On The Dialogue between Continuous and Discrete

Arash Rastegar

Princeton University, Princeton, USA
Sharif University of Technology, Tehran, Iran

Abstract: Analogy between discrete and continuous mathematics is motivation for flow of ideas between these two different approaches to mathematics. This eventually leads to the belief that any piece of continuous mathematics has discrete analogue and vice versa. In this paper, we study the dialogue between these two approaches to mathematics.

Introduction

The first instance when analogy between discrete and continuous come into attention was in my opinion formulation of continuous calculus of integral and derivative by Newton and discovery of the discrete version by Leibnitz at the same time.

It is worthy to note that they were both aware of this analogy. Because discrete version of Taylor expansion was first formulated by Newtown and continuous version was first formulated by Leibnitz. This analogy was discovered in 17th century and was further developed in number theory, numerical analysis and algebraic geometry. One shall regard analogy between algebraic geometry over C and those are $\acute{F}_p$ an analogy between finite and infinite mathematics. Arakelov Theory was another instance when dialogue between discrete and continuous become crucial.

§1 Z as a model for R

Although Z serves well as a model for R but there is hardly any analogy developed between mathematics over Z and mathematics over R in number theory of Z and algebraic geometry of R . But if one treats R as a continuous model for time, Z serves well as its discrete analogue. This can be observed well in analogy between numerical solution of differential equations and continuous analysis. Note that when Z is considered as a model for R as time it is not necessarily then the case that Z embeds in R as a subring. Any embedding as a subset which preserve the order will do.

The points of Z are discrete in R but may converge to a finite point in R or to infinity. Distance between consecutive points could remain constant or exceed linearly or polynomially or exponentially.

Embedding of Z on R as a model for time should preserve order of events and their direction in time, other than that we can have a floating embedding. If one can make calculus for such embedding of Z in R it would be very appreciated for mathematical modeling of the satisfaction function in economics. Z as a subring of R should be studied in the context of Diophantine equations.

§2. Diophantine equations

It looks like Diophantine equations are discrete analogue of algebraic equations. But the phenomena appearing in these two parts of discrete and continuous mathematics is by no means analogous. If we consider rational solution of rational equations, there is a great deal of similarity between the two, but the point is that Z is a ring and not a field. Even considering solutions, which are negative integer, came into consideration after Fermat and Descartes. Algebraic structure of rational solutions to elliptic equations definitely has algebraic continuous analogue. But this phenomenon does not happen over integers. Integer solutions are always finite. This kind of finiteness has not analogue in continuous world. Finiteness of solutions to a polynomial equation of one variable is by no means similar to finiteness of integer solutions. The latter generalize to Bezout's theorem and finiteness in intersection theory over a compact algebraic variety, but this phenomenon has Arakelov theory as its discrete analogue. One could say that functional equations are a continuous analogue of Diophantine problems. But the theory of functional equations is not advanced as much as finiteness theorems appear in it. There is hardly any characterization of functional equations with similar behaviors. Of course, solutions to functional equations could exist in continuous families, but here we are only interested in finiteness results.

§3. Algebraic Lie groups and symmetric spaces

Algebraic Lie groups actioning on symmetric spaces is a field of interactions between discrete and continuous mathematics. The simplest nontrivial example being the actions of $SL_2(R)$ on upper half plane. The phenomena of covering spaces and fundamental group is discussed in a separate section. But the phenomena of congruence subgroups and Hecke operators is another arithmetic phenomena, which hardly has any continuous analogue. Although there is a dialogue between discrete and continuous mathematics in the study of quotients of symmetric spaces under the actions of congruence subgroup but one could hardly find any continuous analogue in arithmetic phenomena. Taking numbers modulo N is in fact the map $Z \to Z / N$ which is analogue of $R \to R/Z = S^1$. Therefore, one could say that tori are analogue of torsions. Some other phenomena in the theory of modular forms, which have analogues in geometric Langlands program, could be regarded as geometric formulation of arithmetic phenomena. Unfortunately

