"On the concept of a cognitive mathematician" or "On the cognitive approach to mathematics"

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Abstract

We will introduce the cognitive progress as a motivation to do mathematics, and then we study implications of this point of view on research, collaboration, evaluation, problem solving, theorizing, teaching, interpreting history, communication and other aspect of being a mathematician. This is in continuation of another paper in which we compare different styles of being a mathematician like problem solvers and theorizers.

Introduction

The motivation of a cognitive mathematician for doing mathematics is cognitive progress. All aspects of mathematical life are to be understood by their affects on the cognitive structure of the mathematician. There are many different aspects of doing mathematics which have consequences on cognitive structure or being affected by the cognitive structure. The strategy of research including problem solving aspect and theorizing aspect is different from mathematicians doing research in other styles. Teaching and educating has a different story in the mind of such a mathematician which should be studied in its own right. History of mathematics is understood quite differently in the eye of a cognitive mathematician. The last but not the least, the role of a mathematician in a society is different in the cognitive approach to doing mathematics. The author acknowledges support by an Oswald Veblen grant from Institute for Advanced Study.

Choice of research field

When I decided to be a number theorist and algebraic geometer, I had extensive background in geometry and dynamical systems, and my cognitive style of learning was pictorial. Every clue suggested that I shall become a geometer, but I chose my path to be different. The reason was that, I felt educated and affected already by geometric material, and I wanted to taste the affect of another kind of mathematics on my cognitive structure. Not only that, but also I knew that number theory and algebraic geometry are the heavens of cognitive style of doing mathematics. In many ways they open the mind of a geometer that geometry cannot. I knew very well that I will never be as good as number theorists and algebraic geometers whose style of learning is verbal, or at least I shall never consider verbal mathematicians as a role model. My adviser Andrew Wiles and almost all other students of his were verbal cognitively. It didn`t bother me being compared with people in an art which is not mine. Because I had a different good in my mind. I was not good in computation and not very strong in logical thinking, but very strong in intuition and pictorial thinking. In the world I was educated such a personality was not appropriate for doing mathematics. But having cognitive motivations this was exactly what I should have done.

Change of research field

I started my research in geometric group theory, then moved to Arakelon Theory and then algebraic geometry and then I started working with Andrew Wiles and dealt with modular forums and their congruences, then started working with Galois-Lie representations and their deformations and then moved to arithmetic dynamics and recently dictionary between primes and knots. This is apart from my interest in philosophy, cognology and education. This path shows my start from land of command then travelling far, and then trying to connect far lands and my land of command. Change of research field for me has a history and goal, and choice of different research fields I come into contact with, has cognitive goals hidden behind it. Eventually going back towards the roots but not forgetting the history of development of mathematical thought seems to be the overall picture of my history of change in research fields. I am not suggesting that everyone should be changing research fields for the same reasons, but I am pointing out that many mathematicians change their field of research for other reasons and less often as a cognitive mathematician. On many occasions the reasons for changing research field are external rather than internal. Some goes for the motivation of people doing mathematics. Many people have external motivation rather than internal motivations.

Collaboration in writing a paper

I have hardly ever collaborated in writing a mathematics paper, but very often collaborated on writing textbooks of mathematics. The reason is that my colleagues in mathematics education agreed with me on the cognitive and philosophical role I could play in writing textbooks, but of colleagues in mathematics did not believe in the style of doing mathematics which I did. But my conception of collaboration is the following: each of the collaborators should take responsibility in one or few cognitive abilities needed for performing the research. This does not mean that people cannot have contributions in aspects they are not responsible, but it means that in each aspect one person is responsible for the final decision.It is hardly possible for a person not with cognitive motivations for doing mathematics to collaborate in this manner, because such a person doesn't think in these terms. Usually collaborators divide according to the conceptual structure, or the problem solving and theorizing skills, or they divide theorems and propositions or different sections between themselves. Sometimes one becomes the leader and others follow him in what to do which I had the luxury of such assistant once with a PhD student were we did not have such a successful paper.

