \leq b$.

Definition 7.6.1.
The *upper limit* of the sequence a_n is b as defined above, and its *lower limit* is c. We write these as

$$c = \liminf_{n\to\infty} a_n, \quad b = \limsup_{n\to\infty} a_n.$$

The significance of these numbers is that whereas $\lim a_n$ need not exist, *the upper and lower limits always exist*. Moreover, as the next result shows, their values determine whether or not the sequence a_n converges.

Theorem 7.6.2.
Let (a_n) be a real bounded sequence. Then (a_n) converges if and only if

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n,$$

and if this is so, then $\lim_{n\to\infty} a_n$ is this common value.

Proof
If $a_n \to \alpha$, say, then for any positive ε,

$$\alpha - \varepsilon \leq a_n \leq \alpha + \varepsilon \tag{7.6.2}$$

for almost every n, so that in the notation of (7.6.1),

$$\alpha - \varepsilon \leq c_n \leq b_n \leq \alpha + \varepsilon. \tag{7.6.3}$$

This shows that $0 \leq b - c \leq 2\varepsilon$, so that $b = c$.

Suppose now that $b = c$, and denote this value by α. Then (7.6.3) holds for almost every n; thus (7.6.2) does also, and this shows that $a_n \to \alpha$. ■

Exercises

1. Show that the sequence a_n given by $a_n = [3 + (-1)^n]/2$ has lower limit 1 and upper limit 2.

2. Suppose that $\alpha = \liminf_{n\to\infty} a_n$ and $\beta = \limsup_{n\to\infty} a_n$. Show that given any positive ε, there is an integer n_0 such that $a_n \in [\alpha - \varepsilon, \beta + \varepsilon]$ whenever $n \geq n_0$. Interpret this when $\alpha = \beta$.

   Show also that if $a_n \in [\alpha_0, \beta_0]$ for all but a finite set of n, then $[\alpha_0, \beta_0] \supset [\alpha, \beta]$.

3. Suppose that a_n is a real sequence that has a subsequence converging to α. Show that

$$\liminf_{n\to\infty} a_n \leq \alpha \leq \limsup_{n\to\infty} a_n.$$

4. Let b_n be a subsequence of a_n. Show that

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty} a_n.$$

5. Show that for any real sequences a_n and b_n,

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

   Give an example in which a strict inequality holds here.

   Formulate and prove the corresponding result for lim inf.

6. Let $a_n = \sin n$. Show that

$$\liminf_{n\to\infty} a_n = -1, \quad \limsup_{n\to\infty} a_n = 1.$$

8 CHAPTER Continuous Functions

Abstract
The main properties of continuous functions are discussed, including the stronger notion of uniform continuity and the continuity of an infinite sum of continuous functions.

8.1 Continuous functions

We have already defined what we mean by a function being continuous (Definition 3.4.7), and this is as follows.

Definition 8.1.1.
Suppose that E is a subset of $\mathbb{R}$, or of $\mathbb{C}$, and let a be a point of E. Then a function $f : E \to \mathbb{C}$ is *continuous* at the point a if given any positive number ε, there is a positive number δ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$ and $x \in E$. If f is continuous at every point of E we say that *f is continuous on E*. If f is not continuous at the point a, we say that f is *discontinuous* at a.

There is one special (but artificial) situation in which every function on E is continuous at a point a. The point a is an *isolated point* of E if there is a positive number ρ such that every point of E other than a is at a distance at least ρ from a; this means that

$$\{x : |x - a| < \rho\} \cap E = \{a\}.$$

In this case, we let $\delta = \rho$ regardless of the choice of ε; then $x \in E$ and $|x-a| < \delta$ implies that $x = a$, so that $|f(x)-f(a)| < \varepsilon$, and f is continuous at a. In short, *every function is continuous at every isolated point of its domain.* In the case when a is not an isolated point of E, a function f on E is continuous at a if and only if $\lim_{x\to a} f(x)$ exists and equals $f(a)$.

EXAMPLE 8.1.2.
This example is a special case of our general definition, but it is worth stating explicitly. Suppose that f is defined on the real interval $[a, b]$; then f is continuous at a if given any positive ε, there is a positive number δ such that $|f(x) - f(a)| < \varepsilon$ whenever $a \leq x < a + \delta$ and $x \in [a, b]$. A similar statement can be made about the endpoint b. □

If f and g are defined on a set E and are continuous at the point a in E, then so are the sum and product functions

$$x \mapsto f(x) + g(x), \quad x \mapsto f(x)g(x).$$

If $g(a) \neq 0$, then the quotient function $x \mapsto f(x)/g(x)$ is also continuous at a, provided that we restrict this function to the set $\{x \in E : g(x) \neq 0\}$. These are trivial if a is an isolated point of E, and otherwise they follow from our earlier general results (for example, Theorem 3.5.1). We have already seen many examples of continuous functions; for example, all polynomials, the exponential function, and the trigonometric functions sin and cos are all continuous on $\mathbb{C}$.

There is a close relation between the continuity of a function f at a point and sequences converging to this point. The next result is a precise statement of this.

Theorem 8.1.3.
Suppose that f is defined on a set E and that $a \in E$. Then f is continuous at a if and only if, for every sequence x_n converging to a, the sequence $f(x_n)$ converges to $f(a)$.

Proof
Suppose first that f is continuous at a. Then, given any positive ε, there is a positive δ such that $|f(x)-f(a)| < \varepsilon$ whenever $|x-a| < \delta$ and $x \in E$. Now take any sequence x_n that converges to a. Then there exists an integer n_0 such that when $n > n_0$, $|x_n - a| < \delta$ and hence $|f(x_n) - f(a)| < \varepsilon$. This shows that $f(x_n) \to f(a)$ as $n \to \infty$.

To complete the proof we show that if f is not continuous at a, then there is some sequence x_n converging to a for which $f(x_n)$ does not converge to $f(a)$. The *denial of continuity at a* is an important statement in its own right, and it means that the assertion 'given any positive ε, there is a positive δ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$ and $x \in E$' is false. In turn, this means that *there is some positive number ε_1 with the* *property that for every positive δ, there is some x satisfying $|x - a| < \delta$, $x \in E$, and $|f(x) - f(a)| \geq \varepsilon$.* As this holds for every choice of δ, we may take δ to be $1, 1/2, 1/3, \ldots$ in turn, and let y_n be any one of the values of x corresponding to the choice $\delta = 1/n$. Then $|y_n - a| < 1/n$, and $|f(y_n) - f(a)| \geq \varepsilon_1$, so that $y_n \to a$ but $f(y_n)$ does not converge to $f(a)$. This completes the proof. ■

We end this section with an example to show that points of discontinuity and the points of continuity of a function need not be isolated. This example has many points of interest in it and is well worth studying.

EXAMPLE 8.1.4.
We shall show that if $r_1, r_2, \ldots$ is any sequence of distinct real numbers, then there is a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at each r_n and continuous at every other point of $\mathbb{R}$. For $n = 1, 2, \ldots$, define

$$f_n(x) = \begin{cases} 0 & \text{if } x < r_n; \\ 1/2^n & \text{if } x \geq r_n, \end{cases}$$

and

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

For each x, the terms in this series satisfy $0 \leq f_n(x) \leq 1/2^n$, so that by the Comparison Test (Theorem 5.1.5), the series converges. Thus f certainly exists as a function from $\mathbb{R}$ to $\mathbb{R}$.

We shall now show that if a is any real number that is not equal to any r_n, then the function f is continuous at a. Take any positive ε, and find an integer N such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N} < \varepsilon.$$

Now choose a positive δ such that none of the numbers $r_1, r_2, \ldots, r_N$ lie in the interval $(a - \delta, a + \delta)$. The significance of this choice is that if $x \in (a - \delta, a + \delta)$, then for $n = 1, \ldots, N$, we have $f_n(x) = f_n(a)$. It follows that if $x \in (a - \delta, a + \delta)$, then

$$\begin{aligned} |f(x) - f(a)| &= \left| \sum_{n=N+1}^{\infty} f_n(x) - f_n(a) \right| \\ &\leq \sum_{n=N+1}^{\infty} |f_n(x) - f_n(a)| \\ &\leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\ &< \varepsilon, \end{aligned}$$

so that f is continuous at a.

We shall now show that f is discontinuous at each r_k. To do this we write

$$f(x) = \left(\sum_{n=1, n\neq k}^{\infty} f_n(x) \right) + f_k(x).$$

The argument given above (applied to the sequence r_n with the single term r_k removed) shows that the infinite sum on the right is continuous at r_k. It follows that f is continuous at r_k if and only if f_k is, and this is not so. □

An amusing special case of this example is that there exists a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational number and discontinuous at every rational number. To see this we have only to show that we can include all of the rational numbers in a sequence $r_1, r_2, \ldots$. First consider the positive rational numbers; each positive rational can be expressed in a unique way as the fraction p/q where p and q are positive, coprime integers. Further, for each integer k there are only finitely many fractions p/q with $p + q = k$. We can now list the positive rationals; we start with those rationals for which $p + q = 2$ (there is only one of these, namely 1), then follow with those with $p + q = 3$ (these are $1/2$ and $2/1$), then those with $p + q = 4$, and so on. Suppose now that this list is $s_1, s_2, s_3, \ldots$; then the sequence

$$0, s_1, -s_1, s_2, -s_2, \ldots$$

lists all rational numbers (and only these), and every rational occurs once and only once in the list.

Exercises

1. Let f be a complex-valued function defined on a set E containing the point a, and let $f = u + iv$. Show that f is continuous at a if and only if both u and v are continuous at a. [Thus we may restrict our attention to discussing the continuity of real-valued functions.]
2. Suppose that the graph of a continuous function $f : \mathbb{R} \to \mathbb{R}$ has a chord with a positive slope (so that for some a and b, $a < b$ and $f(a) < f(b)$) and a chord with a negative slope. Show that the graph of f has a horizontal chord (so that for some a and b, $a < b$ and $f(a) = f(b)$). [Recall Theorem 4.2.1.]
3. The 'ruler' function f is defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 0; \\ 1/2^n & \text{if } x = k/2^n, \text{ where } k \text{ is odd and } n = 0, 1, 2, \ldots; \\ 0 & \text{otherwise.} \end{cases}$$

Show that f discontinuous at every point of the form $k/2^n$. At which points is f continuous? [The name 'ruler function' is given because this is the pattern of marking on rulers marked in inches rather than centimetres.]

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$f(x + y) = f(x) + f(y)$$

for every x and y in $\mathbb{R}$. Suppose also that f is continuous at some point x_0 in $\mathbb{R}$. Show that f is continuous at every point of $\mathbb{R}$, and deduce that $f(x) = xf(1)$. [There exist functions satisfying this equation that are not continuous at any point of $\mathbb{R}$.]

5. Explain how the function $f(p/q) = 2^p 3^q$ enables us to list the positive rational numbers p/q in a sequence.

8.2 Functions continuous on an interval

In this section we consider conclusions that can be drawn simply from the assumption that a function f is continuous on a closed bounded interval. We recall the Intermediate Value Theorem (Theorem 4.2.1), namely that *if $f:[a,b] \to \mathbb{R}$ is continuous, and if y lies between $f(a)$ and $f(b)$, then there is some c in $[a,b]$ such that $f(c) = y$.* The next two theorems are direct consequences of this result.

Theorem 8.2.1.
Suppose that $f:[0,1] \to \mathbb{R}$ is continuous. If either $f([0,1]) \subset [0,1]$ or $f([0,1]) \supset [0,1]$, then f has a fixed point in $[0,1]$.

Theorem 8.2.2.
Let I be an interval in $\mathbb{R}$, and suppose that f is continuous on I. Then the image $f(I)$ is also an interval.

Proof of Theorem 8.2.1.
The function F given by $F(x) = f(x) - x$ is continuous on $[a, b]$, and $F(y) = 0$ if and only if $f(y) = y$ (that is, y is a fixed point of f). If $f([0, 1]) \subset [0, 1]$, then $F(0) \geq 0$ and $F(1) \leq 0$, so that Theorem 4.2.1 implies the existence of a y with $F(y) = 0$. If $f([0, 1]) \supset [0, 1]$, there are points u and v in $[0, 1]$ with $f(u) = 0$ and $f(v) = 1$. Then $F(u) \leq 0$ and $F(v) \geq 0$, and again, there is some y with $f(y) = y$. ∎

Proof of Theorem 8.2.2.
To show that $f(I)$ is an interval, we take any α and β in $f(I)$ with $\alpha < \beta$. Then there are distinct points a and b in I with $f(a) = \alpha$ and $f(b) = \beta$.

We may assume that $a < b$ (for a similar argument holds when $b < a$), and the Intermediate Value Theorem then implies that for any value γ between α and β, there is some c in $[a, b]$ with $f(c) = \gamma$. This shows that if α and β are in $f(I)$ with $\alpha < \beta$, then $[\alpha, \beta] \subset f(I)$, and this completes the proof. ∎

The function $f(x) = 1/x$ defined on the interval $(0, 1)$ shows that a continuous function on a bounded interval need not be bounded there. The function $f(x) = x$ on $(0, 1)$ shows that even if a continuous function on a bounded interval is bounded, it need not take a maximum value there (there is no y in $(0, 1)$ such that $f(x) \leq f(y)$ for every x in $(0, 1)$). In view of these examples, the following theorem is of considerable interest.

Theorem 8.2.3.
Suppose that $f: I \to \mathbb{R}$ is continuous, where I is a closed, bounded interval. Then there are points u and v in I such that for all x in I, $f(u) \leq f(x) \leq f(v)$.

This result is usually referred to by saying that a function that is continuous on a closed, bounded interval is bounded and attains its bounds there.

Proof
We show first that there is some number M_1 such that $f(x) \leq M_1$ for all x in $[a, b]$. Suppose not; then for each n, there must be some x_n with $f(x_n) > n$. The Bolzano-Weierstrass Theorem guarantees that there is a subsequence of the sequence x_n, say $x_{n_1}, x_{n_2}, \ldots$, converging to some point α in $[a, b]$. As f is continuous at a, Theorem 8.1.3 implies that $f(x_{n_j}) \to f(a)$ as $j \to \infty$, and this clearly contradicts the assumption that for all n, $f(x_n) > n$.

Now let

$$M = \sup\{f(x) : x \in [a, b]\}.$$

If $f(x) \neq M$ for any x, then the function $g(x) = 1/[M - f(x)]$ is defined and continuous on I, and so by what we have just proved, there is some m such that, for all x, $g(x) \leq m$. As this implies that

$$f(x) \leq M - (1/m) < M,$$

this is a contradiction. We deduce that for some v, $f(v) = M$. The result about lower bounds can be proved in exactly the same way. ∎

Theorems 8.2.2 and 8.2.3 combine to give the following result.

Theorem 8.2.4.
Suppose that f is real-valued and continuous on the interval $[a,b]$. Then the image $f([a, b])$ is a closed, bounded interval.

We end with the (important) remark that that essentially the same argument as that used in the proof of Theorem 8.2.3 proves a similar result for functions defined on subsets of the complex plane. For example, we have the following result.

Theorem 8.2.5.
Suppose that the function f is complex-valued and continuous on the rectangle $R = \{x + iy : a \le x \le b,\ c \le y \le d\}$. Then there are points w_1 and w_2 in R such that for all z in R,

$$|f(w_1)| \le |f(z)| \le |f(w_2)|.$$

Exercises

1. Complete the remaining part of the proof of Theorem 8.2.3.
2. Give an example of a function described in Theorem 8.2.4 in which neither $f(a)$ nor $f(b)$ is an endpoint of $f([a, b])$.
3. In each of the following cases, give an example of a bounded open interval I and a continuous function $f : I \to \mathbb{R}$ with the given property:
   (i) $f(I)$ is bounded and closed;
   (ii) $f(I)$ is unbounded and closed;
   (iii) $f(I)$ is bounded and open;
   (iv) $f(I)$ is unbounded and open;
   (v) $f(I)$ is bounded, but neither open nor closed.
4. Prove Theorem 8.2.5. Find points w_1 and w_2 when $f(z) = \exp z$ and $a = c = 0$ and $b = d = R$.

8.3 Monotonic functions

A function is *monotonic* on an interval I if it is either increasing on I or decreasing on I. We shall only consider increasing functions, but similar results hold for decreasing functions.

Theorem 8.3.1.
Suppose that $f : I \to \mathbb{R}$ is strictly increasing on the interval I. Then $f^{-1} : f(I) \to I$ is continuous on $f(I)$.

Remark
This is often stated with the assumption that f is continuous in I, but this is not necessary. In particular, $f(I)$ need not be an interval. See Exercise 8.3 for more information.

Proof

As f is strictly increasing on I, it is injective on I, so that each of the functions $f : I \to f(I)$ and $f^{-1} : f(I) \to I$ is a bijection.