$$SL_2(N) \to SL_2(Z / N)$$

has no continuous analogue like $SL_2(R) \to SL_2(R/Z) = SL_2(S^1)$ but you can form $SL_2(R) / SL_2(Z)$ which is close to the concept of symmetric spaces. May be one could put a ring structure on $\oplus S^1(n \in Z)$ Using multiplication on R which serves our purpose. $\oplus S^1 \to S^1(n \in Z)$ Would be the continuous analogue $Z \to Z / N$. $\oplus S^1(n \in Z)$ Is the same as the group ring G[Z], so it is a sort of well-known object.

§4. Z [i] as a model for C

The role Z plays in R is played by Z [i] in C. Note that $Z^2 \to R^2$ is missing the multiplication structure both on continuous and discrete sides. Solutions of Diophantine equations with coefficients in Z [i] could be analogue of complex solutions of complex equations. One may expect that this can be generalized to ring of integers of any number field but some aspect of Z [i] are missing in general case. One can form again ($S^1 \times S^1$)[Z] which is similar to S^1 [Z] in higher dimensions. But there is extra multiplication structure coming from C. Maybe we should use the new notation (S^1)[Z [i]], which is still in group-ring and not a new object. Note that there is a map S^1 [Z] $\to R / Z$ and an analogue S^1 [Z [i]] $\to C / Z$ [i] one could consider polynomials on S^1 [Z] and S^1 [Z [i]] and that could be thought as analogue of real and complex polynomials. Structure on R^n like dot product and cross product on R^3 should have analogues on S^1 [Z]^n and S^1 [Z]^3. One can think of S^1 [Z] as set of functions $Z \to S^1$ or sequence of elements on S^1. In order to accommodate arithmetic simplicity on S^1 [Z] we can assume that there sequences have only finitely many non-zero elements. But without this assumption, we could have defined multiplication also.

§5. Discrete and Continuous dynamical systems

Moving from continuous to discrete dynamical systems is very easy: just replace time R with its discrete analogue. The iverse direction is easy by extending the total space. The discrete dynamics of f: $M \to M$ can be replaced by a continuous analogue which obtained by (Mx I)/Mx {1} $\to$ Mx {0}. If you can consider the discrete analogue of this continuous dynamical system, you reconstruct f if you make appropriate choices. This idea can be used to associate a zeta function to periodic points of a dynamical system, which is analogues to Selberg zeta function associated to closed geodesics. The analogy between concepts in the fields of discrete and continuous dynamical systems are well -know and popular and extended all over these research fields which are developed parallel to each other. In this setting time is discrete or continuous but space is continuous. One can think of discrete and continuous space-time and physics over discrete space-time. The goal is to see continuous physics as a limit of discrete physics and comparing these two features of modeling the natural phenomena. All such analogy started from the discrete and continuous versions of calculus developed by Newton and Leibniz.

§6. Discrete versus continuous calculus

We already noted the fact that the role of Newton and Leibniz in originating discrete and continuous versions of calculus was mixed. During the first half of the 20th century discrete model were used for mathematical modeling of natural problems and were taught in universities. In the second half engineers started to learn that continuous models work as well. Numerical analysis is all based on discrete model and on interaction between discrete and continuous models. In fact, the discrete model of calculus embeds inside the continuity model. Working with discrete space, which does not live inside a continuous realm, is an almost impossible cast. There are physicists working on discrete space-time, which does not include in the dialogue between discrete and continuous we are considering here. These interactions are discussed in sequel sections with the titles, limit of finite structures, the analogy between sum and integral, the analogy between difference and derivative, on the convergence of series, series as functions from Z to R and discrete holomorphic functions.