Collaboration in a research project

Collaboration in research projects are set up much more fluently at any time. They are problems that their importance in some field is established and many people around the world think about them and make progress. Sometimes there in framework to think about the problem which is agreed upon or a theoretical setting is set around the problem, which is well-accepted. Sometimes people try to work around a well-established result and make progress. This could be also thought as a research project. I have been attending a few research projects by now, but my contribution is defined by my cognitive skills. I do not try to make contributions which I cannot cognitively compete in them, and I expect my research be accepted for publications the way it is. Usually referees expect a work to be mature in a number of aspects which are not cognitively related and it is not natural to expect everybody to be good at all of them, and it is not fair to say someone who is not good in all basic aspect of research should not do research and should not contribute. I, because of cognitive motivations am not pleased to try to be good in aspects not related to my cognitive path of progress. So, collaboration in research projects is quite different from the perspective of a cognitive mathematician.

Evaluation of research

According to what is the meaning of doing mathematics for different people, they evaluate research material performed by others differently. For some a paper is unacceptable without problem solving contributions and for others it is worthless without introducing an appropriate theoretical framework to formulate and think about the problem. For a cognitive mathematician a piece of mathematical research is evaluated by its cognitive contribution to the field or to the mathematical thought. An exposition is evaluated by cognitive contribution it could have to the reader`s cognition, or even his cognitive structure. A piece of mathematical research which could be done by the same cognitive structure available before that paper, is hardly of significant value. The value of someone`s research is their breakthroughs in contribution to cognitive structure in mathematical thought. For example, understanding something which was available to mathematical modeling beforehand, or inventing new cognitive skills, or discovering new paths of cognition. One may ask why this is not a research in cognitive psychology but in mathematics. The answer is, such is cognitive mathematics as a motivation for doing research. Of course such papers are hardly found among the papers published every day in mathematics journals.

Strategy of problem solving and theorizing

Cognitive mathematics has something to say in how one should approach a problem to solve it or how one should theorize a method of formulating the problem. The main question is how a problem could lead to creation of new cognitive structure to deal with it. This way it is not important at all if there already exist a satisfactory formulation for the problem or not. So every piece of old mathematics could be motivation for contribution to cognitive structure if it is studied from a new perspective. In this approach, understanding the old mathematics is more urgent than looking for new concepts and new problem and new theorems, which is also important in its own right. Sometimes one makes a cognitive contribution by proving a theorem which is not preceded by a cognitively similar result, or someone may give a proof for a theorem which has cognitive contribution to the concept of proof. Question is: Does the cognitive mathematician look any different solving everyday problems and theorizing formulation to them? The answer is not much, except that he tries extra to be more creative cognitively and learn cognitively from the mathematical experience.

Strategy of research

Under the cognitive approach umbrella, a cognitive mathematician has a different approach in research. The main question is what the cognitive need for doing the research project is. Is it possible to do the research project? Is it possible to do the research in the existing cognitive structure introduced by other mathematicians before? This is a kind of evaluating how difficult the problem is cognitively. Are there cognitive barriers on solving the problem? Are there other problems that are similar to this problem cognitively? Trying to find cognitive dictionaries between the unsolved problems and solved problems. Trying to reduce the problem to a problem in the language of cognition. For this, it is necessary to have a cognitive understanding of history of mathematics. In better words we should understand what are the cognitive steps that have happened during the course of development of mathematics. Of course not any research topics appears to be a cognitive challenge. Solving many problems is cognitively smooth because of previous achievements in history of mathematics. Also, if one makes a cognitive contribution to mathematics it paves the way for solving many new problems in mathematics. An important example would be invention of calculus by Newton and Leibnitz, and invention of (co)homology theories by Poincare, and invention of schemes by Grothendieck.