We shall now show that f^{-1} is continuous at any point γ of $f(I)$, and the reader is urged to interpret the following argument on a diagram. As $\gamma \in f(I)$, there is a point c in I such that $f(c) = \gamma$. We shall assume that γ is not an endpoint of $f(I)$ (the argument is entirely similar when it is), so that c is not an endpoint of I. Now choose a positive number ε such that $[c - \varepsilon, c + \varepsilon] \subset I$, and let

$$\delta = \min\{f(c) - f(c - \varepsilon), f(c + \varepsilon) - f(c)\} > 0.$$

If $y \in f(I)$ and $|f(c) - y| < \delta$, then

$$f(c - \varepsilon) \leq f(c) - \delta < y < f(c) + \delta \leq f(c + \varepsilon),$$

so that $|f^{-1}(y) - f^{-1}(\gamma)| < \varepsilon$, because

$$f^{-1}(\gamma) - \varepsilon = c - \varepsilon < f^{-1}(y) < c + \varepsilon = f^{-1}(\gamma) + \varepsilon.$$

This proves that f^{-1} is continuous at the point γ. ■

The discontinuities of a monotonic function are of a rather special type, and we end this section with a brief discussion of these. Suppose that f is increasing on $[a, b]$, but not necessarily continuous there, and select any c with $a < c < b$ (the argument can easily be modified to include the cases $c = a$ and $c = b$). The function f is increasing on the interval $[a, c]$ and is bounded above there by $f(c)$; thus, by Theorem 3.6.3,

$$f_-(c) = \lim_{x \to c-} f(x) = \sup\{f(x) : a \leq x < c\} \tag{8.3.1}$$

exists. In a similar way,

$$f_+(c) = \lim_{x \to c+} f(x) = \inf\{f(x) : c < x \leq b\}. \tag{8.3.2}$$

Because $f(c)$ is an upper bound of the set in (8.3.1) and a lower bound of the set in (8.3.2), we have

$$f_-(c) \leq f(c) \leq f_+(c).$$

There need not be equality here; both inequalities are strict when, for example, f is defined by $f(x) = -1$ when $x < 0$, $f(0) = 0$, and $f(x) = 1$ when $x > 0$. The significant feature of an increasing function f is that its one-sided limits exist at every point c (this is not true for general functions), and this allows us to define the jump of f at c. The *jump* $j_f(c)$ of f at c is defined by

$$j_f(c) = f_+(c) - f_-(c),$$

and a typical situation is illustrated in Figure 8.3.1.

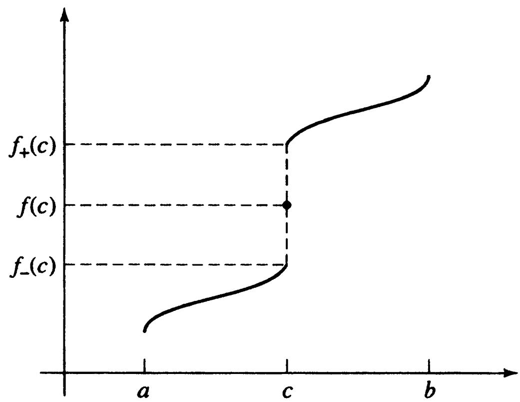

Figure 8.3.1

The following result is now immediate.

Theorem 8.3.2.
Suppose that f is increasing on $[a,b]$, and $a < c < b$. Then f is continuous at c if and only if $j_f(c) = 0$.

Let us now consider the jumps of f at distinct points. We suppose that $a < c < d < b$ and select any x with $c < x < d$. Then

$$f(a) \leq f_-(c) \leq f_+(c) \leq f(x) \leq f_-(d) \leq f_+(d) \leq f(b),$$

so that

$$j_f(c) + j_f(d) \leq f(b) - f(a).$$

More generally, for any distinct points $c_1, \ldots, c_m$, in (a, b)

$$j_f(c_1) + \cdots + j_f(c_m) \leq f(b) - f(a).$$

It follows that for a given f, there is only a finite number of points at which the jumps exceeds any given positive value; again, nothing like this can be said of a general function on $[a, b]$.

Exercises

1. Give the details of the proof of Theorem 8.3.2.

2. Suppose that f is a strictly increasing map of an interval I onto an interval J. Prove that $f : I \to J$ and $f^{-1} : J \to I$ are both continuous.

3. Let $E = [-1, 0) \cup [1, 2]$ and define $f : E \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } -1 \leq x < 0; \\ x - 1 & \text{if } 1 \leq x \leq 2. \end{cases}$$

   Use Theorem 8.3.1 to show that f is continuous on E. Now prove this directly.

4. Suppose that f is increasing on $[a, b]$ and that there are k distinct points where the jump of f is at least t. Show that $k \leq [f(b) - f(a)]/t$. [This means that the discontinuities of f can be listed in a sequence, listing the points x where $j_f(x) \geq 1$ first, and then the x such that $1/2 \leq j_f(x) < 1$, and so on.]

5. Construct a bijection $f : E_1 \to E_2$, where E_1 and E_2 are subsets of $\mathbb{R}$, such that f is not continuous on E_1 and f^{-1} is not continuous on E_2.

8.4 Uniform continuity

Uniform continuity is stronger than continuity. We shall begin by defining uniform continuity; then we shall compare it with continuity and discuss its geometric significance.

Definition 8.4.1.
A function f defined on E is *uniformly continuous on E* if given any positive ε, there is a positive δ such that $|f(x) - f(y)| < \varepsilon$ whenever x and y are in E and $|x - y| < \delta$.

A casual glance may suggest that there is no difference between continuity and uniform continuity, but a simple example will convince the reader that there is.

EXAMPLE 8.4.2.
The function $f(x) = x^2$ is continuous on $\mathbb{R}$, and we shall now show that it is *not* uniformly continuous on $\mathbb{R}$. Let us suppose that it is; then (taking $\varepsilon = 1$) there is a positive number δ such that $|x^2 - y^2| < 1$ whenever $|x - y| < \delta$. Taking $y = x + \delta/2$, this implies that $\delta x < 1$ for all positive x, and this is false. □

What, then, is the difference between a function being continuous on a set E and uniformly continuous on E? Suppose for the moment that a function is continuous at every point of $\mathbb{R}$. Then, given any point a in $\mathbb{R}$ and any positive ε, there is a number δ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Now the value of δ here depends (obviously) on f and ε, *but it also may depend on the point a*. Indeed, if f is differentiable at a, we would expect (roughly) to take δ to be $\varepsilon/|f'(a)|$, and this certainly depends on a. By contrast, the definition of uniform continuity says that given a positive ε there is a positive δ (which can only depend on the data given so far, namely f and ε, but not any points), such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. In fact, *uniform continuity* means that in some sense, if we take any two points x and y whose distance apart is less than δ, and then move them (keeping them the same distance apart) anywhere in the set E, the inequality $|f(x) - f(a)| < \varepsilon$ will remain valid throughout this motion. It is, of course, an immediate consequence of the two definitions that *if f is uniformly continuous on a set, then it is continuous there.*

There are circumstances under which continuity implies the stronger property of uniform continuity, and the most important case of this is given in the next result.

Theorem 8.4.2.

If a function f is continuous on $[a,b]$, then it is uniformly continuous on $[a,b]$.

Proof

Assume that there is some continuous function $f : [a, b] \to \mathbb{R}$ that is not uniformly continuous on $[a, b]$. Then there is some positive ε such that for all positive δ, there are points x and y in E with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. Taking δ to be (in turn) $1, 1/2, 1/3, \ldots$, we find that there are sequences x_n and y_n in $[a, b]$ such that

$$|x_n - y_n| < \frac{1}{n}, \quad |f(x_n) - f(y_n)| \geq \varepsilon. \tag{8.4.1}$$

By the Bolzano-Weierstrass Theorem, there is some subsequence of x_n converging to α, say, and clearly, $\alpha \in [a, b]$. We denote this subsequence by x_n, where $n \in \{n_1, n_2, n_3, \ldots\}$, and then from (8.4.1), the corresponding subsequence of y_n also converges to α. We deduce that

$$f(x_n) - f(y_n) \to f(\alpha) - f(\alpha) = 0,$$

and this contradicts (8.4.1). The proof is complete. ■

Exercises

1. (i) Show that the function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = 1/x$ is continuous but not uniformly continuous on $(0, 1)$.
   (ii) Show that the function $f : [0, +\infty) \to \mathbb{R}$ given by $f(x) = x^2$ is continuous but not uniformly continuous on $[0, +\infty)$.
2. Prove carefully that if a function f is uniformly continuous on a set E, then it is continuous at every point of E.
3. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that for every x in $\mathbb{R}$, $|f'(x)| \leq M$. Show that f is uniformly continuous on $\mathbb{R}$.

4. Show that the function $f(x) = x^{3/2}\sin(1/x)$, and $f(0) = 0$ is uniformly continuous on $[0, 1]$ but has unbounded derivative there.
5. Prove that if a function is continuous on a closed bounded rectangle in $\mathbb{C}$, then it is uniformly continuous there.

8.5 Uniform convergence

Suppose that we have a sequence of functions $f_0, f_1, f_2, \ldots$, all defined on the same set E, and that for each x in E, the infinite series

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

is convergent. Can we conclude that if each f_n is continuous at a point a, then the function f is also continuous at a? Unfortunately, the answer is no. Indeed, if we define $f_0, f_1, \ldots$ on $[0, 1]$ by $f_n(x) = x^n - x^{n+1}$, then

$$f_0(x) + f_1(x) + \cdots + f_n(x) = 1 - x^n,$$

so that

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1; \\ 0 & \text{if } x = 1. \end{cases}$$

We shall now introduce a stronger form of convergence of infinite series of functions, known as uniform convergence, and then show that this stronger form is enough to guarantee that the infinite sum f is continuous at any point at which all of the f_n are. The difference between convergence and uniform convergence is analogous to the difference between continuity and uniform continuity. In the case of continuity, the parameter δ depends, or does not depend, on the point in question; here it is the integer n_0 that depends, or does not depend, on the point.

Definition 8.5.1.
Suppose that $f_1, f_2, f_3, \ldots$ are all defined on E, and that for each x in E the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges. We say that this series is *uniformly convergent on E* to f if given any positive ε, there is an integer n_0 such that for all $n > n_0$ and all x in E,

$$\left|[f_1(x) + f_2(x) + \cdots + f_n(x)] - f(x)\right| < \varepsilon.$$

Similarly, we say that the sequence $F_1, F_2, \ldots$ of functions *converges uniformly to the function F on E* if, given any positive ε, there is an integer n_0 such that for all $n > n_0$, and all x in E,

$$\left|F_N(x) - F(x)\right| < \varepsilon.$$

There is an immediate, and rather obvious, sufficient condition for uniform convergence. As

$$|f_1(x) + f_2(x) + \cdots + f_n(x) - f(x)| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right| \leq \sum_{k=n+1}^{\infty} |f_k(x)|,$$

it is clear, for example, that if $|f_n(x)| \leq 1/2^n$ then $\sum_n f_n$ is uniformly convergent on E, for then we have

$$|f_1(x) + f_2(x) + \cdots + f_n(x) - f(x)| \leq \frac{1}{2^n}.$$

More generally, the same is true if we replace $1/2^n$ here by the nth term of any convergent series, and this leads us to the following result.

Theorem 8.5.2.
Suppose that $f_1, f_2, f_3, \ldots$ are defined on E and that there is a constant M and a convergent series $\sum a_n$ of nonnegative terms such that for all n and for all x in E, $|f_n(x)| \leq Ma_n$. Then the series $\sum_n f_n$ is uniformly convergent on E.

Remark
The inclusion of M in this result is irrelevant (for the series $\sum Ma_n$ converges if and only if $\sum a_n$ does), but we have included it for historical reasons, as the test is often referred to as the Weierstrass M-test.

Proof
Given any positive ε, choose an integer N such that

$$M(a_{N+1} + a_{N+2} + \cdots) < \varepsilon.$$

Then for $n > N$ and all x,

$$\left|f_1(x) + f_2(x) + \cdots + f_n(x) - f(x)\right| < M(a_{N+1} + a_{N+2} + \cdots) < \varepsilon$$

as required. ∎

The next result confirms that uniform convergence is sufficient to ensure that an infinite sum of continuous functions is continuous.

Theorem 8.5.3.
Suppose that $f_1, f_2, f_3, \ldots$ are defined on E, that each f_n is continuous at a point a in E, and that $\sum_n f_n$ is uniformly convergent on E to the function f. Then f is continuous at a.

Proof
It is simplest to write the proof in terms of the functions F_n defined by

$$F_n(x) = f_1(x) + f_2(x) + \cdots f_n(x).$$

Clearly, each F_n is defined on E and is continuous at a. We begin by estimating the difference $|f(x) - f(a)|$, for we want this to be small when x is close to a. Now, for any k,

$$|f(x) - f(a)| \leq |f(x) - F_k(x)| + |F_k(x) - F_k(a)| + |F_k(a) - f(a)|. \quad (8.5.1)$$

Given any positive ε we now choose n_0 such that for all x in E and all n with $n \geq n_0$, $|f(x) - F_n(x)| < \varepsilon/3$. This is possible by the assumption of uniform convergence. We now take k in (8.5.1) to be n_0; then

$$|f(x) - f(a)| \leq |F_{n_0}(x) - F_{n_0}(a)| + 2\varepsilon/3. \quad (8.5.2)$$

Finally, holding n_0 fixed and using the same ε, there is some positive δ such that $|F_{n_0}(x) - F_{n_0}(a)| < \varepsilon/3$ whenever $|x - a| < \delta$. Using this with (8.5.2) now shows that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$, and this proves that f is continuous at a.

There is an interesting way of writing the conclusion of Theorem 8.5.3 that shows why it is not entirely elementary. The conclusion in Theorem 8.5.3 is that

$$\lim_{n\to\infty}\left(\lim_{x\to a} F_n(x)\right) = \lim_{x\to a}\left(\lim_{n\to\infty} F_n(x)\right).$$

■

EXAMPLE 8.5.4.
We illustrate Theorem 8.5.3 by returning to Example 8.1.4, in which we considered the continuity of a series $\sum f_n$ of functions f_n based on a sequence r_n of real numbers. We showed there that the function $\sum_n f_n(x)$ was continuous at all points except the r_n. We recall that in this example, $|f_n(x)| \leq 1/2^n$ throughout $\mathbb{R}$, so that by Theorem 8.5.3, $\sum_n f_n(x)$ converges uniformly on $\mathbb{R}$. It follows that this sum is continuous at every point at which each f_n is continuous; thus the series is continuous at every point except the points r_n. □

Exercises

1. Show that the sequence f_n given by $f_n(x) = x^n$ is uniformly convergent to 0 on any interval of the form $[0, a]$, where $0 < a < 1$. Show that it is not uniformly convergent on the interval $[0, 1)$.
2. Show that the series

$$\sum_{n=0}^{\infty} \frac{\sin n\theta}{n^2}$$

is uniformly convergent, and hence continuous, on $\mathbb{R}$.

3. Let $f_n(x) = x^n(1-x)$ on $[0, 1]$. Show that for each x, $f_n(x) \to 0$ as $n \to \infty$. Does $f_n \to 0$ uniformly on $[0, 1]$?

   [Hint: uniform convergence will occur if and only if the maximum value of f_n on $[0, 1]$ converges to 0.]

4. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^x}$$

converges when $x > 1$. Show also that this series converges uniformly on any interval of the form $[a, b]$, where $a > 1$, but not on any interval of the form $(1, b]$.

5. Show that the series $\sum_{n=0}^{\infty} x(1-x)^n$
   (i) converges when $x \in [0, 2)$,
   (ii) is continuous at each point of $(0, 2)$,
   (iii) is not continuous at 0, and
   (iv) is not uniformly convergent on $[0, 2)$.

6. Show that the series $\sum_{n=0}^{\infty} x^n(1-x^n)$
   (i) is convergent when $x \in [0, 1]$,
   (ii) is uniformly convergent on each interval $[0, a]$, where $0 < a < 1$,
   (iii) is not uniformly convergent on $[0, 1]$.

CHAPTER 9 Derivatives

Abstract
The definition of a derivative is reviewed, and the basic results concerning the Mean Value Theorem, the differentiability of inverse functions, and power series are discussed.

9.1 The derivative

Much of our discussion of derivatives (Definition 3.4.8) applies equally well to real- or complex-valued functions of a real or complex variable. A function f defined in some open interval in $\mathbb{R}$ containing the point a is *differentiable* at a if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{9.1.1}$$

exists. The value of the limit is the *derivative* of f at a and is denoted by $f'(a)$. Similarly, a function f defined in some disc $\{z : |z - a| < r\}$ in the complex plane $\mathbb{C}$ is *differentiable* at a if the limit

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists, and again it is denoted by $f'(a)$.