§7. Discrete holomorphic functions

Dodziak has a number of papers developing the discrete counterpart for the notion of holomorphic functions. They came to my sight by introduction of my teacher Siavash Shahshahani while we have a discussion on the existence of discrete analogues for every continuous idea in mathematics. Dodziak works with functions over Z^2 which have incredible order and geometry. It is difficult to extend this work to arbitrary discrete subsets of C since residing in Z^2 is a feature which should be present in such a generalization. In fact, this feature is a blessing because it allows continuous model to be interpreted as the limit of discrete models. Same principle hold in statistical mechanics, which we will study in another section. Analogue of complex derivation exists in the discrete models and in the limit coincides with classical complex derivation. This limiting principle is an aspect of the dialogue between discrete and continuous. Every continuous concept has a discrete analogue such that in the limit it approaches the original concept. This is called the principle of continuity. In some parts of mathematics, this is called discrete deformation, or discretization of a concept. It is not at all the case that every discrete concept has a continuous analogue in the limit. Many discrete structures do not even fit in a system, which accepts continuous model as a limit.

§8. Sequences as functions from Z to R

One of the main concepts that can be discretized is the concept of a function. Function is an extension of the concept of number. In fact, a function is continuous deformation of or a continuous family of numbers. This concept can be discretized in two steps. First step is to consider function from Z to R and second step is to integer sequence. Analogy between integer sequence and continuous functions is very deep and is hardly explored. But analogy between sequences and functions is well-known. Sum of the elements of the sequence is analogue of integral of functions and difference operator is analogue of derivative polynomials sequences and polynomial functions are analogous objects, which can be induced from R into Z if we fix an embedding of Z in R . We will focus on each of these aspects in a separate section. Integer sequences could also be polynomial or exponential or periodic. But there are aspects in integer sequences, which hardly generalize to continuous model like multiplicative sequences and rise of prime numbers. The arithmetic world is much richer the analytic world. This is why the analogy between knots and primes is a simplifying model in direction of translating to geometry. Complexity of knots is combinatorial but complexity of prime numbers is deeper.

§9. Limit of finite structures

This is a general dialogue between finite and infinite mathematics but in arithmetic, it can be regarded as a transformer of mod p phenomena to p-adic analogues. P-adic analogue in turn can be regarded as analogue of continuous phenomena. Mahler has a p-adic analogue of Taylor series and analogy between mathematics over R and over p-adic completions of Q which are denoted by Q_p. For example, there are physicists trying to do physics over Q_p or algebraic geometers trying to do algebraic geometry over the completion of the algebraic closure Q namely $\hat{\bar{Q}}_p$. This art is called rigid geometry. This is another limiting procedure which embeds discrete objects into p-adic objects and the p-adic objects are analogues to structure over R. All concepts in calculus have p-adic analogues. This is the second route of dialogue between discrete mathematics and continuous mathematics, which passes through dialogue between finite and infinite mathematics. If someone could see beyond the analogy between p-adic numbers and real numbers or beyond the continuous and discrete models or beyond the finite and infinite models of mathematical incarnations of truth, he or she has a very pure understanding of the mathematical truth.

§10. The analogy between sum and integral

This analogy was evident even when the concept of integral was invented. The symbol of integral is a large skew S. Series could be convergent or divergent but analogy between sum and integral is beyond the convergent series and convergent integrals. Archimedes developed the concept of sum of infinitesimals, which was translated to the concept of integral of a function after Newton and Leibniz. The fact that sum of infinitesimals is analogue of a discrete sum of numbers, which is called series. Area and sum of a series could both be finite or infinite. Double integrals are analogues of double sums. Telescope sums and derivative of integral are also analogues, which will be discussed in next section. Measure theory is an attempt to unify these analogous concepts. Also, sum of functions as an analogue of sum of numbers has continuous analogue in terms of integral, which in turn is easily formulated in terms of measure theory. Formula combining infinitesimals and integrals are hard for me to interpret. There are many features of the infinite mathematics, which are unknown to me. This is why I concentrate in convergent sums and convergent integrals and I put aside a huge body of mathematics about divergent series produced by Euler.

