On The Dialogue between Local and Global

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Abstract: The concept of local and global appeared very late in history of mathematics although it could have appeared earlier. We study the dialogue between these two concepts and some results in mathematics which link local concepts to global concepts. Process of moving from local to global is called superposition.

Introduction

Points on a Riemann-surface and primes of ring of integers of a number field parametrize the local structure associated to these objects. There is a dictionary between function field and number field respecting their concepts of local and global. Yet, the idea of local information and global information comes from local invariants and global invariants in differential geometry. Gauss-Bonnet formula is believed to be the first local to global results appearing in differential geometry. Euler characteristic is one of the first global invariants in topology. In the Gauss-Bonnet formula, the principle of superposition is incarnated as an integral of a local invariant. In number theory and algebraic geometry superposition takes different forms. In this paper, we explore different forms that moving from local to global takes place in many branches of mathematics.

Analogy and interpretation

Maxwell equations have a number of symmetries which are very illuminating. The first one $t \rightarrow -t$ which shows that the direction of time and direction of space are of no hidden source of the higher truth. Nothing happens if you think backward in time or backwards in space. A more complicated symmetry is $t \rightarrow 1/t$ and $x \rightarrow 1/x$ which introduces a one-to-one correspondence between small and large phenomena. This symmetry suggeststhat the concept of local and global are at the same level of abstraction. This is against the expectation that global concepts are more abstract than local concepts. We are suggesting here that local and global concepts have their origin in physics and study of nature. For example, a force field could be generated locally but it is a global concept. Objects that live on the whole space or Space-time have their origin in physics. Global invariants are fairly new except the concepts of measurement. I cannot imagine the length of a stick or area of a land or volume of a box be global concepts. Space of finite
volume are definitely dealing with global concepts, by they appeared in 20 th century. The concept of space actually appeared in second half of 19 th century and is very

recent. In short, the concept of local and global are analogous and they are incarnations of the same higher truth.

Incarnation

To understand mathematical phenomena from a local perspective or global perspective should both make independence sense. Every concept should have global and local incarnations. Gauss-Bonnet says that the local concept of curvature and global concept of Euler characteristic come from the same background truth. The theory of field extensions has both local and global versions both incarnating from the same higher truth. The theory of space extensions have local and global incarnations for Riemann surfaces. We have branched covers and local branching in geometric setting. By the analogy between function fields and number fields both these theories of extensions come from a higher truth. The truth has local versus global incarnations, geometric versus algebraic incarnations, continuous versus discrete incarnations and finite versus in finite incarnations. How can we have access to this higher truth? By analogy between finite and infinite, between continuous and discrete, between geometry and algebra and between local and global. Many of these analogies appear in local to global superposition. This is why it is important to study the dialogue between local and global mathematics. Our methodology to access the truth is elevation from analogies to higher level of abstractions.

Elevation

Elevation is moving from less to more abstract. This is a movement in reverse direction of incarnation. As if you put two carbon copies of the same image on top of each other in order to find more details about the actual scene the pictures are taken form. Incarnation, in this analogy is taking picture, and elevation is superposition of all pictures to get close to that truth behind the pictures. Incarnation from truth is automatic and elevation to the truth is a difficult cognitive step. Thinking in terms of incarnation and elevation has heavy background in a platonic philosophy of mathematics which thinks of knowledge as having several layers of abstraction. The path of cognition is to climb this hierarchy of levels of abstractions. Some levels are out of reach and intuition comes to help the cognition. This is the framework in which we study local concepts against global concepts. We start from physics and then move to geometry and then move towards number Theory and algebraic geometry. At the end, we try to arrange what we learn in the perspective of EPH-mathematics. EPH-mathematics refers to classification of Mathematics to elliptic, parabolic, and hyperbolic mathematics.