If f is differentiable at a, then it is continuous at a; this follows directly from the fact that

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a}(x - a)\left(\frac{f(x) - f(a)}{x - a}\right) = 0 \cdot f'(a) = 0,$$

where here, x may be a real or complex variable. Further, we have already found (in Theorems 3.5.5 and 3.5.7) the rules for finding the derivative of a linear combination, a product, and a quotient of differentiable functions.

Let us suppose now that f is defined on some open interval I containing a and that $f'(x)$ exists for all x in I. We ask whether or not f' is differentiable at a, and if it is, its derivative, the *second derivative of f at a*, is denoted by $f''(a)$ or $f^{(2)}(a)$. Higher-order derivatives are defined in the obvious way, and the nth derivative is denoted by $f^{(n)}(a)$. Of course, entirely similar remarks (and notation) apply to the derivatives of complex-valued functions.

Exercises

1. Suppose that f is defined on an open interval I, $a \in I$, and $f(x) \leq f(a)$ for every x in I. Show that

$$f'(a) = \lim_{x\to 0+} \frac{f(a + x) - f(a)}{x} \leq 0$$

and

$$f'(a) = \lim_{x\to 0-} \frac{f(a + x) - f(a)}{x} \geq 0,$$

so that $f'(a) = 0$.Show also that if $f(x) \geq f(b)$ for every x in I, where $b \in I$, then $f'(b) = 0$.

2. Use induction to prove Leibnitz's Theorem that if f and g are n times differentiable at y and if $h(x) = f(x)g(x)$, then so is h, and

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(y)g^{(n-k)}(y).$$

3. Discuss the first, second, and third derivatives of the function

$$f(x) = x(x + |x|)(1 - x + |1 - x|).$$

4. Let n be a positive integer, and suppose that $f(x) = x^n$ when $x \geq 0$ and $f(x) = -x^n$ when $x < 0$. How many times is f differentiable at 0?

5. Suppose that $f_1, \dots, f_n$ are defined and never zero on an open interval I, and let $F(x) = f_1(x)f_2(x)\cdots f_n(x)$. Show that

$$\frac{F'(x)}{F(x)} = \frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}.$$

6. Suppose that we have a mass distribution spread along $\mathbb{R}$ and that we can find the mass $\mu(I)$ of any interval I. A natural definition of the *density* $\rho(x_0)$ of the mass distribution is
$$\rho(x_0) = \lim \frac{\mu(I)}{\ell(I)},$$
where I is an open interval of length $\ell(I)$ that contains x_0, and where this limit is taken over all such I as $\ell(I) \to 0$. Strictly speaking, we let X be the class of open intervals that contain x_0 with the direction $\succ$ given by $I \succ J$ if and only if $I \subset J$, and then $\rho(x_0)$ is the limit of $\mu(I)/\ell(I)$ with respect to the direction $\succ$.
An alternative definition is to take any a with $a < x_0$, let $M(x)$ be the mass in the interval $[a, x]$, and define the density $\rho_1(x_0)$ to be $M'(x_0)$ when this exists.
The first definition is more natural from a physical point of view, for it says that the mass in *any* small interval containing x_0 is approximately the multiple $\rho(x_0)$ of its length. Show that the two definitions are equivalent in the sense that $\rho(x_0)$ exists if and only if $M'(x_0)$ exists, and that when they both exist they are equal.

9.2 The Chain Rule

We shall now discuss the important Chain Rule for the derivative of a composite function $f(g(x))$.

Theorem 9.2.1: the Chain Rule.
Suppose that g is defined in some open interval containing a, that f is defined in some open interval containing $g(a)$, and that $g'(a)$ and $f'(g(a))$ exist. Then the composition $h(x) = f(g(x))$ of f and g is differentiable at a, and
$$h'(a) = f'\big(g(a)\big)g'(a).$$

We might try to prove this by letting x tend to a in the identity
$$\frac{h(x) - h(a)}{x - a} = \left(\frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)}\right)\left(\frac{g(x) - g(a)}{x - a}\right),$$
and it is here that an inherent weakness in our definition of the derivative appears. The problem is that our definition involves division, and we cannot guarantee that $g(x) - g(a) \neq 0$, even when $x \neq a$. Indeed, this 'identity' has no meaning when $g(x) = g(a)$, and such functions do exist (see Exercise 9.2.3). Another very good reason to avoid division is that ultimately (though not in this text) we wish to discuss the derivatives of maps from, say, $\mathbb{R}^3$ to itself, and for this we must seek an alternative definition, for clearly we cannot divide vectors by vectors.
It is apparent, then, that there must be an advantage in rewriting our definition of a derivative so as to avoid division, and this is easily done.

As the derivative $f'(a)$ is the *slope* of the tangent to the graph of f at a, the solution is simple: we reject the slope and instead use the tangent itself. The most general straight line L through the point $(a, f(a))$ is of the form $y = f(a) + A(x - a)$, and this is tangent to the graph of f precisely when $A = f'(a)$. Starting afresh, we can define $f'(a)$ as that value of A that makes L the 'best approximation' to the graph of f, and these comments lead to the following alternative definition of a derivative.

Definition 9.2.2.
Suppose that f is defined on some open interval I containing a. Then f is differentiable at a if and only if there is some constant A and some function $\varepsilon : I \to \mathbb{R}$ such that

$$f(x) = f(a) + A(x - a) + (x - a)\varepsilon(x), \tag{9.2.1}$$

where $\varepsilon(a) = 0$ and $\varepsilon(x) \to 0$ as $x \to a$. Further, (9.2.1) can hold for at most one value of A (see below), and this value of A is defined to be the derivative $f'(a)$ of f at a.

We are obliged to show that our two definitions of the derivative are equivalent. First, if (9.2.1) holds, then

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \big(A + \varepsilon(x)\big) = A, \tag{9.2.2}$$

so that f is differentiable in the sense of (9.1.1) and $f'(a) = A$. This shows that (9.2.1) can hold for at most one value of A. Conversely, if f is differentiable in the sense of (9.1.1), we define the function ε by $\varepsilon(a) = 0$, and for $x \neq a$,

$$\varepsilon(x) = \frac{f(x) - f(a)}{x - a} - f'(a).$$

Then from (9.1.1), $\varepsilon(x) \to 0$ as $x \to a$, and then (9.2.1) follows with $A = f'(a)$. The two definitions (9.1.1) and (9.2.1) are therefore equivalent. Of course, an entirely similar discussion holds for complex functions of a complex variable.

EXAMPLE 9.2.3.
Suppose that $f(x) = x^{-1}$ for $x \neq 0$. If $a \neq 0$ then

$$f(x) = \frac{1}{a} - \frac{(x - a)}{a^2} + \frac{(x - a)^2}{a^2 x},$$

and this is of the form (9.2.1) with $A = -1/a^2$ and $\varepsilon(x) = (x - a)/a^2 x$. We deduce that f is differentiable at a with $f'(a) = -1/a^2$. □

We are now in a position to prove the Chain Rule.

Proof of the Chain Rule.
We write $A = g'(a)$ and $B = f'(g(a))$ and restrict x to an interval about a on which $f(g(x))$ is defined. Definition 9.2.1 implies that there are functions δ and ε with

$$\lim_{x \to a} \delta(x) = 0 = \delta(a), \qquad \lim_{y \to g(a)} \varepsilon(y) = 0 = \varepsilon(g(a))$$

such that

$$f(y) = f(g(a)) + B(y - g(a)) + (y - g(a))\varepsilon(y),$$
$$g(x) = g(a) + A(x - a) + (x - a)\delta(x).$$

We now replace y in the first expression by $g(x)$ and use the second expression; this gives (after a little simplification)

$$h(x) = h(a) + BA(x - a) + (x - a)E(x),$$

where

$$E(x) = B\delta(x) + A\varepsilon(g(x)) + \delta(x)\varepsilon(g(x)).$$

As $E(a) = 0$ and $E(x) \to 0$ as $x \to a$, the Chain Rule has been proved. We remark that when we substitute $g(x)$ for y, we need to know that $\varepsilon(g(x))$ is defined, and it is for this reason that we have insisted that $\varepsilon(y)$ be defined when $y = g(a)$. ∎

Exercises

1. Let $f(x) = \exp(t \log x) = x^t$, where $x > 0$ and t is real. Show that $f'(a) = ta^{t-1}$.
   Suppose now that $t > 0$. What is the derivative of g, where $g(x) = t^x$?
2. Suppose that $f'(x) = Af(x)$ on $\mathbb{R}$ and that $f(0) = 1$. By considering $f(x)\exp(-Ax)$, show that $f(x) = \exp(Ax)$.
3. Let

$$g(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that g is differentiable at 0 and that $g'(0) = 0$. Show further that $g'(x)$ exists for every x but that $g'(x)$ is *not* continuous at the origin.

Show also that every open interval containing 0 contains a point x with $x \neq 0$ but $g(x) = g(0)$.

9.3 The Mean Value Theorem

In this section we discuss the Mean Value Theorem and some of its immediate consequences. Geometrically, the Mean Value Theorem says that given the straight line segment joining the points $(a, f(a))$ and $(b, f(b))$, there is some point c between a and b such that the tangent to the graph of f at c is parallel to the segment. The special case when the segment is horizontal (that is, when $f(a) = f(b)$) is known as *Rolle's Theorem*, and the conclusion is then that $f'(c) = 0$ for some c in (a, b). As we shall see, these two versions are equivalent to each other. We remark that the Mean Value Theorem is stronger than Theorem 4.3.1, but whereas Theorem 4.3.1 extends to higher dimensions, the Mean Value Theorem does not. For example, the function $f(z) = \exp z$ satisfies $f(0) = f(2\pi i)$, yet there is no z (anywhere) for which $f'(z) = 0$.

Theorem 9.3.1: the Mean Value Theorem.

Suppose that $f:[a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there is some point c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{9.3.1}$$

Proof

First we prove Rolle's Theorem. Suppose that f satisfies the hypotheses of the Mean Value Theorem and that $f(a) = f(b)$. By Theorem 8.2.3, f assumes its maximum value at some point, say v, of $[a, b]$, and also its minimum value at, say u. If $f(u) = f(v)$, then f is constant and (9.3.1) holds, so we may suppose that $f(u) \neq f(v)$. In this case one of the values $f(u)$ and $f(v)$ is not equal to $f(a)$, and we may suppose that $f(v) \neq f(a) = f(b)$. Then $a < v < b$, so that $f'(v)$ exists, and as v gives a maximum value of f, we have $f'(v) = 0$ (see Exercise 9.1.1). As (9.3.1) holds with $c = v$, we have now proved Rolle's Theorem.

Now consider the case in which $f(a) \neq f(b)$. The function

$$F(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$$

satisfies the hypotheses of the Mean Value Theorem and $F(a) = F(b)$. We deduce that there is some c in (a, b) with $F'(c) = 0$, and as this is equivalent to (9.3.1), the proof is complete. ■

An important corollary of the Mean Value Theorem is that if $f'(x) = 0$ throughout (a, b), then f is constant on (a, b) (see Theorem 4.3.2). There is a stronger Mean Value Theorem that contains the first version as a special case (with $g(x) = x$).

Theorem 9.3.2: the Second Mean Value Theorem.

Suppose that f and g are continuous on $[a,b]$ and differentiable on (a,b). Then there is some point c in (a,b) such that

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c). \tag{9.3.2}$$

Proof

We define the function F by

$$F(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x).$$

As Rolle's Theorem is applicable to F, there is some c with $F'(c) = 0$, and this is (9.3.2). ■

We end this section with an application of the Mean Value Theorem (see Exercise 9.3.5 for a similar application).

Theorem 9.3.3.

Let I be an open interval containing a, and suppose that $f: I \to \mathbb{R}$ is continuous and that f is differentiable on I except possibly at a. If $f'(x) \to A$ as $x \to a$, then f is differentiable at a and $f'(a) = A$.

Proof

Suppose that $x \in I$ and $x \neq a$. Then by the Mean Value Theorem,

$$\left| \frac{f(x) - f(a)}{x - a} - A \right| = |f'(c) - A|$$

for some c between a and x. The conclusion is now obvious, for $|f'(c) - A| < \varepsilon$ when x is sufficiently close to a. ■

Exercises

1. Use the Mean Value Theorem to show that if $p > 0$ then the polynomial $x^3 + px + q$ has at most one real zero. Now show that it has exactly one real zero.

2. Let

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f'(x)$ exists for every x. Show also that $f'(0) = 1$, but that f is not increasing in any interval containing the origin.

3. Suppose that the functions f, g, and h are differentiable at each point of $\mathbb{R}$, and define the function F by

$$F(x) = \begin{vmatrix} f(b) & f(a) & f(x) \\ g(b) & g(a) & g(x) \\ h(b) & h(a) & h(x) \end{vmatrix}$$

(a 3×3 determinant). Show that $F(a) = F(b)$. What is the result of applying Rolle's Theorem to F? What does this become in the special cases (i) $h(x) = 1$ and (ii) $h(x) = 1$ and $g(x) = x$?

4. Show that the polynomial

$$p(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n}$$

has exactly one real zero if n is odd, but none if n is even.

5. Suppose that f and g are differentiable on some open interval containing the origin, that $f(0) = 0 = g(0)$, and that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \alpha.$$

Show that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \alpha.$$

Illustrate this with a discussion of the function $x^{-1} \sin x$.

Discuss the existence or otherwise of the two limits when $g(x) = x$ and

$$f(x) = \begin{cases} x + x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

6. Suppose that f is differentiable at each positive x and that $f'(x) \to a$ as $x \to +\infty$. Show that $f(x)/x \to a$ as $x \to \infty$ and interpret this geometrically.

7. Suppose that f is twice differentiable on $\mathbb{R}$ and that for all x, $|f(x)| \leq A$ and $|f^{(2)}(x)| \leq B$. Show that, for all x, $|f'(x)| \leq 2\sqrt{AB}$.

9.4 Inverse functions

Throughout this section we suppose that a function f is strictly increasing and differentiable throughout an open interval I in $\mathbb{R}$. In this case, $f(I)$ is an open interval, and there is a strictly increasing continuous inverse function $f^{-1} : f(I) \to I$ (see Theorems 8.2.2 and 8.3.1). If f^{-1} is differentiable at the point $f(a)$, then using the identity $f^{-1}(f(x)) = x$ and the Chain Rule,

$$(f^{-1})'(f(a))f'(a) = 1. \tag{9.4.1}$$

This shows that f^{-1} *cannot be differentiable* at any point $f(a)$ where $f'(a) = 0$, and $f(x) = x^3$ and $a = 0$ is the classic example of this situation. Of course, (9.4.1) does *not* imply that $(f^{-1})'(f(a))$ exists when $f'(a) \neq 0$, for its existence was assumed when deriving (9.4.1). However, it does, as we shall now show.

Theorem 9.4.1.

Let I be an open interval, and let $f:I \to \mathbb{R}$ be strictly increasing and differentiable throughout I. If $a \in I$ and $f'(a) \neq 0$, then f^{-1} is differentiable *at* $f(a)$ *and*

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

Proof
For all x in I,

$$f(x) = f(a) + (x - a)[f'(a) + \varepsilon(x)], \tag{9.4.2}$$

where $\varepsilon(a) = 0$ and $\varepsilon(x) \to 0$ as $x \to a$. Let $b = f(a)$, and take any y in $f(I)$ with $y \neq f(b)$. Then $y = f(x)$ for some x, so writing $g = f^{-1}$, (9.4.2) gives

$$\frac{g(y) - g(b)}{y - b} = \frac{1}{f'(a) + \varepsilon(g(y))}$$

Now $f'(a) \neq 0$ and $\varepsilon(g(y)) \to 0$ as $y \to b$ (because g is continuous), so the result follows. ■

Exercises

1. Explain carefully the last step in the proof of Theorem 9.4.1, namely that as $y \to b$, $\varepsilon(g(y)) \to 0$.
2. Find the derivative of $x^{1/n}$ by regarding it as the inverse of the function $x \mapsto x^n$.
3. The function $\sin : (-\pi/2, \pi/2) \to (-1, 1)$ is strictly increasing and so has an inverse $x \mapsto \sin^{-1} x$. Show that $\sin^{-1}$ is differentiable on $(-1, 1)$ and that its derivative is $1/\sqrt{1 - x^2}$.
4. Let $\tan x = \sin x/ \cos x$ when $|x| < \pi/2$. Show that $\tan^{-1}$ is a strictly increasing, continuous map of $\mathbb{R}$ onto $(-\pi/2, \pi/2)$, and that

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2}$$

   Let a_n be a real sequence. Show that $a_n \to +\infty$ if and only if $\tan^{-1} a_n \to \pi/2$.