Choice of research topic

Choice of research topic is tricky because of two reasons. The first important aspect is that it should be compatible with the cognitive abilities of the researcher. In better words, the researcher must be able to be cognitively creative at the level of research problem. The second aspect is that research topic should be quicker natively non trivial but still accessible to the cognitive progress already made in the field of research. Choice of research topic also depends on the cognitive Help available to the researcher. One may use contribution of colleagues and felow mathematicians to have progress in the problems. For better choice of research topic one needs to be cognitively up-to-date this means that one should be translating the research done by other mathematician to the cognitive language. One cannot be up to date in research done in all different fields of mathematics. Therefore one should only follow particular research in fields of mathematics which are cognitively relevant to the research problem chosen by the researcher. Make a note that, cognitively relevant mathematics, is different from conceptually related mathematics, or theoretically similar mathematics. For example, if there exist a dictionary between the research project and other branches of mathematics, this is a reason why research in other related areas of mathematics must also be followed up.

Following popular research on a given topic

Suppose that a cognitive mathematician set his mind on following a particular research and a given research problem. He shall summarize their different attempts and attacks to the research problem given. Some people summarize research in terms of conceptual structure, some according to theorems proven in these papers, some according to problem solving contents and other according to theoretical the structure of papers. A cognitive mathematician tries to understand the structure of relation between papers in a given field according to cognitive contents and cognitive contributions of these papers. Many important paper have no cognitive contribution and they will not occupy a central place in the map a cognitive mathematician has in mind. Everything is understood with respect to papers which have proven to be cognitively creative. This is actually a smart way to summarize the data. Because, the cognitive contribution of research material is usually very brief. But the conceptual structures or theorems proven are rarely that brief. Also, in the mind of a cognitive mathematician, there are fewer cognitive relations among papers, rather than conceptual relations or referring relations. This is why the politics of referring to other research is very different in the hands of a cognitive mathematician. This will certainly affect the way such a mathematician writes a paper. This is what we shall investigate in the next section.

How to write a paper

Introduction of any paper should include the cognitive set up of a problem and summarize the paper's cognitive contributions. Cognitive background should be the contents of the first chapter and cognitive implications should form the contents of the final chapter of the paper. The structure of the paper should not be arranged according to conceptual background or theoretical the structure, but in terms of cognitive relations between different parts of the paper. It is best that each chapter should have cognitive definition and cognitive reasons for existence. For example, computations and proofs should not be left in a different chapter. Theorems and preposition should be also stated in the cognitive language and results which have cognitive interpretations should be made Bolded. Cognitive relations between chapters should be bolded. Beginning of any chapter should contain the material of the chapter from the cognitive perspective. The cognitive language of doing mathematics is a new style and it is best shown by examples. For example, you could make a list of proofs of Pythagoras theorem and decide which groups are cognitive and which proofs are not. Cognitive proofs do not necessarily have the same cognitive content. The most cognitive proof of the Pythagoras theorem I know is the proof by similarity of triangles:

How to read a paper

In order to relate with the material in a paper a cognitive mathematician is bound to translate context to cognitive language and this is not fully possible if he has no information about papers surrounding the paper under consideration. The first generation of related papers are those papers referred to and those papers referring to the paper to be studied. Not all papers in the first generation are related to the main one cognitively. For the second generation or shall not restrict to papers which are cognitively related to the main one. All papers in second hand relation for the main paper should be in the study pool, but only those which have cognitive contributions should be studied. Usually two generation are enough to be studied and the third generation papers relating to the main one are too many and studying them is a lifetime project. Except if the paper is very recent and not many papers have referred to it yet. Only in light of rewriting many of related papers one can fully translate a Paper to cognitive language. Otherwise one has to keep himself happy with some cognitive comments and perspective. This indicates that becoming a cognitive researcher in some area, is not a one night decision and it takes years of hard work to understand a field of mathematics in the cognitive language.