Maxwell equations

Heard from local to global symmetry of Maxwell equations from C. Vafa. He referred to this while introducing models of space-time in which expansion of the Universe from some point on, was equivalence unification and contraction. Because in this model infinitely large is the same as infinity small. He declared that he got the idea from symmetries of the Maxwell equation when $t \rightarrow 1/t$ and $x \rightarrow 1/x$. I don't know who noticed the symmetry first, but it is an important observation because it has a groundbreaking interpretations about the truth behind the concept of local and global. Local concepts and global concepts are at the same level of abstraction. Thinking about mathematical truths locally or globally, analytically or holistically, does not affect the level of abstraction of our cognition of the truth. In other words, there should exist a symmetry of truth in which interchange local and global concepts happens, so that after application of these symmetries some local concepts become global and some global concepts become local. The example of local function fields and global function fields is very illuminating. They are in one to one correspondence with points on a curve, where some points are assumed to be infinite points and some others assumed to be finite points. But the concept of finite or infinite point art interchangeable.

Euler characteristics

Euler characteristics is calculated by superposition of curvature, therefore it has information about global curvature of a space. Euler characteristics indeed characterized elliptic, parabolic and hyperbolic surfaces of constant positive, zero and negative curvature respectively. Gauss-Bonnet generalizes to Riemannian manifolds of even dimension and it is called Gauss-Bonnet-Chern theorem, which is a special case of theory of characteristic classes and Atiyah- Singer index theory which asserts that the analytic index does not chang when one varies the elliptic operator smoothly and in fact is equal to a topological index. Riemann-Roch, Hisebrukh Riemann-Roch and Grothendieck-Riemann-Roch are theorems in the same direction which can be understood under the shadow of local to global philosophy. All these theorems have interpretations like Gauss-Bonnet which introduce local counterparts for global characteristics classes. The Euler characteristics first appeared from a triangulation of space and then showing that this invariant is independent of the choices made. We are not able to formulate a local to global picture similar to Gauss-Bonnet in this setup. In fact, this suggests why Gauss-Bonnet is a valuable formula.

Constant curvature spaces

For a Riemann surface being constant curvature is a local condition, but it has global implications. Universal cover of a constant curvature compact Riemann surface is either a sphere, a plain, or Poincare disc, which means that the geometry of the universal covering space is elliptic, parabolic, or hyperbolic. This limits the topologies of constant positive curvature and flat Riemann surfaces to quite a few cases. The spaces of locally constant curvature have the significance that solids could be moved in such spaces without breaking, and this is a significant physical property. Study of constant curvature spaces could lead to a classification of the topology of a space which is a global information. Assuming local characteristics and then trying to give global characterizations is a 20-th century tradition is geometry. The concept of manifold, vector bundle and locally trivial fiber bundle are introduced locally, but obtaining global characterizations of such objects is in demand. Not all assumptions in geometry are necessarily local, and not all conclusions in geometry are necessarily global. But moving from local to global is a popular trend in geometry. It is hard in geometry to visualize why local and global formulations of concepts are at the same level of abstraction. Moving from global to local in geometry looks impossible almost always. For example, one can claim that given such and such global conditions a metric of locally constant curvature exists on that space.

Adelic formulation

In number field case, we have finite and infinite varieties which play the role of finite and infinite points. Adelic formulation is a superposition of local data varying over all completions of the number field. Similar formulation of superposition is applicable to all valuations on the function field. Many arguments in algebraic number theory and algebraic functions theory can be formulated in the language of Adels. A famous example is Adelic proof of Riemann-Roch theorem. There are many examples of superposition in mathematics which lead to global object and global invariants, but they are not necessarily moving from local to global. Homology and cohomology and homotopy are examples of such theories. Adelic formulation is important because there was a global to local approach on the other direction in Tate's thesis which provided us local counterparts of a global object such that the global object can be constructed again by superposition of local counterparts. This is the main issue which gave credit to Adelic philosophy. This approach to L-functions proved to be promising in general theory of automorphic forms, developed by Langlands and motivating his pioneering conjectures about relations between infinite dimensional representations of linear groups, number theory and algebraic geometry.

Number field-function field analogy

The local to global concepts in number fields were originated in the function field theory and then developed for arbitrary commutative ring. The analogy between number fields and function fields originated first from geometry toward algebra, but then eventually the two fields contributed to each other enormously. One of the instances which this concept of local and global in the number fields played an important role was Hasse principle for quadratic forms. Another contribution of the idea that maximal ideals of a commutative ring should correspond to points of a geometric space, was the appearance of Von Neumann rings where there is no concept of a minimal geometric object like a point. This was the origin of the idea that quantum mechanics makes sense in a model for geometry of space where there is no minimal geometric object like a point. This eventually lead to Hilbert space formulation of quantum mechanics that people eventually tried to develop similar theories to number field and function fields for finitely generated field over Q. They made progress in many Diophantine questions and theorems. This gives us hope to be able to develop a similar theory for finitely generated fields over finite fields. Another contribution was the appearance of Arakelov theory which was a milestones in history of connection between algebraic geometry and number theory.