9.5 Power series

Motivated by the discussion of exp, sin, and cos in Chapter 6, we turn now to discuss the general *power series*

$$\sum_{n=0}^{\infty} a_n z^n. \tag{9.5.1}$$

Our first task is to determine where the series (9.5.1) converges.

Theorem 9.5.1.
There is a number R, satisfying $0 \le R \le +\infty$, *such that the series* (9.5.1) *converges absolutely when* $|z| < R$ *and diverges when* $|z| > R$.

A comment about the inequality $0 \le R \le +\infty$ is required. When $R = 0$, this result means that the series diverges whenever $z \neq 0$ (and it clearly converges when $z = 0$). When $R = +\infty$, the result is to be interpreted as saying that the series converges *for all* z. This is the case, for example, for the series for $\exp z$.

Definition 9.5.2.
R is the *radius of convergence* of the series (9.5.1).

Proof of Theorem 9.5.1.
First, we define R. Let B be the set of nonnegative numbers r for which the sequence $|a_0|, |a_1|r, |a_2|r^2, \ldots$ is bounded above. Obviously, $0 \in B$, and if $r_1 \in B$ then $r \in B$ whenever $0 \le r \le r_1$. Let $R = \sup B$, where we write $R = +\infty$ when B is not bounded above.

Suppose now that z satisfies $|z| < R$. Then $|z|$ is not an upper bound of B, so there is some r in B with $|z| < r \le R$. As $r \in B$, there is some M such that for all n, $|a_n|r^n \le M$. It follows that

$$|a_n z^n| \le M \left(\frac{|z|}{r}\right)^n, \tag{9.5.2}$$

so that by the Comparison Test, the series (9.5.1) is absolutely convergent.

Conversely, suppose that (9.5.1) is convergent. Then $a_n z^n \to 0$, so certainly the sequence $|a_0|, |a_1||z|, |a_2||z|^2, \ldots$ is bounded. Thus $|z| \in B$, so that $|z| \le R$. This shows that if $|z| > R$, then (9.5.1) diverges, and this completes the proof. ■

Theorem 9.5.1 makes no claim about the convergence or divergence of the series (9.5.1) at points on the circle $\{z : |z| = R\}$, and simple examples show that a great variety of possibilities exist. Consider, for example, the following three series:

$$\sum_{n=0}^{\infty} z^n, \quad \sum_{n=0}^{\infty} \frac{z^n}{n^2}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n}.$$

All three series have radius of convergence 1; however,

(1) the first series diverges at every point of $|z| = 1$ (because when $|z| = 1$, $|z^n|$ does not tend to 0);
(2) the second series converges at every point of $|z| = 1$ (by comparison with the series $\sum 1/n^2$);

(3) the third series converges when $z = 1$ and diverges when $z = -1$.

We turn now to the question of continuity and differentiability of a function given by a power series. The argument leading to (9.5.2) shows that if R is the radius of convergence of the series (9.5.1) and if $R' < R$, then the series converges uniformly on the smaller disc $\{z : |z| \leq R'\}$ (see Theorem 8.5.2). Thus, by Theorem 8.5.3, the series is continuous there and hence is continuous at every point z_0 with $|z_0| < R$. We omit the details, for the continuity will follow from differentiability, which we establish below. There is, however, a general point to be made here. It is *not* in general true that if a series is uniformly convergent on each of the sets $E_1, E_2, \ldots$, then it is uniformly convergent on their union $E_1 \cup E_2 \cup \cdots$ (by analogy, a function may be bounded on each E_i but not on their union). It follows, then, that although the series (9.5.1) is uniformly convergent on each disc $\{z : |z| \leq R'\}$, where $R' < R$, *it may not be uniformly convergent on the disc* $\{z : |z| < R\}$.

We have already seen that term-by-term differentiation is valid for the power series for exp, sin, and cos, and we now show that for the general series (9.5.1),

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=1}^{\infty} n a_n z^{n-1}. \tag{9.5.3}$$

In particular, the series (9.5.1) is continuous at every point in the circle $|z| < R$. There is, however, a question here: how do we know that the series on the right-hand side of (9.5.3) converges when $|z| < R$? The following theorem answers all these matters.

Theorem 9.5.3.

Suppose that the series (9.5.1) *has radius of convergence R. Then the power series*

$$\sum_{n=1}^{\infty} n a_n z^{n-1} \tag{9.5.4}$$

also has radius of convergence R, and the function defined by (9.5.1) *is differentiable in the disc* $\{z : |z| < R\}$ *with derivative given by* (9.5.3).

Proof

First, for any positive numbers r and δ,

$$\frac{1}{r}|a_n| r^n \leq n|a_n| r^{n-1} \leq \frac{1}{\delta}|a_n|(r+\delta)^n. \tag{9.5.5}$$

Now let R' be the radius of convergence of (9.5.4). If $r < R'$ then the central sequence in (9.5.5) is bounded; hence so is the sequence on the left, so that $r \leq R$. Thus $r < R'$ implies that $r \leq R$, so that $R' \leq R$. Suppose now that $r < R$, and choose a positive δ such that $r + \delta < R$.

Then the sequence on the right, and hence also the central sequence, in (9.5.5) is bounded. Thus $r < R$ implies that $r \le R'$, so that $R \le R'$. Thus $R = R'$.

We now show that term-by-term differentiation of the series (9.5.1) is valid. First, exactly as in the proof of Theorem 6.1.1, we see that if $|z| < \delta$, then

$$\begin{aligned}|(w+z)^n - w^n - nzw^{n-1}| &= \sum_{k=2}^{n} \binom{n}{k} |z|^k |w|^{n-k} \\ &\le \sum_{k=2}^{n} \binom{n}{k} |z|^2 \delta^{k-2} |w|^{n-k} \\ &\le \frac{|z|^2}{\delta^2} (|w| + \delta)^n.\end{aligned}$$

Now denote the series in (9.5.1) by $f(z)$, and choose any w with $|w| < R$. Next, choose a positive δ such that $|w| + \delta < R$, and then restrict z so that $|z| < \delta$. This implies that $|w + z| < R$, and for all such z,

$$\begin{aligned}\left| f(w+z) - f(w) - z\sum_{n=1}^{\infty} na_n w^{n-1} \right| &= \left| \sum_{n=2}^{\infty} a_n \left[(w+z)^n - w^n - nzw^{n-1} \right] \right| \\ &\le \sum_{n=2}^{\infty} |a_n| |(w+z)^n - w^n - nzw^{n-1}| \\ &\le \left(\frac{|z|}{\delta} \right)^2 \sum_{n=2}^{\infty} |a_n| (|w| + \delta)^n.\end{aligned}$$

As the series $\sum a_n(|w| + \delta)^n$ is absolutely convergent, say to the positive number W, it follows that if $|z| < \delta$, then

$$\left| f(w+z) - f(w) - z\sum_{n=1}^{\infty} na_n w^{n-1} \right| \le |z|^2 (W/\delta^2),$$

so that

$$\frac{f(w+z) - f(w)}{z} \to \sum_{n=1}^{\infty} na_n w^{n-1}$$

as $z \to 0$. The proof is complete. ■

Of course, we can apply Theorem 9.5.3 again and again to the same power series, and in this way we obtain, for each positive integer k,

$$\frac{d^k}{dz^k} \left(\sum_{n=0}^{\infty} a_n z^n \right) = \sum_{n=1}^{\infty} n(n-1)\cdots(n-k+1) a_n z^{n-k}.$$

Exercises

1. Show, by using Theorem 9.5.3, that

$$\left(\frac{1}{1-z}\right)^2 = 1 + 2z + 3z^2 + \cdots.$$

2. Find the radius of convergence of each of the following power series:

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+1)^5}, \quad \sum_{n=0}^{\infty} \frac{z^n}{n\sqrt{n}}, \quad \sum_{n=0}^{\infty} \left(\frac{2^n}{n^2} + \frac{3^n}{n^3} + \frac{5^n}{n^5}\right) z^n,$$

$$\sum_{n=0}^{\infty} \frac{z^n}{5^n + 7^n}, \quad \sum_{n=0}^{\infty} \left(\frac{n+1}{n}\right)^n z^n, \quad \sum_{n=0}^{\infty} \left(\sqrt{n}\right)^n z^n.$$

3. Suppose that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R. Show that for every positive integer k, the series

$$\sum_{n=0}^{\infty} n^k a_n z^n$$

also has radius of convergence R.

4. Suppose that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R. Given a positive integer k, find the radius of convergence of each of the series

$$\sum_{n=0}^{\infty} a_n z^{kn}, \quad \sum_{n=0}^{\infty} (a_n)^k z^n.$$

5. Show that

$$\sum_{n=0}^{\infty} \frac{z^{n!}}{n+1}$$

has radius of convergence 1. Show also that every arc of positive length on the circle $|z| = 1$ contains points at which this series diverges.

9.6 Taylor series

Our task here is to find conditions under which we can express a real-valued function f by its *Taylor series*; that is, we ask, When is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n? \tag{9.6.1}$$

There are two matters to be resolved here; first, does the Taylor series converge (we assume, of course, that all derivatives of f exist), and second, if it does, is its sum equal to f? The second question should not be dismissed as an excessive concern for detail as there *are* functions for which the Taylor series converges, but *not* to the value $f(x)$.

First, we consider an expansion of f in a *finite* Taylor series. It is sufficient to take $a = 0$ and to consider a function f defined on some open interval $I = (-r, r)$. We shall assume that *all derivatives of f of all orders exist throughout I*, and initially it will simplify matters greatly if we assume that

$$f(0) = f^{(1)}(0) = \ldots = f^{(n)}(0) = 0. \tag{9.6.2}$$

As we shall see shortly, this is harmless. With this assumption, we take any *nonzero* x in I and create the function $F(t)$ by the formula

$$F(t) = f(t) + (x-t)f^{(1)}(t) + \cdots (x-t)^n \frac{f^{(n)}(t)}{n!} + \lambda(x-t)^{n+1}, \tag{9.6.3}$$

where λ is a constant uniquely determined by the condition $F(x) = F(0)$. This and (9.6.2) yield

$$f(x) = \lambda x^{n+1}.$$

We now apply the Mean Value Theorem to the function $t \mapsto F(t)$ and conclude that there is some c, lying strictly between x and 0, such that $F'(c) = 0$. After a little elementary algebra, this yields

$$\lambda = \frac{f^{(n+1)}(c)}{(n+1)!},$$

so finally,

$$f(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}. \tag{9.6.4}$$

This simple form is due to the assumption (9.6.2).

We are now ready to generalise (9.6.4) to functions that are not constrained by (9.6.2). Suppose, then, that all derivatives of f of all orders exist throughout I, and define the function g by

$$g(x) = f(x) - \left[\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k\right].$$

As the condition (9.6.2) applies to g, it follows that g satisfies (9.6.4) (with f replaced by g), and this shows that there is some c between x and 0 such that

$$f(x) = f(0) + \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}. \tag{9.6.5}$$

This is a *finite Taylor expansion* of f, and it is a finite sum whose last term may be considered as the error term containing an 'unknown' point c. Note that this may be considered as an extension of the Mean Value Theorem that corresponds to the case $n = 0$.

Clearly, the same result holds at any point a. Indeed, if f is defined on an open interval I and if all derivatives of f exist throughout I, then for any a in I, we can define h by $h(x) = f(a + x)$ and then apply (9.6.5) to h. Thus we obtain the finite Taylor expansion of f as described in the following theorem.

Theorem 9.6.1.

Suppose that f is defined on some open interval I containing a and that all derivatives of f exist throughout I. If $x \in I$ and $x \neq a$, then there is some c between x and a such that

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \qquad (9.6.6)$$

This is the major step in proving our next result, namely that certain functions can be expanded in an infinite Taylor series.

Theorem 9.6.2.

Suppose that f is defined on some open interval I containing a and that all derivatives of f exist throughout I. Suppose also that there is some M such that for all integers n and all t in I, $|f^{(n)}(t)| \leq M$. Then (9.6.1) holds.

Proof

The error term (containing c) in the expansion (9.6.6) satisfies

$$\left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right| \leq M \frac{(x-a)^{n+1}}{(n+1)!},$$

and this upper bound tends to 0 as $n \to \infty$, for it is the nth term in the (convergent) series for $M \exp(x-a)$. Letting $n \to \infty$ in (9.6.6), we obtain (9.6.1). ∎

We end this section by using a different technique to prove the Binomial Theorem for a nonintegral power.

Theorem 9.6.3: the Binomial Theorem.

Suppose that t is a real number and $-1 < x < 1$. Then

$$(1+x)^t = 1 + tx + \frac{t(t-1)}{2!}x^3 + \frac{t(t-1)(t-2)}{3!}x^3 + \cdots. \qquad (9.6.7)$$

We write the Binomial coefficents used in this expansion as

$$\binom{t}{n} = \frac{t(t-1)\cdots(t-n+1)}{n!}, \quad \binom{t}{0} = 1,$$

and as

$$(n+1)\binom{t}{n+1} = (n+1)\frac{t(t-1)\cdots(t-n)}{(n+1)!} = \binom{t}{n}(t-n),$$

these satisfy the identity

$$(n+1)\binom{t}{n+1} + n\binom{t}{n} = t\binom{t}{n}. \tag{9.6.8}$$

Proof of Theorem 9.6.3.
The Ratio Test shows that the series in (9.6.7) converges absolutely when $|x| < 1$, and we denote its sum by $s(x)$; thus

$$s(x) = \sum_{n=0}^{\infty} \binom{t}{n} x^n.$$

Theorem 9.5.3 implies that $s'(x)$ is obtained by term-by-term differentiation, and this with (9.6.8) implies that

$$\begin{aligned}
(1+x)s'(x) &= \sum_{n=1}^{\infty} n\binom{t}{n} x^{n-1} + \sum_{n=1}^{\infty} n\binom{t}{n} x^n \\
&= t + \sum_{n=2}^{\infty} n\binom{t}{n} x^{n-1} + \sum_{n=1}^{\infty} n\binom{t}{n} x^n \\
&= t + \sum_{n=1}^{\infty} \left[(n+1)\binom{t}{n+1} + n\binom{t}{n}\right] x^n \\
&= t\sum_{n=0}^{\infty} \binom{t}{n} x^n \\
&= ts(x).
\end{aligned}$$

Now define $g(x) = (1+x)^{-t}s(x)$. A straightforward calculation shows that $g'(x) = 0$ whenever $|x| < 1$, so that g is constant on $(-1, 1)$. This constant must be $g(0)$, which is 1, so finally, we see that as claimed, for all such x, $s(x) = (1+x)^t$. ■

Exercises

1. Show that the radius of convergence of the series in (9.6.7) is 1.
2. Show (as stated in the proof of Theorem 9.6.3) that $g'(x) = 0$ whenever $|x| < 1$.
3. Suppose that the functions $s : \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}$ satisfy

$$s'(x) = c(x), \quad c'(x) = -s(x), \quad s(x)^2 + c(x)^2 = 1.$$

Show that for all x,

$$s(x) = s(0)\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + c(0)\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

4. Use (9.6.6) to show that for $0 \leq x < 1$,

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n,$$

and also that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots.$$

See Exercise 10.6.2.

10 CHAPTER Integration

Abstract

We introduce the integral as the limit of approximating sums and then use the upper and lower sums to give a criterion for integrability. Properties of the integral are then discussed.

10.1 The integral

We recall Definition 3.4.10 of the integral of $f : [a, b] \to \mathbb{R}$ as the limit of sums of the form

$$S_f(X, T) = \sum_{j=0}^{n} f(t_j)(x_{j+1} - x_j), \tag{10.1.1}$$

where $X = \{x_0, x_1, \dots x_n\}$, $T = \{t_0, t_1, \dots, t_{n-1}\}$, and

$$a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b, \quad x_j \le t_j \le x_{j+1}. \tag{10.1.2}$$

This limit is taken with respect to the direction $\succ$ defined on the set of all pairs (X, T) by $(X_1, T_1) \succ (X, T)$ if and only if $X_1 \supset X$.

We call any pair (X, T) a *partition* of $[a, b]$ (although often we simply refer to X as the partition), and the sum in (10.1.1) is called the *approximating sum* for the partition (X, T). To be quite clear, let us rewrite this definition without any reference to the direction $\succ$.

Definition 10.1.1.
The bounded function $f : [a, b] \to \mathbb{R}$ is *integrable* on $[a, b]$ with integral I if given any positive ε, there is a partition X_0 of $[a, b]$ such that $|S_f(X, T) - I| < \varepsilon$ for every partition (X, T) such that $X \supset X_0$.

We denote the integral I of f by expressions such as

$$\int_a^b f, \quad \int_a^b f(x)\,dx, \quad \int_a^b x^5\,dx.$$

Because the integral is defined as a limit, the following three results follow immediately from Theorems 3.5.3 and 3.6.1 and Corollary 3.6.2.