§11. On the convergence of series

The concept of a two- dimensional region in plane, which is of finite or infinite area, and finite or infinite boundary and the concept of a three-dimensional region in space of finite or infinite volume and of finite or infinite boundary area, could be modeled in discrete mathematics. Boundary operator is analogue of derivation. There could be that a series is divergent or convergent and the derivative of the series is divergent or convergent. For a Taylor series, these complications do not happen. A Taylor expansion is convergent if its derivative is convergent. But the fact is that a Taylor series is a very simple series of functions. The idea of telescope has continuous analogue, which is in fact the fundamental theorem of calculus. Fundamental theorem of calculus generalizes to Stokes theorem. I am not aware that how deep discrete analogues of Stokes theorem are proved or even formulated but it should all be under the shadow of analogy between difference and derivative. I think discrete analogue of Stokes theorem should involve a discrete analogue of differential forms, which according to my limited knowledge is not formulated yet.

§12. The analogy between difference and derivative

This analogue leads to solution of difference equations, which is discrete analogue of solutions of continuous differential equations. This theory is an important part of numerical analysis. Finite element method is also a similar analogous finite analogue of infinite systems. The latter analogy fits into the dialogue of finite and infinite. But here we are concerned about the dialogue between discrete and continuous. Stokes theorem suggest that counting on a discrete realm on arbitrary manifold should be possible to be formulated. I propose to work on lattices on affine manifolds. The complex analogue of affine manifold is hard to deal with since the operator $Z \rightarrow \frac{1}{Z}$ does not preserve a lattice like Z [i] which brings complications. If we work with logarithmic embedding of Z into $R^{¿}$ we may get into somewhere, but how to formulate the higher dimensional analogue is not clear to me. In other words, what is the complex analogue of lattices on an affine manifold? The quotient of $C^{¿}/Z$ where Z is multiplicatively generated by a non-unit element is compact and this has higher dimensional analogue. May be we can patch together some open subsets of C^{n}/Z^{n} in a way that we get a lattice of locally compact quotients.

§13. Statistical mechanics

One can do statistical mechanics over a Riemann surface or over any manifold. One can do physics over arbitrary manifold and make the statistical computation converge in the limit to continuous way of doing mechanics. Usually this model is used to understand the local to global behavior of gas and get hands on concepts of thermodynamics but it can be done for any piece of classical mechanics. Although this approach originated under the philosophy of local to global but it can be definitely viewed under the shadow of dialogue between discrete and continuous. You can consider solids and liquids as superposition of large number of points and assuming local correlations between them and deduce global formulas of classical mechanics by superposition. Same formulas can be used for models of continuous objects and this forms a dialogue between discrete and continuous by forming an analogy. The same goes between any two discrete and continuous models of any phenomena in nature. There are abstract analogues of statistical thinking on Riemann surfaces or on higher dimensional spaces, which deal with this concept, which are not necessarily generated from nature. The sources of ideas are both from nature and metaphysics. It is impossible for human to create a concept from nothingness.

§14. Fundamental group and covering spaces

This is one the most important dialogues between discrete and continuous. In some special cases, this happens in Lie groups and symmetric spaces also. This theory was originated by Poincare. Fundamental group is the discrete object approximating the quotient space. For example, hyperbolic, elliptic and parabolic Riemann surfaces can be determined completely by their fundamental group and in fact, this is how the concept of hyperbolic groups was originated by Gromov. The hierarchy of covering spaces is very similar to hierarchy of finite field extensions. One can think of this analogy in the formalism of analogy between function fields and number fields, but the analogy between discrete and continuous is not present in this formulation. The general format is properly discontinuous action of a group on a space, where one considers the group as being a discrete analogue of the space. In fact, the relative space obtained by quotient of the original space by the action of the group can be thought as geometric analogue of group acting. It is not that any distribution of discrete points on a continuous space could serve as an analogue between discrete and continuous objects. This distribution should be optimal in some sense.