Arakelov theory

The main idea in the Arakelov theory is to treat finite primes and infinite primes differently. One does algebraic geometry over a scheme in finite piece and does differential geometry and complex geometry over in the in finite places. This formulation works for a variety defined over ring of integers of a number field. Arakelov Produced a way to formulate an intersection theory on these varieties which was not available before because is scheme over a spectrum of ring of integers of a number field does not carry a nice intersection theory since it is not compact. Arakelov's idea was to compactify this scheme by doing differential geometry over the infinite places. Arakelov used specifically Green functions to define intersection of two points on a Riemann surface which was revolutionary idea. Arakelov's theories were generalized to higher dimensions by Gillet and Soule and may other collaborators. Eventuallyanalogies of Grothendieck's standard conjectures were formulated and also the theory was used for development of heights which satisfy Northcott's condition in many different cases. Arakelov's idea was a revolution in number theory philosophically and technically. Divisor theory and sections of sheaves were playing an essential role in Arakelov's ideas. These ideas originally come from geometry and it is related to characteristic classes.

Section of vector bundles and cycles

To an algebraic variety, one associates a function field which consists of more or less global functions on the variety. There is a function field formulation of algebraic geometry due to Andrew Weil. Rational functions are global objects, which have divisors associated to them called Weil divisors. Holomorphic sections of vector bundle and divisors associated to them are in a way generalization of the rational function as a global object and associating local divisors to them. Using this setup one defines Chow group and equivalence classes on the set of divisors. Chow groups play an important role in intersection theory on algebraic varieties. Chow groups are global objects but like cohomology theories they are not obtained by superposition of local objects. Only in case of Riemann surfaces a divisor is a finite sum of points which are regarded as local objects. This gives us the idea that may be the set of all irreducible cycles could be thought of analogous of primes and one could develop local to global principal using them. This could be a step ahead in function field-number field analogy extending the theory to finitely generated field over. An example of such theorems are Riemann-Roch theorem which deal with global objects and global environments and the relations between each other.

Riemann-Roch theorems

Many local to global principles was originated by ideas from Riemann. Riemann's mind was pictorial and holistic. Riemann-Roch was originally an inequality and Roch interpreted the difference of the sides of inequality as dimension of a vector space and Riemann-Roch came into being. Riemann-Roch was the content of thesis of Roch under the direction of Riemann. So Riemann has definitely influenced him in the idea that such an equality should be true. Riemann-Roch was generalized to surfaces and then to arbitrary dimension by Hirzebruch and then Grothendieck and Atiyah-Singer gives independent generalizations in K-theory and operator theory. Arakelov came with the idea of arithmetic Riemann-Roch then was generalized by Faltings and Gillet-Soule to higher dimensions. Arakelov noticed that in dimension zero, Riemann-Roch for the spectrum of a number field boils down to Minkowsky's theorem and Dirichlet theory of units. It is a challenge to make Riemann-Roch look like a local to Global result. At the moment it is a relation between global objects which are obtained by superposition of objects which are not necessarily local at the moment. Similar to the case of Chow groups we suggest a formulation of Riemann-Roch which is local and gives the Riemann-Roch theorem as a superposition of local Riemann-Roch formulas.

Universal mapping theorem

Universal mapping theorem states that any simply connected Riemann surface is biholomorphic to sphere, complex plane, or unit disc. This is the second example of trichotomy of elliptic, parabolic, hyperbolic appearing in history after the rise of elliptic, parabolic and hyperbolic geometry. The local to global nature of this theorem is evident. Riemann surfaces which are covered by sphere admit a complete metric with constant positive curvature. There is only the Riemann sphere satisfying this condition. Those Riemann surfaces with complex plane as universal cover admit a complete metric with constant curvature zero. These are complex plane and punctured complex plane and elliptic curves which should have been called parabolic curves. These are many and all have algebraic structure of a group. The Riemann surfaces with Poincare disc as universal cover should all admit a complete metric of constant negative curvature. All this suggest that maybe there is local to global principle holdings for elliptic, parabolic, hyperbolic mathematics. Now we will search for incarnations of local to global principle in each of the elliptic, parabolic, hyperbolic subsections of mathematics.