Theorem 10.1.2.
Suppose that f and g are integrable on $[a,b]$. Then for any real constants α and β, the function $\alpha f + \beta g$ is integrable on $[a,b]$ and

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g. \qquad (10.1.3)$$

Theorem 10.1.3.
Suppose that f and g are integrable on $[a,b]$ and that, for all x in $[a,b]$, $f(x) \le g(x)$. Then

$$\int_a^b f \le \int_a^b g. \qquad (10.1.4)$$

Theorem 10.1.4.
Suppose that f is integrable and that for all x in $[a,b]$, $m \le f(x) \le M$. Then

$$m(b-a) \le \int_a^b f \le M(b-a).$$

Next, we reassure ourselves by proving the following result, which will be familiar to all those who have studied integrals before.

Theorem 10.1.5.
For all constants $a_0, \ldots, a_k$, we have

$$\int_0^y \left[a_0 + a_1 x + \cdots + a_k x^k\right] dx = a_0 y + a_1 \frac{y^2}{2} + \cdots + a_k \frac{y^{k+1}}{(k+1)}.$$

Proof
It is obvious that if f is a constant function, say with value k, on $[a, b]$, then $S_f(X, T) = k(b-a)$ for every (X, T). Thus the constant function k is integrable with integral $k(b-a)$. By virtue of Theorem 10.1.2 we have

only to prove that for any nonnegative integer m,

$$\int_0^y x^m\,dx = \frac{y^{m+1}}{m+1}. \tag{10.1.5}$$

The proof is based on the following inequality, which follows directly from two applications of the Mean Value Theorem: if $0 \le u \le t \le v$, then

$$\left|\frac{v^{m+1}-u^{m+1}}{m+1} - (v-u)t^m\right| \le m(v-u)^2 v^{m-1}.$$

Now consider any partition (X, T) of $[0, y]$, so that (10.1.2) holds with $a = 0$ and $b = y$, and let $f(x) = x^m$. Then

$$y^{m+1} = y^{m+1} - 0^{m+1} = \sum_{j=0}^{n}(x_{j+1}^{m+1} - x_j^{m+1}),$$

so that

$$\begin{aligned}
\left|\frac{y^{m+1}}{m+1} - S_f(X,T)\right| &= \left|\frac{y^{m+1}}{m+1} - \sum_{j=0}^{n}(x_{j+1}-x_j)t_j^m\right| \\
&\le \left|\sum_{j=0}^{n}\left[\frac{x_{j+1}^{m+1}-x_j^{m+1}}{m+1} - (x_{j+1}-x_j)t_j^m\right]\right| \\
&\le \sum_{j=0}^{n} m(x_{j+1}-x_j)^2 y^{m-1} \\
&\le my^m \max\{x_{j+1}-x_j : j = 0,\dots,n\}.
\end{aligned}$$

We are now finished, for given any positive ε, we choose a partition X_0 of $[a, b]$ such that for this partition, $x_{j+1} - x_j < \varepsilon/my^m$. Then, for any partition (X, T) for which $X \supset X_0$, we have

$$\left|\frac{y^{m+1}}{m+1} - S_f(X,T)\right| < \varepsilon$$

as required. ■

Next, we show that the integral is unaffected by changing the value of a function f at a finite number of points.

Theorem 10.1.6.

Suppose that f and g are bounded real-valued functions defined on $[a,b]$ and that $f(x) = g(x)$ except at a finite set of points. If f is integrable on $[a,b]$, then so is g, and the two integrals are equal.

Proof

We begin by showing that the function

$$h(x) = \begin{cases} 1 & \text{if } x = c, \\ 0 & \text{if } a \leq x \leq b \text{ and } x \neq c, \end{cases}$$

where $a < c < b$, is integrable on $[a, b]$ with integral 0. Take any positive ε with $a < c - \varepsilon < c + \varepsilon < b$, and let X_0 be the partition $\{a, c-\varepsilon, c+\varepsilon, b\}$. As $h(x) = 0$ when $x \neq c$,

$$0 \leq S_h(X_0, T) = 2\varepsilon h(t_1) \leq 2\varepsilon.$$

Clearly, the same inequality holds for $S_h(X, T)$ whenever $X \supset X_0$; thus h is integrable with integral 0. The given result now follows immediately, because for suitable constants a_j and suitable functions h_j of this type,

$$g(x) = f(x) + a_1 h_1(x) + \cdots + a_k h_k(x).$$

■

We end with an example of a function that is *not* integrable.

EXAMPLE 10.1.7.
We shall show that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is not integrable on $[0, 1]$. We suppose that it is, and let its integral be I. Then there is a partition X_0 of $[0, 1]$ such that $|S_f(X, T) - I| < 1/2$ whenever X is a partition with $X \supset X_0$. Thus, for any choices, say T' and T'' of T, we have

$$|S_f(X_0, T') - S_f(X_0, T'')| \leq |S_f(X_0, T') - I| + |I - S_f(X_0, T'')| < 1.$$

However, by taking every t_j in T' to be rational we have $S_f(X_0, T') = 1$, and by taking every t_j in T'' to be irrational we have $S_f(X_0, T'') = 0$. This is a contradiction, so f is not integrable on $[0, 1]$. □

Exercises

1. Show that $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = 0$ if $0 \leq x \leq 1$ and $f(x) = 3$ if $1 < x \leq 2$ is integrable on $[0, 2]$, and its integral is 3.
2. Suppose that f is continuous and nonnegative on $[a, b]$. We shall see later (Theorem 10.3.1) that f is then integrable on $[a, b]$. Assuming this, show that if $\int_a^b f = 0$, then $f(x) = 0$ for every x in $[a, b]$.

3. Suppose that $a < c < b$. Show that if f is integrable on $[a, c]$ and on $[c, b]$, then it is integrable on $[a, b]$.

4. Suppose that $g : [0, 1] \to \mathbb{R}$ is defined by $g(x) = 0$ if x is irrational, and $g(x) = 1/q$ if $x = p/q$, where p and q are positive coprime integers. Show that g has infinitely many discontinuities in $[0, 1]$, but that g is integrable there. What is the integral of g?

10.2 Upper and lower integrals

We now take a different view of the integral of a function f, namely as the area under its graph, and obtain this by approximating it from above and below by collections of rectangles. Suppose that $f : [a, b] \to \mathbb{R}$ satisfies $|f(x)| \leq M$ on $[a, b]$. For each partition (X, T) given, say, by (10.1.2), we define

$$m_j = \text{glb}\{f(x) : x_j \leq x \leq x_{j+1}\},$$
$$M_j = \text{lub}\{f(x) : x_j \leq x \leq x_{j+1}\};$$

see Figure 10.2.1.

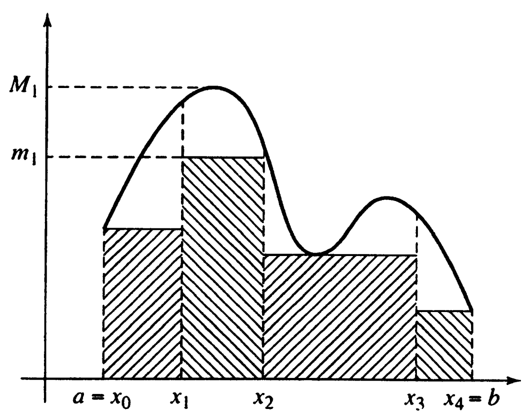

Figure 10.2.1

Definition 10.2.1.
The *lower sum* $L(f, X)$ and the *upper sum* $U(f, X)$ for the partition X of $[a, b]$ are given by

$$L(f, X) = \sum_j m_j(x_{j+1} - x_j),$$

$$U(f, X) = \sum_j M_j(x_{j+1} - x_j).$$

Observe that as $m_j \leq f(t_j) \leq M_j$ for each j, we have the inequalities

$$-M(b-a) \leq L(f,X) \leq S_f(X,T) \leq U(f,X) \leq M(b-a). \qquad (10.2.1)$$

Further, if we add a single new partition point to a partition X to form a new partition X', it is clear that

$$L(f,X') \geq L(f,X), \quad U(f,X') \leq U(f,X).$$

By adding a finite number of partition points one at a time, we find that if $X \subset X'$, then

$$L(f,X) \leq L(f,X') \leq U(f,X') \leq U(f,X). \qquad (10.2.2)$$

The lower sum $L(f,X)$ gives a lower bound of the area under the graph of f (see Figure 10.2.1, where $L(f,X)$ is the area of the shaded region), while $U(f,X)$ gives an upper bound of this area. As $L(f,X)$ is bounded above, for varying X, by $M(b-a)$, we can take the supremum over all values of $L(f,X)$ for varying X, and this will represent the best possible approximation *from below* to the area under the graph of f. This supremum is sufficiently important to warrant its own name, and there is a parallel discussion with $U(f,X)$.

Definition 10.2.2.
The *upper integral* and the *lower integral* of f over $[a,b]$ are defined by

$$\overline{\int_a^b} f = \inf_X U(f,X), \quad \underline{\int_a^b} f = \sup_X L(f,X),$$

respectively.

It is important to understand that the *upper and lower integrals always exist* (even when f is not integrable), and they represent the best upper and lower approximations to the area under the graph of f between a and b. Moreover, it is easy to see from (10.2.2) that they always satisfy the inequality

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f. \qquad (10.2.3)$$

Indeed, given any partitions (X,T) and (X',T'), we see from (10.2.2) that

$$L(f,X) \leq L(f,X \cup X') \leq U(f,X \cup X') \leq U(f,X').$$

Thus $U(f,X')$ is an upper bound of all $L(f,X)$ (as X varies), so that

$$\underline{\int_a^b} f \leq U(f,X').$$

This means that the lower integral is a lower bound of all upper sums, so that (10.2.3) follows. The significance of the upper and lower integrals is described in the following important result.

Theorem 10.2.3.

Suppose that f is bounded on $[a,b]$. Then the following are equivalent:

(i) *f is integrable;*
(ii) *the upper and lower integrals of f are equal;*
(iii) *given any positive ε, there is a partition X_0 of $[a,b]$ such that*

$$|L(f,X_0) - U(f,X_0)| < \varepsilon.$$

In addition, when f is integrable, its integral is the common value of the upper and lower integrals.

Proof

Suppose first that (ii) holds, and denote the common value of the upper and lower integrals by I. Thus, given any positive ε, there are partitions X_1 and X_2 such that

$$I - \varepsilon/2 < L(f, X_1) \leq I \leq U(f, X_2) < I + \varepsilon/2.$$

Then, from (10.2.2),

$$I - \varepsilon/2 < L(f, X_1 \cup X_2) \leq U(f, X_1 \cup X_2) < I + \varepsilon/2,$$

so that (iii) holds when $X_0 = X_1 \cup X_2$. This shows that (ii) implies (iii).

Now suppose that (iii) holds. Given a positive ε, let X_0 be the partition described in (iii). Now, (10.2.2) implies that for any partitions (X_1, T_1) and (X_2, T_2) with $X_1 \supset X_0$ and $X_2 \supset X_0$, we have

$$L(f, X_0) \leq L(f, X_1) \leq S_f(X_1, T_1) \leq U(f, X_1) \leq U(f, X_0),$$

and similarly for X_2, so that

$$\left|S_f(X_1, T_1) - S_f(X_2, T_2)\right| \leq U(f, X_0) - L(f, X_0) < \varepsilon.$$

As this is the Cauchy Criterion for the existence of the integral (as a limit), we deduce, from Theorem 4.4.1, that f is integrable. This shows that (iii) implies (i).

Finally, suppose that (i) holds. Then, given any positive ε, there is a partition X such that

$$\left|S_f(X, T) - \int_a^b f\right| < \varepsilon.$$

Now choose the points t_j in T such that for each j, $f(t_j) > M_j - \varepsilon$. Then

$$S_f(X, T) = \sum_{j=0}^{n} f(t_j)(x_{j+1} - x_j) > U(f, X) - \varepsilon(b - a),$$

so that

$$\int_a^{-b} f(x)\,dx \leq U(f, X) < S_f(X, T) + \varepsilon(b - a) < \int_a^b f(x)\,dx + \varepsilon(1 + b - a).$$

As the upper integral and the integral are numbers (independent of ε), we may let ε tend to zero and conclude that

$$\int_a^{-b} f(x)\,dx \le \int_a^b f(x)\,dx.$$

An entirely similar argument gives

$$\int_a^b f(x)\,dx \le \int_{-a}^b f(x)\,dx,$$

and these, with (10.2.3), yield (ii). The proof is now complete. ■

Remark

Because condition (iii) is so important, we rewrite it in a slightly more explicit form. Theorem 10.2.3 implies that f is integrable if and only if

(iii) for every positive ε, there is a partition X_0 of $[a, b]$ such that

$$\sum_{j=0}^{n} (M_j - m_j)(x_{j+1} - x_j) < \varepsilon.$$

Exercises

1. Let $f(x) = x^2$ on $[0, 1]$. Construct a partition X of $[0, 1]$ such that $0 \le U(f, X) - L(f, X) < 10^{-3}$ and verify that this is so.
2. Use Theorem 10.2.3 to give a direct proof of Theorem 10.1.6.
3. Use Theorem 10.2.3 to show that the function in Exercise 10.1.7 is not integrable.
4. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ (-1)^n & \text{if } 1/(n+1) < x \le 1/n. \end{cases}$$

   Show that f is integrable on $[0, 1]$.
5. Let $f : [0, 1] \to \mathbb{R}$ be the function defined by $f(x) = 0$ when x is irrational, and $f(x) = 1/q$ when $x = p/q$, where p and q have no common factors (except 1). Show that f is integrable on $[0, 1]$, and that $\int_0^1 f = 0$.
   [*Hint*: take any positve integer m and consider the set of x where $f(x) > m$.]

10.3 Integrable functions

We begin with two theorems that guarantee a plentiful supply of integrable functions.

Theorem 10.3.1.
Suppose that f and g are integrable on [a,b]; then so are the functions $|f(x)|$ *and* $f(x)g(x)$. *Further,*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Theorem 10.3.2.
(i) *If* $f:[a,b] \to \mathbb{R}$ *is increasing on* $[a,b]$ *then it is integrable on* $[a,b]$.
(ii) *If* $f:[a,b] \to \mathbb{R}$ *is continuous on* $[a,b]$ *then it is integrable on* $[a,b]$.

We shall need the following preliminary result (which we leave the reader to prove).

Lemma 10.3.3.
Suppose that $f:[c,d] \to \mathbb{R}$ *satisfies* $m \le f \le M$. *Then for all x and y in* $[c,d]$,

(i) $|f(x)| - |f(y)| \le M - m$;
(ii) $f(x)^2 - f(y)^2 \le 2(|m| + |M|)(M - m)$.

Proof of Theorem 10.3.1.
If we apply Lemma 10.3.3 to any interval $[x_j, x_{j+1}]$ arising from any partition X of $[a, b]$, we obtain the inequalities

$$0 \le U(|f|, X) - L(|f|, X) \le U(f, X) - L(f, X);$$
$$0 \le U(f^2, X) - L(f^2, X) \le 2(|m| + |M|)[U(f, X) - L(f, X)],$$

and the integrability of $|f|$ and f^2 now follows directly from Theorem 10.3.1. Suppose now that f and g are integrable. Then so too are $f + g$ and $f - g$, and hence also

$$\left(\frac{f(x) + g(x)}{2}\right)^2 + \left(\frac{f(x) - g(x)}{2}\right)^2.$$

As this function is $f(x)g(x)$, this too is integrable.

The inequality given in Theorem 10.3.1 is a straightforward consequence of the inequality $-|f(x)| \le f(x) \le |f(x)|$, for this implies that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx,$$

and this is equivalent to the given inequality. ■

Proof of Theorem 10.3.2.
Take any partition (X, T) of $[a, b]$. Then

$$0 \leq U(f, X) - L(f, X) = \sum_{j=0}^{n}(M_j - m_j)(x_{j+1} - x_j), \tag{10.3.1}$$

and, from Theorem 10.2.3, it suffices to show that by choosing a suitable (X, T) this can be made less than any given ε. Roughly speaking, the proof of (i) corresponds to making each term $(x_{j+1} - x_j)$ small, while the proof of (ii) corresponds to making each term $M_j - m_j$ small.