How to teach mathematics

The key to teaching mathematics in the cognitive style Is not only treating the material cognitively and insist on the cognitive story behind the material but the point is to correlate with the cognition structure of students which could be treated personally and in groups. The question is what the cognitive background of the students is and the material to be taught makes what kind of cognitive influence to such a mind. This needs cognitive curriculum development, deciding about the global cognitive path that a student is prepared to take. One shall keep in mind single recipe could not work for all cognitive structures, and cognitive histories. So there should be flexibility in the educational system and also flexibility in the treatment of the students being taught in the same class. Therefore teachers should perform multiple roles and multiple personalities in the class, or it is best fit that two or three teachers teach simultaneously in the same class. The students should also learn by interaction so that they understand how the material is treated in a different mind. So it is better that they work in groups or they sit in a way that they could see all students in the class in action. Technology could fundamentally help such a classes to be practiced.

How to understand history of mathematics

Understanding history of the field in cognitive language doesn’t change the perspective much. Mathematical giants will remain important figures, although most of them will not be of the same height or importance. Plato, Aristotle, Archimedes, Khayyam, Toosi, DesCartes, Fermat, Newton, Leibnitz and more recent giants remain Important mathematicians but more or less with comparable influences. I cannot make an accurate list of all such figures, and I refuse to do so. There are a number of great mathematicians who do not seem that influential but studying their research with cognitive perspective reveals that they have been as important as the first group in development of mathematics. Many important figures will prove to be not much influential in the cognitive perspective, and it turns out that, many are the soldiers of machinery set up buy minds of leaders of mathematical thought. Many modern figures will turn out to be as important as ancient giants, like Gerothendieck, Poincare, Hilbert, Weyl, Weil, and some kings like Euler, Riemann & Gauss appear to be only as important as other giants if we consider only their cognitive contributions. There are mathematicians with limited research material who become as important figures like Arakelon, Roch, and others. There are also some mathematicians alive with comparable legacy.

How to communicate to mathematicians in other fields

To communicate cognitively with other mathematicians it is not necessary that the two sides of the communication, work on the same field, or the material be mathematically related. Many cognitive achievements have counterpart in many different fields of research. If you want to put this communication in mathematics setting, I propose that people in different fields of research propose problems for the colleagues to solve which are in the language of their colleagues mathematical fields, using the concepts they are comfortable with and about objects they think they have extensive knowledge about. The prerequisite of such a communication is development of culture of writing expository papers and valuing mathematicians which have encyclopedic personality. Of course, not all cognitive personalities allow such a luxury. An encyclopedic cognition is holistic, which is statistically pictorial more often. This means that special kinds of mathematics are in better hands for communication. Because pictorial mathematicians have preferences according to the style of thinking. Note that, it is not the case that, pictorial mathematicians necessarily study only pictorial mathematics. So the key to communication of cognitive mathematicians is to propose research problems for each other.

How to propose research questions

A cognitive mathematician should be blessed with the art of proposing research questions according to predetermined cognitive implications. Research proposals which try to copy cognitive methods from one branch of mathematics to the other branch will also be of significant importance. For example, using the method of (co)homology in many different branches of mathematics is such an example of cognitive contribution to other fields of mathematics. One shall note that it is not the method of computation and proof which is communicated, but the cognitive contribution of shapes being treated as numbers in a number system which is extended to other branches of mathematics. A recent example, is the method of homotopy theory applied to algebraic geometry and type theory by Voevodsky, which are also cognitive contributions, because the method of homotopy has cognitive formulation and that is understanding deformation of space over the continuum. I shall insist that Hilbert`s style of proposing problem, although made many cognitive contributions, but I cannot say that they were an effort of communication of cognitive achievements between different fields. But they were in the line of cognitive contribution of a philosopher of mathematics to mathematics of the mathematics of future, which is initiated by being concerned about fundamentals of mathematics. Hilbert was highly taken by the question of "what is mathematics?"