§15. Distributions of points on a space

What we mean here is distribution of finitely many points or a discrete set of points on a space. This is different from equidistribution of an infinite set. This is very similar to a discrete realm inside a continuous realm as was theorized by Riemann on his paper on philosophy of spaces. One example would be distribution of internet nods on the surface of earth and all the problems of communication related to that. Another example is distribution of data to mobile phones of customers using a network of correlative antennas. An abstract example would be if we want to distribute n points on a unit sphere, an asking under which pattern the minimum distance of two of the points is maximum. The question is how well the discrete space serves as a model for continuous space. One can ask the same question for infinite in a non-compact space. Suppose every unit disc contains a point of the discrete space, for what patterns of the discrete space the minimum distance between its points is maximum. Imagine this question over a hyperbolic Riemann surface, so that you see how non-trivial it may geometrically look. One can use physical intuition to understand these examples better. If n points on a unit sphere repel each other what are the stable patterns. Can you formulate the analogue for infinitely many points.

§16. Packing and circle packing

Packings of centrally symmetric objects inside centrally symmetric spaces a nice example of well distribution set of discrete points on a continuous realm. For example, packing of sphere by equal balls, packing of circle by equal disks, or packing of square by equal squares. Packing by non-equal objects is much more complicated and much more interesting. It would be more interesting if we consider mobile case when a few points are given inside a circle and they grow at the same time and push each other when these grown circles touch and we ask what is the final pattern of such a packing. One can introduce an analogue problem of packing for squares inside a square or rectangles inside a rectangle. Of course, this problem makes sense for any set of small particles on ε-neighborhood growing around them and letting them push each other away. This problem is mathematically interesting but not related to the dialogue between discrete and continuous. A particular example of packing is tessellation of a square by not necessarily equal squares or tessellation of a square by not necessarily equal rectangles. The centers of tile are much better approximation to the ambient space rather than those given by packings, which are not tessellations.

§17. Modular forms

Modular forms are related to symmetric spaces, which are related to the dialogue between discrete and continuous, but there are other hidden aspects of this dialogue in the world of modular forms. Modular forms, which are eigenforms of hecke operators, serve as arithmetic objects, which are highly rigid and discrete. Part of this rigidity is due to q-expansion and part of this rigidity can be understood in the language of L-functions, which we will study in corresponding sections. But the main feature we want to mention here is the Shimura-Taniyama-Weil conjecture proved by Andrew Wiles. Langlands was the first person who believed that all motives over number fields are modular and Shimura-Taniyama-Weil conjecture was trivial to him. Motives over number fields are arithmetic objects and one cannot have a continuous family of them. If we assume they are defined over ring of integers of a number field it is more clear why these arithmetic objects are discrete. On the other hand, modular forms form a vector space, which is generated, by arithmetic modular form. By arithmetic modular form, we mean those having q-expansions with integer coefficient. This is a dialogue between discrete and continuous. A lattice in a vector space are made exactly those elements, which are defined over ring of integers.

§19. q-expansions

All versions of modular forms that I know, elliptic modular forms, Hilbert modular form, Siegel modular forms, Jacobi modular forms, have a q-expansion formulations. Modular forms have many different formulations. I know of automorphic, moduli, q-expansion, symmetric space, Galois representation formulations. q-expansion among other formulations has the feature of being local. These modular forms with integer coefficients in q-expansion generate the vector space of all modular forms. But they are not necessarily the same as modular forms defined over Z in the moduli space. Finding a geometric lattice generating the space of modular forms is a challenge. Action of Hecke correspondences on modular forms can be translated to the language of q-expansion, which is the case in all different formulations of modular forms. Of course, these correspondences are arithmetic so they also act on modular forms with q-expansion with integer coefficients. This action can be extended to modular forms with arbitrary coefficient. A vector space admitting invariant lattices under group actions is a dialogue between discrete and continuous. In this setting, it says that the study of the discrete actions precedes the study of continuous actions because it has arithmetic structure.