We consider (i) first. Given any positive ε, choose a partition X such that each term $(x_{j+1} - x_j)$ is at most $\varepsilon/[f(b) - f(a)]$. As f is increasing, we have $f(x_j) \leq f(t) \leq f(x_{j+1})$ when $x_j \leq t \leq x_{j+1}$, so that $m_j = f(x_j)$ and $M_j = f(x_{j+1})$. In this case,

$$\begin{aligned} U(f, X) - L(f, X) &= \sum_{j=0}^{n}[f(x_{j+1}) - f(x_j)](x_{j+1} - x_j) \\ &\leq \frac{\varepsilon}{f(b) - f(a)} \sum_{j=0}^{n}[f(x_{j+1} - f(x_j)] \\ &= \varepsilon. \end{aligned}$$

Thus if f is increasing on $[a, b]$, then it is integrable there. If f is decreasing, then $-f$ is integrable and so too is f.

Now suppose that f is continuous on $[a, b]$. Then f is uniformly continuous there (Theorem 8.4.2), so that given any positive ε, there is a positive δ such that if x and y are in $[a, b]$ and if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon/(b - a)$. Now select a partition X_0 such that for each j, $x_{j+1} - x_j < \delta$. Then for every partition X with $X \supset X_0$, we have

$$\begin{aligned} 0 &\leq U(f, X) - L(f, X) \\ &= \sum_{j=0}^{n}(M_j - m_j)(x_{j+1} - x_j) \\ &\leq \frac{\varepsilon}{b - a} \sum_{j=0}^{n}(x_{j+1} - x_j) \\ &= \varepsilon, \end{aligned}$$

so that f is integrable. ■

Theorem 10.3.1 has the following corollary.

Theorem 10.3.4.

Suppose that f is integrable on $[a,b]$, and $a < c < b$. Then f is integrable on both $[a,c]$ and $[c,b]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof

It is easy to see that the functions g and h defined by

$$g(x) = \begin{cases} 1 & \text{if } a \leq x \leq c, \\ 0 & \text{if } c < x \leq b. \end{cases}, \quad h(x) = 1 - g(x),$$

are integrable on $[a, b]$. Thus $f(x)g(x)$ and $f(x)h(x)$ are integrable on $[a, b]$, and as $f(x) = f(x)g(x) + f(x)h(x)$, we have

$$\int_a^b f(x)\,dx = \int_a^b f(x)g(x)\,dx + \int_a^b f(x)h(x)\,dx.$$

Further, it is easy to see that as

$$f(x)g(x) = \begin{cases} f(x) & \text{if } a \leq x \leq c, \\ 0 & \text{if } c < x \leq b \end{cases}$$

is integrable on $[a, b]$, $f(x)$ is also integrable on $[a, c]$, with

$$\int_a^c f(x)\,dx = \int_a^b f(x)g(x)\,dx.$$

Remembering that we can change the value of a function at one point without altering its integral (so that we may take $h(c) = 1$), we can argue for h as we have argued for g and obtain the given result. ■

Exercises

1. Prove Theorem 10.3.4 by using the equivalence of (i) and (iii) in Theorem 10.2.3.

2. Show that if f is bounded on $[a, b]$ and is continuous at all but a finite set of points in $[a, b]$, then f is integrable on $[a, b]$.

3. For $k = 0, 1, \ldots, n$ let $x_k = a + k(b-a)/n$. Illustrate these points on a diagram.
   Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Use the uniform continuity of f together with Theorem 10.2.3 to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) = \int_a^b f(x)\,dx.$$

Deduce that for each positive integer r,

$$\lim_{n\to\infty} \frac{1^r + 2^r + \cdots + n^r}{n^{r+1}} = \frac{1}{r+1}.$$

4. Show that if f is continuous on $[a, b]$ and that

$$\int_a^b f(x)g(x)\,dx = 0$$

for all functions g that are continuous on $[a, b]$, then $f(x) = 0$ for all x in $[a, b]$.

5. Suppose that g is increasing on $[a, b]$ and that f is continuous on $[g(a), g(b)]$. Show that $x \mapsto f(g(x))$ is integrable on $[a, b]$.

6. Suppose that f is nonnegative and continuous on the interval $[a, b]$, and let M be the least upper bound of $\{f(x) : x \in [a, b]\}$. Show that

$$\lim_{n\to\infty} \left[\int_a^b f(x)^n\,dx\right]^{1/n} = M.$$

10.4 Integration and differentiation

In this section we examine the interaction between integration and differentiation. The first two results describe the precise sense in which integration and differentiation act as inverse operations.

Theorem 10.4.1.
Suppose that $f:[a,b] \to \mathbb{R}$ is continuous, and for each x in $[a,b]$, define

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is differentiable on $[a,b]$ and $F'(x) = f(x)$.

Remark
Any function F with $F' = f$ is said to be an *indefinite integral* of f.

Theorem 10.4.2.
Suppose that f is differentiable, with a continuous derivative f', at each point of $[a,b]$. Then

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

Proof of Theorem 10.4.1.
We know from Theorem 10.3.4 that F is defined on $[a, b]$ with, of course, $F(a) = 0$. Next, if $a \le x < x + h \le b$, then (from Theorem 10.3.4)

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\left(\int_x^{x+h} f(t)\,dt\right) - f(x)$$
$$= \frac{1}{h}\int_x^{x+h} [f(t) - f(x)]\,dt.$$

As f is continuous at x, given any positive ε there is a positive δ such that if $|t - x| < \delta$, then $|f(t) - f(x)| < \varepsilon$. It follows that if $0 < h < \delta$, then $|f(t) - f(x)| < \varepsilon$ when $0 \le t \le h$, so that

$$\left|\frac{1}{h}\int_x^{x+h} [f(t) - f(x)]\,dt\right| \le \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|\,dt \le \varepsilon.$$

This shows that if $0 < h < \delta$, then

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| < \varepsilon.$$

The same argumemt holds when $a \le x - h < x \le b$ and when $x = a$ or $x = b$, and the result follows. ■

Proof of Theorem 10.4.2.
This is easy. We define a function F by

$$F(y) = [f(y) - f(a)] - \int_a^y f'(t)\,dt.$$

Theorem 10.4.1 implies that $F'(y) = 0$ for every y in (a, b), and this means that $F(y)$ is constant on $[a, b]$. We deduce that $F(b) = F(a) = 0$, and this proves the result. ■

Theorem 10.4.2 provides a method, known as *integration by parts*, for integrating products of functions. Given suitable functions u and w, we know that

$$\big(u(x)w(x)\big)' = u'(x)w(x) + u(x)w'(x),$$

so that from Theorem 10.4.2,

$$\big[u(x)w(x)\big]_a^b = \int_a^b \big(u(x)w(x)\big)'\,dx$$
$$= \int_a^b u'(x)w(x)\,dx + \int_a^b u(x)w'(x)\,dx.$$

Replacing w' by v, and letting $V = w$ (so that V is an indefinite integral of v), this leads directly to the following result,

Theorem 10.4.3: integration by parts.

Suppose that u is differentiable and that v is continuous on $[a,b]$, *and let V be an indefinite integral of v. Then*

$$\int_a^b u(x)v(x)\,dx = \Big[u(x)V(x)\Big]_a^b - \int_a^b u'(x)V(x)\,dx.$$

Finally, we mention the rule for a change of variable inside the integral sign. The Chain Rule is that for suitable functions f and g,

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x),$$

and this is the basis for the next result.

Theorem 10.4.4.

Suppose that the functions f and g satisfy

(i) f' *exists and is continuous on* $[a,b]$;
(ii) g *maps* $[c,d]$ *into* $[a,b]$ *with* $g(c) = a$ *and* $g(d) = b$;
(iii) g' *exists and is continuous on* $[a,b]$.

Then

$$\int_a^b f(x)\,dx = \int_c^d f(g(t))g'(t)\,dt.$$

Proof

Consider the function

$$F(y) = \int_a^{g(y)} f(x)\,dx - \int_c^y f(g(t))g'(t)\,dt$$

defined on $[c, d]$. The hypotheses imply that $F'(y) = 0$ for each y (after using the Chain Rule on the first integral), so that F is constant on $[c, d]$. This gives $F(d) = F(c) = 0$, and this is the required result. ■

Exercises

1. Show that

$$\int_a^b \exp x\,dx = \exp(b/a).$$

2. Use Theorem 10.4.2 to show that

$$\int_1^x \frac{dt}{t} = \log x.$$

3. Use Theorem 10.4.2 to provide a proof of Theorem 10.1.5.

4. Regard $\log x$ as the product $1 \times \log x$, and hence find

$$\int_1^x \log t \, dt.$$

5. Find

$$\frac{d}{dx}\int_x^b t^n \, dt.$$

Making appropriate assumptions about the functions f, g, and h, find and verify a formula for

$$\frac{d}{dx}\int_{g(x)}^{h(x)} t^n \, dt.$$

Check that your formula reduces to your answer in the first part of the question.

6. Evaluate $\int_0^1 xe^x \, dx$.

10.5 Improper integrals

In this section we shall briefly consider integrals of the type

$$\int_a^{+\infty} f(x)\, dx. \tag{10.5.1}$$

Our first task is to attach a meaning to this (for clearly we cannot subdivide the interval $\{x : a \leq x\}$ into a finite number of intervals of finite length). The definition is the obvious one; namely, we say that the *improper integral* (10.5.1) *exists*, or is *convergent*, if and only if the limit

$$\lim_{R\to+\infty}\int_a^R f(x)\, dx$$

exists, and when it does, the value of the integral in (10.5.1) is this limit. Notice that this defines the integral in (10.5.1) as a 'double limit'; the first limiting process gives rise to the integral from a to R, and the second is needed to let $R \to +\infty$. The first two results follow immediately from Theorem 3.6.3 and Theorem 4.4.1., respectively.

Theorem 10.5.1.
Suppose that f is integrable on each interval $[a,R]$, where $R > a$, and that $f(x) \geq 0$ when $x \geq a$. Then the indefinite integral (10.5.1) *exists if and only if there is a constant M such that for all R,*

$$\int_a^R f(x)\, dx \leq M.$$

Theorem 10.5.2.

Suppose that $f(x)$ is defined on $\{x: x \geq a\}$. Then the indefinite integral (10.5.1) *exists if and only if given any positive ε, there is a number ρ such that*

$$\left|\int_r^R f(x)\,dx\right| < \varepsilon$$

whenever $R > r > \rho$.

There is one more result in this section, and this has an immediate application to infinite series.

Theorem 10.5.3: the Integral Test for series.

Suppose that $f(x)$ is decreasing and nonnegative on $\{x: x \geq 0\}$. Then the indefinite integral and the infinite series

$$\int_0^{+\infty} f(x)\,dx, \qquad \sum_{n=1}^{+\infty} f(n),$$

are both convergent or both divergent.

Proof

Because f is monotonic, it is integrable on every interval $[a, b]$, where $a \geq 0$. If $n \leq x \leq n+1$, then $f(n) \geq f(x) \geq f(n+1)$; so that

$$f(n+1) \leq \int_n^{n+1} f(x)\,dx \leq f(n).$$

We now sum the terms in this inequality for $n = 1, 2, \ldots, N$ and obtain

$$f(2) + f(3) + \cdots + f(N+1) \leq \int_1^{N+1} f(x)\,dx \leq f(1) + f(2) + \cdots + f(N).$$

The result now follows, since the infinite sum converges if and only if the partial sums are bounded above, and by Theorem 10.5.1, the same is true of the integral. ■

EXAMPLE 10.5.4.

For any positive number t, we have, for all sufficiently large n,

$$\frac{1}{n^{1+t}} < \frac{1}{n\log n} < \frac{1}{n}. \tag{10.5.2}$$

Now, $\sum 1/n$ diverges, whereas $\sum 1/n^{1+t}$ converges, so it is of interest to determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\log n}$$

converges or diverges. In fact, a simple application of Theorem 10.5.3 shows that it *diverges*, for we need only note that (by making the substitution $y = \log x$)

$$\begin{aligned}\int_2^R \frac{dx}{x \log x} &= \int_{\log 2}^{\log R} \frac{dy}{y} \\ &= \log(\log R) - \log(\log 2) \\ &\to +\infty\end{aligned}$$

as $r \to \infty$. □

Exercises

1. Show that if $t > 0$, then (10.5.2) holds for all sufficiently large n.

2. Show that if $\varepsilon > 0$, the first of the two series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{1+\varepsilon}}, \quad \sum_{n=3}^{\infty} \frac{1}{n \log n \log \log n}$$

converges. Does the second series converge or diverge?

3. Does

$$\int_0^{\infty} \frac{dx}{x^x}$$

converge?

4. A function $f(x)$ is *absolutely integrable* on $\{x : x \geq 0\}$ if the integral

$$\int_0^{\infty} |f(x)|\, dx$$

converges. Show that if this is so, then the integral in (10.5.1) converges.[This mimics the difference between convergent and absolutely convergent series.]

Give an example of a function for which the first, but not the second, of the following two integrals converges:

$$\int_0^{+\infty} f(x)\, dx \quad \int_0^{+\infty} |f(x)|\, dx.$$

10.6 Integration and differentiation of series

This section is concerned with the important question of deciding when the assertions

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \left(\int_a^b f_n(x)\, dx\right), \qquad (10.6.1)$$

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \left(\frac{d}{dx} f_n(x)\, dx\right) \qquad (10.6.2)$$

are valid. To impress upon the reader the need for care, we begin with examples in which (10.6.1) and (10.6.2) fail. First, we note that if

$$F_n(x) = f_1(x) + \cdots + f_n(x),$$

then (10.6.1) and (10.6.2) are equivalent to

$$\int_a^b \left(\lim_{n\to\infty} F_n(x)\right) dx = \lim_{n\to\infty} \left(\int_a^b F_n(x)\, dx\right) \qquad (10.6.3)$$

and

$$\frac{d}{dx}\left(\lim_{n\to\infty} F_n(x)\right) = \lim_{n\to\infty} \left(\frac{d}{dx} F_n(x)\right).$$

Of course, we can change from the version containing infinite series to that containing sequences, or back again, at will.

EXAMPLE 10.6.1.
For $n \geq 2$, let F_n be defined by

$$F_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/n, \\ n & \text{if } 1/n < x < 2/n, \\ 0 & \text{if } 2/n \leq x \leq 1. \end{cases}$$

The reader should sketch the graph of this function and confirm that

$$\int_0^1 F_n(x)\, dx = n\left(\frac{2}{n} - \frac{1}{n}\right) = 1\ ;$$

thus the right-hand side of (10.6.3) is 1. However, for each x,

$$\lim_{n\to\infty} F_n(x) = 0, \qquad (10.6.4)$$

and this shows that (10.6.3) fails. We now verify (10.6.4). First, if $x = 0$, then $F_n(x) = 0$ for all n, so (10.6.4) holds when $x = 0$. Now suppose that $0 < x \leq 1$, and choose an integer N such that $N > 2/x$. If $n > N$ then

$2/n < x$, so that $F_n(x) = 0$. This shows that (10.6.4) holds for all x in $[0, 1]$, so we conclude that (10.6.1) *is not always true.* □

EXAMPLE 10.6.2.
Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$F_n(x) = \begin{cases} -1/n & \text{if } x \leq -1/n, \\ x & \text{if } -1/n \leq x \leq 1/n, \\ 1/n & \text{if } x \geq 1/n. \end{cases}$$

It is clear that $F'(0) = 1$ for every n and that $F_n(x) \to 0$ for every x, so that at $x = 0$,

$$\frac{d}{dx}\left(\lim_{n\to\infty} F_n(x)\right) = 0 \neq 1 = \lim_{n\to\infty}\left(\frac{d}{dx}F_n(x)\right).$$

In this example, (10.6.2) fails. □

Of course, we want to obtain conditions under which (10.6.1) and (10.6.2) are true, and we now turn to these. The problem with the sequence F_n in Example 10.6.1 is that although $F_n(x) \to 0$ for each x, there is no value of n for which the function F_n is close to 0 for all x. Indeed, as $F_n(3/2n) = n$, we see that although $F_n(x) \to 0$ for each x,

$$\lim_{n\to\infty}\left(\sup_{0\leq x\leq 1} F_n(x)\right) = +\infty.$$

This problem is overcome by the use of uniform convergence (see Definition 8.5.1 and also Exercise 10.6.1).

Theorem 10.6.3.
Suppose that the real-valued functions f_n on $[a,b]$ converge uniformly to the function f on $[a,b]$, and that $f, f_1, f_2, \ldots$ are all integrable on $[a,b]$. Then

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

Proof
This is the sequential version of (10.6.1), and the proof is very easy. Let

$$\delta_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$$

so that $\delta_n \to 0$. As f and each f_n are integrable, so is $|f_n(x) - f(x)|$, and

$$\begin{aligned}\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| &= \left|\int_a^b [f_n(x) - f(x)]\,dx\right| \\ &\le \int_a^b |f_n(x) - f(x)|\,dx \\ &\le \int_a^b \delta_n\,dx \\ &\le \delta_n(b-a).\end{aligned}$$

Thus, given any positive ε, we may choose n_0 such that if $n > n_0$, then $\delta_n < \varepsilon/(b-a)$. Then, if $n > n_0$,

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| < \varepsilon,$$

and this is the required result. ■

We shall only prove the corresponding theorem for (10.6.2) in its simplest form.