LG-principle in elliptic mathematics

There is not a well-defined border between elliptic mathematics and parabolic mathematics. For example, are circle and three-sphere elliptic or parabolic? On one hand they are spherical and on the other hand they are Lie groups. But in dimension one the only connected manifolds are circle and real line, so we don't see the EPH-trichotomy. But in dimension three there are definitely concepts of elliptic, parabolic and hyperbolic mathematics. But actually there is Thurston classification of few geometries in dimension tree which could be illuminative. These geometries in dimension 3 are the following: 3-sphere, 3-euclidean space, 3-hyperbolic space, 2-sphere times real line, 2-hyperbolic plane times the real line, SL(2,R), Nil geometry and Solv geometry. Under the Ricci flow 3-sphere collapse to a point, 3-Euclidean space remain invariant, 3-hyperbolic space expands, 2-sphere times real line and Solv geometry converge to a 1-manifold, and 2-dimensional hyperbolic plane times the real line and Nil geometry converge to a 2-manifold. This means that EPH-trichotomy is still present in dimension Three. 3-sphere is in fact elliptic. Seems that all parabolic mathematical objects should have Abelian algebraic structure. In dimensions one all objects are Euclidean hence parabolic. One can see that any point on 3-sphere could be the one point at infinity of a one point compactification of Euclidean space. If we regard finite mathematics as part of elliptic world you could say that counting in a way is local to global process.

LG-principle in parabolic mathematics

All parabolic objects in mathematics have Abelian group structure and therefore choice of a generating set in a way is a local to global principle. Ring of integers should be regards as discrete analogue of and hence also parabolic and same for all higher powers. By this arrangement of concepts elliptic curves and Abelian varieties and semi-Abelian varieties are all parabolic objects. There are many local to global results which come from arithmetic of these objects. For example L-function of elliptic curves decomposes to local points which are defined locally in arithmetic Sense. Integers generates rationals and these have local analoges p-adic integers and p-adic fields. On the other hand, it seems that being parabolic is a global concept and finding local incarnation for that is difficult. Maybe one can say that flat mathematics is parabolic. In this case, Calabi-Yau manifold are Ricci-flat and we expect them to be related strongly to the realm of arithmetic like elliptic curves. I believe that there is a dictionary between knot theory in three-manifolds and curves on Calabi-Yau manifolds extending the analogies between knots and primes. I am suggesting that they should be a theory of modular forms and Langlands program developed for knot theoryand also for Calabi-Yau manifolds. In the case of knot theory, I have already formulated such a theory, and for Calabi-Yau analogue of this is work in progress.

LG-principle in hyperbolic mathematics

Hyperbolic mathematics first showed up in dimension two. There is no zero-dimensional or one-dimensional analogies of hyperbolic space. Even formulation of the global definition for hyperbolic objects is very challenging. Algebraic varieties of general types are a candidate to hyperbolic concepts introduced in algebraic geometry and the theory of hyperbolic groups developed by Gromov is an analogous incarnation in group theory. Free groups are important examples of hyperbolic groups. Universal cover of a connected graph is a point, or a line or a hyperbolic object. Thinking of hyperbolic mathematics as all mathematics in the complement of elliptic and parabolic mathematics is a misleading thought. Hyperbolic mathematics has a character of its own which, in the limit converges to parabolic mathematics. Just as elliptic mathematics converges to parabolic mathematics. To search for a local to global formulation of hyperbolicity is something like search for analogues of the concept of negative curvature is some local format. It defenitely has something to do with deformation of objects. For graphs it is easy to formulate this, but in many parts of mathematics it is a challenge. For example, I have had unsuccessful attempts to develop a profinite theory of hyperbolic groups. I shall stop here by announcing the most difficult open problem I have ever encountered. How to locally characterize phenomena in mathematics which show hyperbolic behavior. This is a question which will make many mathematicians busy for centuries.