§20. L-functions and Zeta functions

L-functions and zeta functions are arithmetic objects. We cannot have a continuous family of L-functions with Euler product, or else the Riemann hypothesis cannot be true. This discrete nature of L-functions does have continuous analogue or anything to do with continuity in the same sense as q-expansions. L-functions are not a discrete set inside a continuous realm. On the other hand l-functions are analytic functions with zeros and poles and this is a continuous feature these objects. Special values is again a link and a dialogue between continuous and discrete, especially when one tries to interpret the special value in terms of the motive or modular form the L-function is associated to. There should be on the other hand zeta functions associated to all arithmetic objects for example finitely generated function fields over Q or over finite fields or any other arithmetic structure, through associating a arithmetic variety on the way or not. For example, one associate zeta functions to Galois representations and so on, even if they don't come from geometry and we expect special value be interpreted in terms of invariants of arithmetic objects. This is the general dialogue between continuous and discrete associated to all arithmetic objects.

§21. Homology and cohomology over Z with torsion

Homology and cohomology with integer coefficients gives a lattice inside the corresponding homology and cohomology with coefficients in R and in a sense it is the same analogy between discrete and continuous we discussed before, but there is a new feature here and that is the torsion part which is eliminated where one tensors with R. Although torsion is also corresponding to a geometric concept, but is cannot be considered a discrete object approximating a continuous realm. This brings us to a new realm of finite objects which are discrete and do not necessarily have continuous analogue. Linear groups over finite fields have analogues in other fields, but this analogy is not of the same type of the analogies we discussed before. There are many finite groups for example that are not related to continuous objects directly. So, this is a philosophy to find continuous objects analogues to every piece of finite structure, and in this practice many new kinds of analogy between discrete(finite) and continuous will arise. Cyclic groups are analogues to S^1 and any groups can be thought as symmetry of geometric objects or finite subgroups of symmetries of objects with more symmetry. There is in fact an ambiguity in the claim that cyclic are approximating S^1 , because direct sum of cyclic groups should be analogues to direct sum of S^1 's but sometimes direct sum of cyclic group is cyclic.

§22. Z_m as a model for circle S^1

We know that if $m=p_1^{\alpha_1}\dots p_k^{\alpha_k}$ is prime decomposition of m then Z_m = $\bigoplus Z_{p^{\alpha_i}}$. Therefore, we have to understand $Z_{p^{\alpha_i}}$. The question is $\bigoplus Z_{p^{\alpha_i}}$ is always cyclic, but what about $Z_{p^\alpha}\bigoplus Z_{p^\beta}$. For example what shall we say about $Z_p\bigoplus Z_p$? Is it analogue of S^1 or analogue of $S^1\bigoplus S^1$ = T^1 ? Or both? The latter seems unreasonable. We can solve this riddle by considering $Z_{p^\alpha}\bigoplus Z_{p^\beta}$ as a subgroup of the smallest Z_{p^γ} containing both. Then we get a map $Z_{p^\alpha}\bigoplus Z_{p^\beta}\to Z_{p^\gamma}$. There is another approach as and that would be embedding finite Abelian group $\bigoplus Z_{p^{\alpha_i}}$ as a subgroup of the least number of cyclic groups possible and then we will have a new concept of rank of a commutative finite group. I think the best choice will be starting with the assumption that any cyclic group is analogues to S^1 and extensions of cyclic groups are analogues to extension of S^1 . One can relate this to covering spaces and fundamental groups, which we discussed before and also relate them to symmetric space and also there is an analogue of hecke operators. On all these formulations, the corresponding dialogue between discrete and continuous holds.