Theorem 10.6.4.
Suppose that

(i) *the functions $f_n:[a,b] \to \mathbb{R}$ are differentiable on $[a,b]$, and each f_n' is continuous there;*
(ii) *the sequence $f_n(a)$ converges to some value α;*
(iii) *the sequence f_n' converges uniformly to some function g in $[a,b]$.*

Then the sequence f_n converges to some function f on $[a,b]$, and

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

Proof
As f_n' converges uniformly to g on $[a, b]$, and as each f_n' is continuous on $[a, b]$, we deduce that g is continuous, and hence integrable, on $[a, b]$. Now let

$$G(x) = \int_a^x g(t)\,dt.$$

The uniform convergence of f_n' to g, and the continuity of the f_n and f_n', now guarantee that

$$f_n(x) - f_n(a) = \int_a^x f_n'(t)\,dt \to \int_a^x g(t)\,dt = G(x).$$

We deduce that

$$f_n(x) \to \alpha + G(x) = f(x)$$

and also that

$$f_n'(x) \to g(x) = f'(x)$$

as required. ■

We end with the remark that the version of Theorem 10.6.3 that applies to infinite series rather than sequences leads to the result that providing one stays within the region of convergence of the power series, we have

$$\int_0^x \left(\sum_{n=0}^{\infty} a_n t^n \right) dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}.$$

The corresponding result for derivatives of power series has already been established (in Theorem 9.5.3).

Exercises

1. Suppose that the f_n are integrable on $[a, b]$ and that $f_n \to f$ uniformly on $[a, b]$. Use (i) and (iii) of Theorem 10.2.3 to show that f is also integrable on $[a, b]$.[Thus the hypothesis that f is integrable can be deleted from Theorem 10.6.3.]

2. By considering $1 - x + x^2 - x^3 + \cdots$, show that if $|x| < 1$ then

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

3. By appealing to Theorem 10.6.4 (and giving details), verify that

$$\frac{d}{dx} \sin x = \cos x.$$

4. Using the Binomial Theorem (Theorem 9.6.3) and Theorem 10.6.4 (and giving details), verify that for all real t, and $|x| < 1$,

$$\frac{d}{dx}(1 + x)^t = t(1 + x)^{t-1}.$$

5. For each real x and positive integer n define $f_n(x) = x/(1 + nx^2)$. Find functions g and h such that for each x,

$$\lim_{n\to\infty} f_n(x) = g(x), \qquad \lim_{n\to\infty} f_n'(x) = h(x).$$

For which values of x is $h(x) = g'(x)$?

6. For each real x and positive interger n let $f_n(x) = 1/(1 + n^2x^2)$. Find a function f such that for each x, $f_n(x) \to f(x)$ as $n \to \infty$. Show that

$$\int_0^1 f_n(x)dx \to \int_0^1 f(x)dx$$

but that f_n does not converge uniformly to f on $[0, 1]$.

7. Show that for each positive integer q,

$$\int_0^\pi (\cos x)^{2q}\, dx = \pi \frac{(2q)!}{2^{2q}(q!)^2}$$

(see (11.2.2)). What is the value of the integral if $2q$ is replaced by $2q + 1$?
Deduce that

$$\frac{1}{\pi}\int_0^\pi e^{2\cos x}\, dx = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\,.$$

8. For each real x and positive integer n let $f_n(x) = x^{2n}/(1 + x^{2n})$. Show that for each x, $f_n(x)$ converges to some value $f(x)$ as $n \to \infty$, and draw the graphs of f and (for large n) f_n. On which intervals $[a, b]$ does f_n converge uniformly to f?

11 CHAPTER π, γ, e, and $n!$

Abstract
We prove various interesting formulas concerning π, γ, e, and $n!$.

11.1 The number e

We recall the number e defined by

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Theorem 11.1.1.
The number e is irrational, and $e = 2{\cdot}7182\ldots$.

Proof
As

$$\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n\,n!},$$

we have the double inequality

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{n\,n!}. \quad (11.1.1)$$

This enables us to estimate e to any desired accuracy, and we can show, for example, that $e = 2{\cdot}7182\ldots$. Now suppose that $e = p/q$, where p and

q are coprime integers. Then $q \geq 2$ and

$$q!\left(1+\frac{1}{1!}+\cdots+\frac{1}{q!}\right) < q!\,e < q!\left(1+\frac{1}{1!}+\cdots+\frac{1}{q!}\right)+\frac{1}{q}.$$

However, this cannot be so, as both the term on the left and $q!e$ are integers, and their difference is less than $1/q$. ■

Perhaps the most striking formula involving e and π is

$$e^{i\pi} = -1,$$

but this must be interpreted with care. While the (less impressive) formula $\exp(i\pi) = -1$ is true, the number e ($= 2{\cdot}718\ldots$) raised to the power $i\pi$ has infinitely many values, only one of which is -1 (see Exercise 6.5.6).

Exercises

1. Use (11.1.1) to show that $2{\cdot}7182 < e < 2{\cdot}7183$.
2. Show that if each a_n is 0 or 1, and if $a_n = 1$ for infinitely many n, then
$$a_0 + \frac{a_1}{1!} + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots$$
is irrational.
3. Show that
$$\int_0^\infty x^n e^{-x}\,dx = n!$$

11.2 The number π

We shall now show how to evaluate π, which, we recall, was defined (in Chapter 6) as the smallest positive zero of sin. We shall also prove that π is irrational, and we relate π to the length of a circle by the usual formula. We begin with the following formula (due to John Wallis, 1616–1703).

Wallis's formula:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots. \tag{11.2.1}$$

First, we must explain what we mean by this infinite product. If $x \geq 0$ then $1 + x \leq \exp x$, so that given a sequence a_n of nonnegative numbers such that $\sum a_n$ converges to A, we have the inequality

$$(1+a_1)(1+a_2)\cdots(1+a_n) \leq \exp(a_1 + a_2 + \cdots a_n) \leq \exp A.$$

It follows that the sequence $(1+a_1)(1+a_2)\cdots(1+a_n)$ is increasing and bounded above by $\exp A$, and so converges. We use the symbol $\prod$ for the product (as we use $\sum$ for the sum), and we write

$$\prod_{k=1}^{n}(1+a_k) \to \prod_{k=1}^{\infty}(1+a_k).$$

To obtain (11.2.1), we define the a_n by

$$1+a_n = \frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = 1+\frac{1}{4n^2-1},$$

so that $\sum a_n$ converges, and the right-hand side of (11.2.1), namely

$$\prod_{k=1}^{\infty}\left(\frac{2k}{2k-1}\cdot\frac{2k}{2k+1}\right),$$

exists as a real number. Wallis showed that this real number is $\pi/2$.

To prove (11.2.1), we consider, for $n = 0, 1, 2, \ldots$, the integrals

$$I_n = \int_0^{\pi/2} \sin^n x\, dx. \tag{11.2.2}$$

First,

$$I_0 = \pi/2, \quad I_1 = 1. \tag{11.2.3}$$

Next, if $n \geq 1$, then

$$\begin{aligned} 0 &= \int_0^{\pi/2} \frac{d}{dx}\big(\sin^n x\cos x\big)\, dx \\ &= \int_0^{\pi/2} \big[n\sin^{n-1} x(1-\sin^2 x) - \sin^{n+1} x\big]\, dx, \end{aligned}$$

so that the I_n satisfy the relation

$$(n+1)I_{n+1} = nI_{n-1}.$$

In conjunction with (11.2.3), this yields, for $k = 1, 2, \ldots$,

$$I_{2k} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}\cdot\frac{\pi}{2} \tag{11.2.4}$$

and

$$I_{2k+1} = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2k}{2k+1}. \tag{11.2.5}$$

Now, for $0 \leq x \leq \pi/2$, we have

$$\sin^{n+2} x \leq \sin^{n+1} x \leq \sin^n x,$$

so that

$$I_{n+2} \leq I_{n+1} \leq I_n = \frac{n+2}{n+1} I_{n+2},$$

and hence $I_{n+1}/I_{n+2} \to 1$. It follows that $I_{2n+1}/I_{2n} \to 1$, and this is (11.2.1). ■

Next, we define the length of the circle C given by $x^2 + y^2 = 1$ to be the limit, as $n \to \infty$, of the length L_n of a regular n-gon inscribed in C.

Theorem 11.2.1.
The length $|C|$ of C is 2π.

Proof
A simple calculation shows that for any real θ,

$$|1 - \exp(i\theta)| = 2\sin\theta/2,$$

so that $L_n = 2n\sin(\pi/n)$. As

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) = 1$$

(for when $|x| < 1$ the term in brackets differs from 1 by at most ex^2), we see immediately that

$$|C| = \lim_{n\to\infty} L_n = \lim_{n\to\infty} 2\pi\left(\frac{\sin\pi/n}{\pi/n}\right) = 2\pi.$$

■

We now describe a method of calculating π used by Archimedes (circa 260 B.C.). We begin with a regular hexagon h_1 inscribed in C and a regular hexagon H_1 circumscribed on C: see Figure 11.2.1.

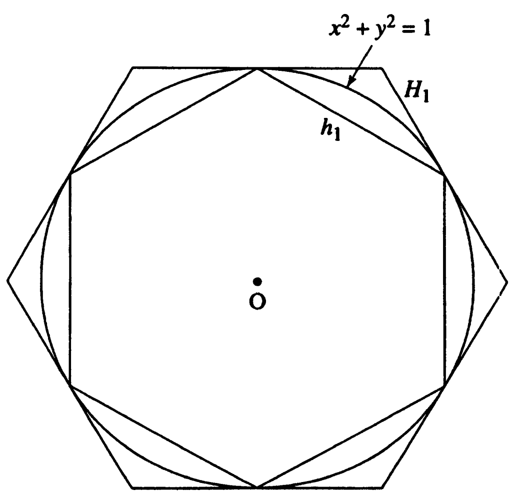

Figure 11.2.1

For each n, where $n \geq 2$, we construct (i) a regular polygon h_n with 3×2^n sides, inscribed in C, and such that the vertices of h_n are among the vertices of h_{n+1}, and (ii) a regular polygon H_n with 3×2^n sides, circumscribed on C, and such that the points of tangency of H_{n+1} to C are among those of H_n. For brevity, we also use h_n and H_n for the lengths of these polygons.

Now, by Theorem 11.2.1, $h_n \to 2\pi$, and (see Exercise 11.2.1)

$$h_1 < h_2 < h_3 < \cdots < H_3 < H_2 < H_1. \tag{11.2.6}$$

Next, we shall show that

$$h_{n+1} = h_n\sqrt{\frac{2H_n}{h_n + H_n}}, \quad H_{n+1} = \frac{2h_nH_n}{h_n + H_n}. \tag{11.2.7}$$

As $h_n \to 2\pi$ and, say, $H_n \to H$, (11.2.7) shows that $H_n \to 2\pi$, and π can now be estimated from above and below by implementing these iterative formulae on a computer.

The proof of (11.2.7) is straightforward. Writing $k_n = 3 \times 2^n$ and $\theta = \pi/k_n$, the polygons h_n and H_n consist of k_n segments of lengths $2\sin\theta$ and $2\tan\theta$, respectively, so that

$$h_n = 2k_n \sin\theta, \qquad H_n = 2k_n \tan\theta,$$

$$h_{n+1} = 4k_n \sin(\theta/2), \qquad H_{n+1} = 4k_n \tan(\theta/2).$$

With these, (11.2.7) follows immediately from the trigonometric identities

$$\frac{\sin\theta\tan\theta}{\sin\theta + \tan\theta} = \frac{\sin\theta}{1 + \cos\theta} = \tan(\theta/2)$$

and

$$\frac{2\tan\theta}{\tan\theta + \sin\theta} = \frac{2}{1 + \cos\theta} = \frac{1}{\cos^2(\theta/2)}.$$

We end this section with a proof that π is irrational.

Theorem 11.2.2.
The number π is irrational.

Proof
For each real t and each nonnegative integer n, let

$$I_n(t) = \int_{-1}^{1} (1 - x^2)^n \cos(tx)\, dx.$$

If we integrate by parts twice (treating the cases $t = 0$ and $t \neq 0$ separately), we obtain, for $n \geq 2$, the difference relation

$$t^2 I_n(t) = 2n(2n - 1)I_{n-1}(t) - 4n(n - 1)I_{n-2}(t) \tag{11.2.8}$$

Although (11.2.8) enables us to find an explicit form of any particular $I_n(t)$, the following information is sufficient. ■

Lemma 11.2.3.

There are polynomials P_n and Q_n, each with integer coefficients and of degree at most $2n$, such that

$$\frac{t^{2n+1}I_n(t)}{n!} = P_n(t)\sin t + Q_n(t)\cos t.$$

Proof

The proof is by induction using (11.2.8) and is left to the reader. ■

It is now easy to see that π is irrational, for suppose that $\pi/2 = b/a$, where a and b are positive integers. Then

$$\begin{aligned}\left(\frac{b^{2n+1}}{n!}\right) I_n(b/a) &= a^{2n+1}\big[P_n(b/a)\sin(b/a) + Q_n(b/a)\cos(b/a)\big]\\ &= a^{2n+1}P_n(b/a)\end{aligned}$$

because $\sin(b/a) = 1$ and $\cos(b/a) = 0$. By considering the series for $\exp b^2$, we see that $b^{2n}/n! \to 0$, and as $|I_n(t)| \leq 2$ for all t, we see that for all sufficiently large n,

$$|a^{2n+1}P_n(b/a)| < 1.$$

However, as $a^{2n+1}P_n(b/a)$ is an integer, for these n we have

$$0 = a^{2n+1}P_n(b/a) = \left(\frac{b^{2n+1}}{n!}\right) I_n(b/a),$$

so that $I_n(b/a) = 0$. This is a contradiction, for

$$I_n(b/a) = I_n(\pi/2) = \int_{-1}^{1}(1 - x^2)^n\cos(\pi x/2)\,dx > 0,$$

so finally, we conclude that π is irrational.

Exercises

1. Verify (11.2.6). The Triangle Inequality gives $h_n < h_{n+1}$. Use the formula for $\tan 2\theta$ in terms of $\tan\theta$ to show that $H_{n+1} < H_n$, and hence that $H_n \to H$, say. Now use (11.2.7) to show that $H_n \to 2\pi$, and deduce that for all n, $h_n < 2\pi < H_n$.

2. Show that if $|x| < 1$, then

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}.$$

Expand the term on the right as a power series and integrate term by term to derive Gregory's formula:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.$$

Combine this with Question 6.4.2 to estimate π.

11.3 Euler's constant γ

In this short section we define Euler's constant γ.

Theorem 11.3.1.
The sequence

$$\gamma_n = 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n} - \log n$$

converges to a number known as Euler's constant γ, and $\gamma = 0{\cdot}5772\ldots$.

Proof
For $n = 2, 3, \ldots$ let

$$a_n = \log\left(\frac{n}{n-1}\right) - \frac{1}{n}. \tag{11.3.1}$$

Then $a_2 + \cdots + a_n = 1 - \gamma_n$, so it is sufficient to show that $\sum a_n$ converges. However, as

$$\int_0^1 \frac{t}{n(n-t)}\,dt = \int_0^1 \left(\frac{1}{n-t} - \frac{1}{n}\right) dt = a_n,$$

we have

$$0 \leq a_n = \int_0^1 \frac{t}{n(n-t)}\,dt \leq \frac{1}{2n(n-1)} \leq \frac{1}{n^2},$$

so that $\sum a_n$ converges as required. ■

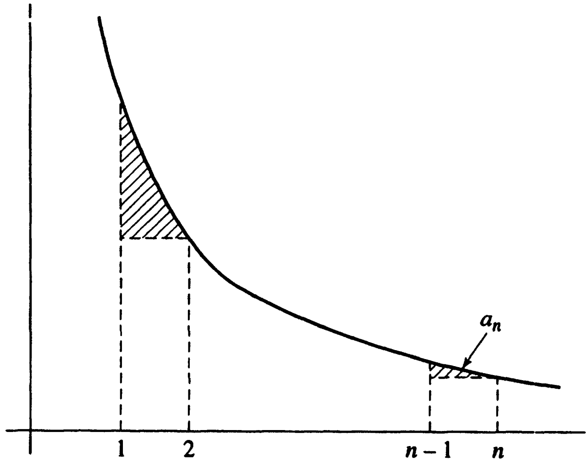

Figure 11.3.1

Exercises

1. Show that a_n is the area of the shaded region in Figure 11.3.1, and deduce that $0{\cdot}5 < \gamma_n < 1$.

11.4 Stirling's formula for n!

For any sequences a_n and b_n, we write $a_n \sim b_n$ when $a_n/b_n \to 1$, and with this we can state

Stirling's formula:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

We begin with an alternative proof (see Theorem 7.2.4) that

$$(n!)^{1/n} \sim \frac{n}{e}.$$

This is weaker than (and is implied by) Stirling's formula, but its proof will serve as an introduction to our proof of Stirling's formula.

To estimate $n!$ we observe that

$$\log n! = \log 2 + \log 3 + \cdots + \log n,$$

and that

$$\begin{aligned}\int_1^n \log x\,dx &= \int_1^2 \log x\,dx + \cdots + \int_{n-1}^n \log x\,dx \\ &< \log 2 + \log 3 + \cdots + \log n \\ &< \int_2^3 \log x\,dx + \cdots + \int_n^{n+1} \log x\,dx \\ &= \int_1^n \log x\,dx + \int_n^{n+1} \log x\,dx - \int_1^2 \log x\,dx.\end{aligned}$$

As

$$\int_1^n \log x\,dx = \Big[x\log x - x\Big]_1^n = \log\left(\frac{n}{e}\right)^n + 1, \qquad (11.4.1)$$

we see that

$$1 + \log\left(\frac{n}{e}\right)^n < \log n! < \log\left(\frac{n}{e}\right)^n - \log\left(\frac{2}{e}\right)^2 + \int_n^{n+1} \log x\,dx.$$

This simplifies to

$$\frac{1}{n} < \log\left[\frac{(n!)^{1/n}e}{n}\right] < \frac{1}{n}\left[\log 4 + \int_n^{n+1} \log x\,dx\right]$$

Finally, as

$$0 \le \frac{1}{n}\int_n^{n+1} \log x\,dx \le \frac{\log(n+1)}{n} \to 0$$

(see Exercise 11.4.1), we find that

$$\log\left[\frac{e(n!)^{1/n}}{n}\right] \to 0,$$

and applying the exponential function (which is continuous), we obtain the desired result.

Proof of Stirling's formula.
We have to prove that the sequence v_n defined by

$$v_n = \frac{e^n n!}{n^n\sqrt{n}}$$

converges to $\sqrt{2\pi}$, and the proof is divided into the following three steps.

Lemma 11.4.1.
The sequence v_n converges to some v, where $v \ge 0$.

Lemma 11.4.2.
$v > 0$.

Lemma 11.4.3.
$v = \sqrt{2\pi}$.

Proof of Lemma 11.4.1.
A computation shows that $v_1 > v_2 > v_3$, and this suggests that we try to prove that v_n is decreasing. Now, the inequality $v_n > v_{n+1}$ is equivalent to

$$\frac{e^n n!}{n^n\sqrt{n}} > \frac{e^{n+1}(n+1)!}{(n+1)^{n+1}\sqrt{n+1}},$$

which, after a little simplification, is equivalent to

$$\log\left(1+\frac{1}{n}\right) > \frac{1}{n+1/2}. \tag{11.4.2}$$

To prove this, we consider the function

$$\varphi(x) = \log(1+x) - \frac{2x}{2+x},$$

which is defined when $x > -1$. As

$$\varphi'(x) = \frac{1}{1+x} - \frac{4}{(2+x)^2} = \frac{1}{1+x} - \frac{1}{1+x+x^2/4} > 0$$

when $x > 0$, and as $\varphi(0) = 0$, we see that $\varphi(x) > 0$ when $x > 0$. In particular, $\varphi(1/n) > 0$, and this is (11.4.2). We have now shown that $v_n > v_{n+1}$; thus v_n converges to some v, where $v \geq 0$. ■

Proof of Lemma 11.4.2.
We write $u_n = \log v_n$ and show that u_n is bounded below; then $v > 0$. As the graph of $\log x$ lies underneath its tangent at any point, we see that for all positive x and a,

$$\log x < \log a + \frac{x-a}{a}. \tag{11.4.3}$$

An analytic proof of this is suggested in Exercise 11.4.2. Integrating both sides of (11.4.3) over the interval $[k-\frac{1}{2}, k+\frac{1}{2}]$, where k is an integer with $k \geq 2$, yields

$$\int_{k-1/2}^{k+1/2} \log x\,dx \leq \log k,$$

and hence

$$\int_{3/2}^{n+1/2} \log x\,dx \leq \log 2 + \cdots + \log n = \log n!\,.$$

Using (11.4.1), this shows that

$$\begin{aligned}\log n! &\geq \log\left(\frac{n}{e}\right)^n + \int_n^{n+\frac{1}{2}} \log x\,dx + 1 - \int_1^{3/2} \log x\,dx \\ &\geq \log\left(\frac{n}{e}\right)^n + \frac{1}{2}\log n + A,\end{aligned}$$

say, so that $u_n \geq A$, and $v \geq \exp A > 0$. ∎

Proof of Lemma 11.4.3.
We recall the formulae (11.2.4) and (11.2.5) for the integrals I_n defined in (11.2.2). Now

$$I_{2k} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}\cdot\frac{\pi}{2} = \frac{(2k)!}{2^{2k}(k!)^2}\cdot\frac{\pi}{2}$$

and

$$I_{2k+1} = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2k}{2k+1} = \frac{2^{2k}(k!)^2}{(2k+1)!}.$$

As $I_{2k+1}/I_{2k} \to 1$, these show that

$$\frac{v_n^2}{v_{2n}} = \frac{(n!)^2 2^{2n}}{(2n)!}\sqrt{\frac{2}{n}} = \sqrt{\frac{I_{2n+1}}{I_{2n}}}\sqrt{\frac{(2n+1)\pi}{n}} \to \sqrt{2\pi},$$

so that $v = \sqrt{2\pi}$. ∎

Exercises

1. Show that $x/e^x \to 0$ as $x \to +\infty$. Deduce that $n^{-1}\log(n+1) \to 0$ as $n \to \infty$.
2. For positive x and a, let

$$\varphi(x) = \frac{x-a}{a} + \log a - \log x.$$

Show that φ has a minimum at $x = a$, and hence derive (11.4.3).

3. Use the proof of Theorem 7.2.4 given above to show that if $n \geq 2$, then

$$\frac{e}{(n+1)^{1/n}}\cdot\frac{n}{n+1} < \frac{n}{(n!)^{1/n}} < \frac{e}{4^{1/n}}.$$

11.5 A series and an integral for π

In this section we sketch proofs of the following two results:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{11.5.1}$$

and

$$\int_0^{\infty} e^{-x^2/2}\, dx = \sqrt{\frac{\pi}{2}}. \tag{11.5.2}$$

In fact, it is known that for each positive integer k,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = r_k \pi^{2k},$$

where r_k is rational, but no such formula is known for odd powers. In this, the last section in the book, we merely sketch the proofs of these results.

It is possible to prove (11.5.1) by using real functions of a real variable, but we prefer the following method which introduces ideas that are of interest in their own right. We recall that

$$\sin z = z - \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots.$$

Now, the zeros of $\sin z$ are the points $n\pi$, where $n \in \mathbb{Z}$, and by using the theory of infinite products, *it is possible to express* $\sin z$ *as an infinite product*, namely

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).$$

Assuming this, we see that $P_n(z) \to \sin z$ as $n \to \infty$, where

$$\begin{aligned} P_n(z) &= z \prod_{m=1}^{n} \left(1 - \frac{z^2}{m^2\pi^2}\right) \\ &= z - \frac{z^3}{\pi^2}\left(\frac{1}{1^2} + \cdots + \frac{1}{n^2}\right) + \cdots + a_n z^{2n+1}. \end{aligned}$$

Now, P_n converges uniformly to S in some neighbourhood of the origin, so that (from the theory of functions of a complex variable) for each m, the mth derivative of P_n at the origin converges to the mth derivative of sin at the origin. Taking $m = 3$, we obtain (11.5.1).

In order to prove (11.5.2) we need the theory of double integrals (which can be handled much like double sums). In fact, we need to know that

$$\int\int e^{-(x^2+y^2)/2}\,dxdy = \int_{y=0}^{\infty}\left(\int_{x=0}^{\infty} e^{-(x^2+y^2)/2}\,dx\right)\,dy$$
$$= \int_{y=0}^{\infty}\left(e^{-y^2/2}\int_{x=0}^{\infty} e^{-x^2/2}\,dx\right)\,dy$$
$$= \left(\int_0^{\infty} e^{-x^2/2}\,dx\right)^2,$$

where the first integral is taken over the first quadrant. In order to evaluate this integral, we regard it as a repeated integral with respect to polar coordinates (r,θ), where $dxdy = rdrd\theta$. We integrate with respect to θ first, and then with respect to r (by making the substitution $s = r^2$), and in this way we obtain (11.5.2).

Exercises

1. Complete the following steps to show that

$$\int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

(i) Let

$$I_p = \int_0^{\pi/2}\frac{\sin px}{\sin x}\,dx, \quad J_p = \int_0^{\pi/2}\frac{\sin qx}{\sin x}\cos x\,dx.$$

By considering $\sin(2n+1)x$ and $\sin(2n-1)x$, show that

$$\frac{\pi}{2} = I_1 = J_2 = I_3 = J_4 = \cdots.$$

(ii) Show (using by integration by parts) that for each positive integer n,

$$\lim_{n\to\infty}\int_0^{\pi/2}\sin nx\left[\frac{1}{x} - \frac{\cos x}{\sin x}\right]\,dx = 0.$$

(iii) Deduce that

$$\frac{\pi}{2} = \lim_{n\to\infty}\int_0^{\pi/2}\frac{\sin 2nx}{x}\,dx = \lim_{n\to\infty}\int_0^{n\pi}\frac{\sin y}{y}\,dy.$$

(iv) Finally, show that

$$\frac{\pi}{2} = \lim_{R\to\infty}\int_0^{R}\frac{\sin y}{y}\,dy.$$

Appendix: Mathematical Induction

The construction of the real numbers usually starts with Peano's axioms for the integers. Next, one develops the arithmetic of the integers; then the rational numbers are introduced as ordered pairs of integers; and then, finally, one constructs the real numbers, either as suitable sequences of rational numbers or by the so-called Dedekind cuts of rational numbers. In any event, the end product is a set of objects that is an ordered field in which the Least Upper Bound Axiom holds and that is denoted by $\mathbb{R}$. In this construction (and this is the usual one) Mathematical Induction appears as one of Peano's axioms.

In this text (as in many others) we have chosen to avoid these foundational matters and start with the assumption that we have a set, $\mathbb{R}$, which we call the set of real numbers, which is an ordered field in which the Least Upper Bound Axiom holds. The question now is whether or not we have implicitly included Mathematical Induction in our assumptions. In short, do we still have to incorporate it as an *axiom*, or is it now a *theorem*? The answer is that it is a theorem, and our aim here is to sketch its proof.

We say that a set A of real numbers is **inductive** if $1 \in A$ and if $x+1 \in A$ whenever $x \in A$. Such sets exist, for example, the set of positive real numbers. It is clear that the intersection Ω of all inductive sets is itself an inductive set; thus Ω is inductive, $1 \in \Omega$, and if A is inductive, then $\Omega \subset A$. The Principle of Induction *for the set* Ω is now a triviality.

The Principle of Mathematical Induction.

Suppose that $A \subset \Omega$, $1 \in A$, *and that* $x + 1 \in A$ *whenever* $x \in A$. *Then* $A = \Omega$.

Proof

Let A satisfy the given hypotheses. Then A is inductive, so that $\Omega \subset A$. But by assumption, $A \subset \Omega$; thus $A = \Omega$. ■

Of course, this is not yet the familiar Principle of Induction, but our problem has shifted. We have *proved* the Principle of Induction for Ω, and it now only remains to identify Ω as the set of positive integers. Note that because $\{x \in \mathbb{R} : x \geq 1\}$ is inductive, it contains Ω, so that every element y in Ω satisfies $y \geq 1$. In particular, this already shows that 1 is the smallest member of Ω.

The set $\mathbb{Z}$ of integers is, by definition, the additive subgroup of the real numbers that is generated by 1 or, more precisely, the intersection of all additive subgroups of $\mathbb{R}$ that contain the element 1. To derive the Principle of Mathematical Induction for $\mathbb{N}$, we have to show that $\Omega = \mathbb{N}$, and this is a consequence of the following result (where $-\Omega$ is defined by $x \in -\Omega$ if and only if $-x \in \Omega$).

Theorem A.

$\mathbb{Z} = \Omega \cup \{0\} \cup (-\Omega)$. *In particular,* 1 *is the smallest positive integer.*

For this, we need two preliminary results.

Lemma 1.

$\Omega = \{1\} \cup \{x + 1 : x \in \Omega\}$.

Proof

Let $A = \{1\} \cup \{x + 1 : x \in \Omega\}$. We begin by showing that A is inductive. First, $1 \in A$. Now suppose that $a \in A$. Then either $a = 1$ or $a = x + 1$, where $x \in \Omega$. In both cases, $a \in \Omega$, so that

(i) $A \subset \Omega$, and
(ii) $a + 1 \in A$.

Now, (ii) shows that A is inductive, and hence $\Omega \subset A$. We have now proved that $A = \Omega$. ■

Lemma 2.

Let $\Gamma = \Omega \cup \{0\} \cup (-\Omega)$. *Then*

(a) Γ *is inductive;*
(b) *if* $x \in \Gamma$ *and* $y \in \Omega$, *then* $x + y \in \Gamma$;
(c) *if* $x \in \Gamma$ *and* $y \in \Omega$, *then* $x - y \in \Gamma$.

Proof

We prove (a). First, by definition, $1 \in \Gamma$. Now take any x in Γ; then either $x \in \Omega$ or $x = 0$ or $-x \in \Omega$. In the first case, $x + 1 \in \Omega$ (because Ω

is inductive). In the second case, $x + 1 = 1 \in \Omega$. By Lemma 1, the third case implies that either $-x = 1$ (in which case $x + 1 = 0 \in \Gamma$) or $-x = r + 1$ for some r in Ω (in which case $x + 1 = -r \in \Gamma$). Thus in all cases, $x + 1 \in \Omega \cup \Gamma = \Gamma$, so that (a) holds.

We now prove (b). Take x in Γ and let $A = \{y \in \Omega : x + y \in \Gamma\}$. We want to show that $A = \Gamma$. By Lemma 1, $1 \in A$. Now take y in A; then $x + y \in \Gamma$, so that $x + y + 1 \in \Gamma$, whence $y + 1 \in A$. Thus A is inductive, so that $\Omega \subset A$; whence $A = \Omega$. The proof of (c) is similar. ■

Proof of Theorem 1.

As $\mathbb{Z}$ is inductive, $\Omega \subset \mathbb{Z}$. Thus (because $\mathbb{Z}$ is an additive group) $\Gamma \subset \mathbb{Z}$. It remains to prove that Γ is an additive group, for then, as $1 \in \Gamma$, we must have $\mathbb{Z} \subset \Gamma$, and hence $\mathbb{Z} = \Gamma$, which is the desired conclusion.

As $0 \in \Gamma$ and as $-x \in \Gamma$ whenever $x \in \Gamma$, in order to show that Γ is a group we have only to show that it is closed under addition. Take any x and y in Γ. Now, either $y \in \Omega$, $y = 0$, or $-y \in \Omega$. If $y \in \Omega$, then $x+y \in \Gamma$ by Lemma 2(b). If $y = 0$, then $x+y = x \in \Gamma$. If $-y \in \Omega$, then by Lemma 2(c), $x - (-y) \in \Gamma$. The proof of Theorem 1 is complete. ■

Finally, we have

Theorem B.

Every nonempty set of positive integers has a smallest member.

Proof

Let A be a nonempty subset of Ω, and define

$$B = \{x \in \Omega : \text{ for all } a \text{ in } A, x \leq a\}.$$

Choose any element a_1 of A. Then $a_1 + 1 \notin B$ (for $a_1 + 1 \leq a_1$ is false). However, as a_1 is in A, it is in Ω, so that $a_1 + 1 \in \Omega$. It follows that $B \neq \Omega$, and hence (as $B \subset \Omega$) that B is not inductive.

As B is not inductive and as $1 \in B$, there is some b in B with $b + 1 \notin B$. Now, $b + 1 \notin B$ implies that there is some α in A with $\alpha < b + 1$, and because $b \in B$, $b \leq \alpha$; thus $b \leq \alpha < b+1$. However, $\mathbb{Z}$ is an additive group that respects addition, and as $b \in B \subset \Omega \subset \mathbb{Z}$, we see that $b - \alpha$ is an integer in the range $[0, 1)$. By (1), $b = \alpha$, whence $\alpha \in B$, and this implies that α is the smallest member of A. This completes the proof. ■

References

We recommend the following references to readers interested in directed sets.

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